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Energy of Graph: A Review

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Abstract:

Graph energy is a concept that has attract\ted a lot of interest in graph theory since it can describe the fundamental structural properties and connectivity patterns of graphs. In the present study, the author reviewed various articles and concluded the exponential rise of graph energy research from the year 2000 to the year 2024, showing how it varies from being a specialized discipline to a prominent one in science. Its rapid development and expanding uses are marked by a growing global interest and importance across scientific and commercial sectors, as mentioned in the present research.

Keywords: Graph Energy, Eigen values, Adjacency Matrix, Laplacian Matrix, Graph Theory

1. Introduction

In 1978, one of the current authors (I.G.) established a new graph spectral term called graph energy [1]. Let G be a simple graph of order n. Let A(G) represent its adjacency matrix.
The eigenvalues of **A**(G), denoted by $\lambda_1, \lambda_2, \lambda_3, \ldots\ldots, \lambda_n$, form the spectrum of G [2].

Definition 1.1. The energy of the graph G is

$$E(G)=\sum_{i=1}^{n} |\lambda_i|$$

This definition was inspired by previous findings for the Huckel molecular orbital total-π electron energy [3]. The author of Definition 1.1 presented it with the expectation that the mathematical community would realize its relevance, sparking future research and leading to the discovery of several other discoveries. Despite the author's repeated attempts to popularize the graph energy concept, there remained a lack of enthusiasm. Other mathematicians mostly ignored the graph energy hypothesis for more than twenty years.

The adjacency matrix is a fundamental representation that makes connectivity patterns simpler. Graph theory offers a strong framework for modeling links and connections inside systems through graph structures [4]. Because it provides a clear and flexible means of expressing the interactions between vertices in a network, this matrix is essential to the study of graph theory. The square adjacency matrix's rows and columns match the vertices of the graph. Its components show if there is an edge between two vertices; an edge is present when its value is 1, and it is absent when its value is 0. The adjacency matrix in undirected networks is symmetric, mirroring the symmetry of the edges connecting the vertices. On the other hand, the asymmetry of the matrix in directed graphs might represent the directed character of the edges. The adjacency matrix and Laplacian matrix diagonal entries take self-loops—edges that join a vertex to itself—into account as well. The analysis becomes more difficult as a result of these self-loops, which can affect different matrix representations and graph features [5].

Owing to its unambiguous and binary representation of graph connection, the adjacency matrix has gained significant traction across multiple fields, including computer science, network analysis, and operations research. It is crucial for defining common graph operations such as vertex degree, connected components, and pathfinding algorithms because to its simplicity of calculation and manipulation using algorithms.

Graph theory explores relationships using graph structures; one important tool for modelling connectivity patterns is the adjacency matrix. For undirected graphs, the binary values (0 or 1) in this square matrix indicate the presence of edges between vertices; for directed graphs, the elements are asymmetric. It is a key component of network research and computer science, enabling basic graph operations like component identification, pathfinding, and vertex degrees [6]. Because of its compactness and algorithmic efficiency, the adjacency matrix serves as the foundation for a number of algorithms and analyses that help with understanding and optimizing graph features across disciplines. Because of its simplicity and adaptability, it is a vital tool for manipulating, analysing, and visualising graph structures. As a result, it has made a substantial contribution to the field of graph theory and its practical applications.

2. Literature Review

Over the years, academics have made great strides in the study of graph energy, examining a variety of graph energy-related topics and characteristics. This survey of the literature highlights important discoveries and advancements in the subject of graph energy research, demonstrating the growth of our understanding.

- **Akbaria et al. (2023), [7]** In this paper author discuses about the formula $\sum_{uv\in E(G)}\sqrt{d_u^2+d_v^2}$ for each edge u and v in a graph G gives the Sombor index, where du and dv are the degrees of vertices u and v, respectively. E(G) represents the energy of G, which is the total of all the eigenvalues' absolute values in its adjacency matrix. $E(G)$ is less than the Sombor index, $SO(G)$, for graphs with at least three orders, as previous studies show. By proving that $E(G)$ is smaller than or equal to $SO(G)$ for a connected graph G of rank n (apart from pathways $P_n(n \leq 8)$ this study improves on the previous result $E(G) \leq \frac{SO(G)}{2}$
- **Zahir & Mohammadi (2023), [8]** An adjacency matrix's total of a graph G's singular values is its energy, which is equal to the sum of its absolute eigenvalues. Let $x + y = z$ be the sum of the matrices x, y, and z. An inequality between the sum of the singular values of z and the whole sum of the singular values of x and y is established by the Ky- Fan theorem. This theorem produces several new inequalities and offers new evidence for a few previously known ones when applied to the notion of graph energy.
- **Firdowasi (2023), [9]** The use of graph energy in Data Science received greater attention in the last few years. The graph that's related to the district map of Tamil Nadu is the main subject of this study. After the district-associated graph's adjacency matrix is built, a 32 × 32 square matrix is produced. This adjacency matrix is used to calculate the energy graph, and the effects of its energy values are carefully investigated. This analysis clarifies the characteristics of the routes that link the districts that are being studied.
- **Tanga et al. (2023), [10]** This research paper focus on the E(G), denoted as $E(G)$ is determined by the sum of the absolute values of all of its eigenvalues. An intriguing and challenging problem in graph theory, introduced by Gutman in "E(G): old and new results," seeks to characterize the graphs G and their edges e for which the E(G) with the edge removed, $E(G - e) < E(G)$. In this research, author present a new sufficient condition for $E(G - e) < E(G)$, where the edge e is not required to be a cut-edge set or a cut edge. This development has the potential to extend some well-established findings in the field.
- **Rathera & Imranb (2023), [11]** An "adjacency matrix" $A(G)$ of a graph G with $V(G) = \{v_1, v_2, \dots, v_n\}$ and degree sequence $(d_{v1}, d_{v2}, \dots, d_{vn})$ is a (0,1) square matrix of order n with ij^{th} entry 1, if v_i is adjacent to v_J, and 0, otherwise. Sombor matrix $S(G) = (s_{ij})$ is a square matrix of order n, where $s_{ij} = \sqrt{d_{vi}^2 + d_{vj}^2}$, whenever v_i is adjacent to v_j, and otherwise 0. Energy of graph is equal to the sum of the absolute values of the eigenvalues of $A(G)$, and the Sombor energy of G is equal to the sum of the absolute eigenvalues of $S(G)$. In this notes author provide counter examples of the upper bound.
- **Gutman, et al. (2023), [12]** The Sombor index, denoted as SO, is a topological index for a graph G that relies on vertex degrees. It is calculated as the sum of the squares of the degrees of all vertices in G, paired with the sum of the squares of the degrees of the vertices connected by edges. In mathematical terms, SO is defined as $SO = \sum_{uv\in E(G)}\sqrt{d_u^2+d_v^2}$ where d_u represents the degree of vertex u, and $E(G)$ is the edge set of G. Bounds for the Sombor index are established in relation to graph energy, the size of the minimum vertex cover, the matching number, and the induced matching number.
- **Popat & Shingala (2023), [13]** Self-loops are attached to the σ vertices of the graph G to create the graph G_σ. The formula used for the energy $E(G)_\sigma$ of the graph G_σ with order n eigenvalues $\lambda_1, \lambda_2, \dots\dots, \lambda_n$ is $E(G_\sigma) = \sum_{i=1}^{n}$

Lλ $-\frac{\sigma}{n}$L$_i$. It has been demonstrated

that $E(G) = E(G_\sigma)$ if $\sigma = 0$ or n. The obvious question is raised: Is there a graph where $0 < \sigma < n$ and $E(G) = E(G_\sigma)$ exists? The author found a yes response to this query and has provided a graph family that meets this condition.

- **Ulker et al (2022), [14]** The sum of the absolute values of the eigenvalues in a graph's adjacency matrix is known as the graph's energy. A new topological index of a graph is called the Sombor index. In this paper, Author explored the relationships between a graph energy and Sombor index in terms of its degrees. A regular graph's upper bound is also provided in terms of its Sombor index

- **Ghodrati (2022), [15]** In this paper author show some upper bounds on the energy of a network in terms of degrees, average 2-degrees, and number of common neighbours of its vertices using Hadamard's inequality and its generalization. Additionally author prove an inequality that connects the energy of a graph to that of an arbitrary subgraph contained within it.

- **Akbari et al. (2022), [16]** The total absolute value of each eigenvalue in the "adjacency matrix" of a graph G is its energy, or $E(G)$. For every graph G with maximum degree
$\Delta(G)$ and lowest degree $\delta(G)$ whose adjacency matrix is non-singular, the following equality is conjectured to apply: $E(G) \geq \Delta(G) + \delta(G)$, required and only if G is a complete graph. Let G be a graph that is linked and has an edge set $E(G)$. In this study, they first prove that, for any given G, $E(L(G)) \geq |E(G)| + \Delta(G) - 5$. Here, $L(G)$ stands for G's line graph. Next, the author uses this finding to demonstrate the conjecture's validity for any connected graph's line with at least seven orders of magnitude.

- **Gutman et al. (2021)**, **[17]** Author considered the energy of self-loop-containing graphs. If the graph G of order n has self-loops, then its energy is defined as $E(G) =$
$\sum|\lambda_i - \sigma / n|$where $\lambda_1, \lambda_2, \ldots \ldots, \lambda_n$are the eigenvalues of sic features of $E(G)$ are established, and a number of unsolved issues are mentioned or predicted.

- **Filipovski & Jajcay (2021), [18]** The Graph G has n vertices and m edges, with the highest degree denoted as $\Delta(G)$ and the lowest degree as $\delta(G)$. The Adjacency matrix A represents the connection in G, and the eigenvalues of G are $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$. The energy of G, $E(G)$, is the sum of the absolute values of these eigenvalues, meaning
$E(G) = |\lambda_1| + \ldots + |\lambda_n|$. It is known that $E(G)$ is at least twice the minimum degree,
$E(G) \geq 2\delta(G)$. Akbari and Hossein Zadeh proposed that for graphs where the adjacency matrix is non-singular, the energy is greater than or equal to the sum of the maximum and minimum degrees, that is $E(G) \geq \Delta(G) + \delta(G)$. In this paper author offer a proof of this conjecture for hyperenergetic graphs and establish an inequality that supports the conjecture. Author also derived various bounds for E(G), using elementary inequalities and their application.

- **Paramadevan & Sotheeswaran (2021), [19]** This research paper focuses on the examination of a graph's adjacency matrix, which is a fundamental matrix linked to every graph. Investigating the properties and theorems related to the adjacency matrix is essential, especially considering that computers store graphs in terms of their adjacency matrices.

- **Gowtham & Narasimha (2021), [20]** This paper focus on Sombor index ($SO(G)$), introduced by Gutman in the context of chemical graph theory, is a topological index based on vertex degree denoted by $SO(G)$In this study, we develop a new kind of graph energy called Sombor energy ($ES(G)$) and provide a novel matrix for a graph G, which they call the Sombor matrix. Interestingly, there is a significant relationship between the Sombor matrix and known degree-based topological indices that are sometimes called forgotten indices. A good correlation coefficient (r = 0.976) is obtained by comparing the total π-electron energy of some compounds containing heteroatoms with their $ES(G)$ values. The work also provides characterizations and constraints for the Sombor energy of graphs and the maximum eigenvalue of the Sombor matrix.

- **Vaidya & Popat (2017), [21]** A graph $G's$ eigenvalue is equal to the adjacency matrix's eigenvalue. Summing the absolute values of these eigenvalues provides the energy of
G, or $E(G)$. This introduces an obvious question: What is the best way to relate the energy of a given graph G to a graph that is generated from G using a particular graph operation? Author have looked at two specific graphs to answer this question: the splitting graph, which is represented by the symbol $S'(G)$, and the shadow graph, which
is designated as $D_2(G)$. The conclusive results of the study show that $E(D_2(G)) = 2E(G)$ and $E(S'(G)) = \sqrt{5}E(G)$.

3. STATISTICS

The number of articles on E(G)began to rise sharply around the year 2013. Table 1 and Figure 1, which display the distribution1 of graph-energy-papers by year over the previous two decades, provide evidence of this pattern.

Table 1. Number of papers on graph energies published in the last twenty-five years.

Year	No. of Research	Year	No. of Research
2000	5	**2013**	107
2001	6	**2014**	117
2002	9	**2015**	164
2003	11	**2016**	170

2004	13	**2017**	209
2005	29	**2018**	280
2006	17	**2019**	315
2007	23	**2020**	410
2008	28	**2021**	434
2009	37	**2022**	528
2010	31	**2023**	563
2011	95	**2024**	520 till now
2012	95		

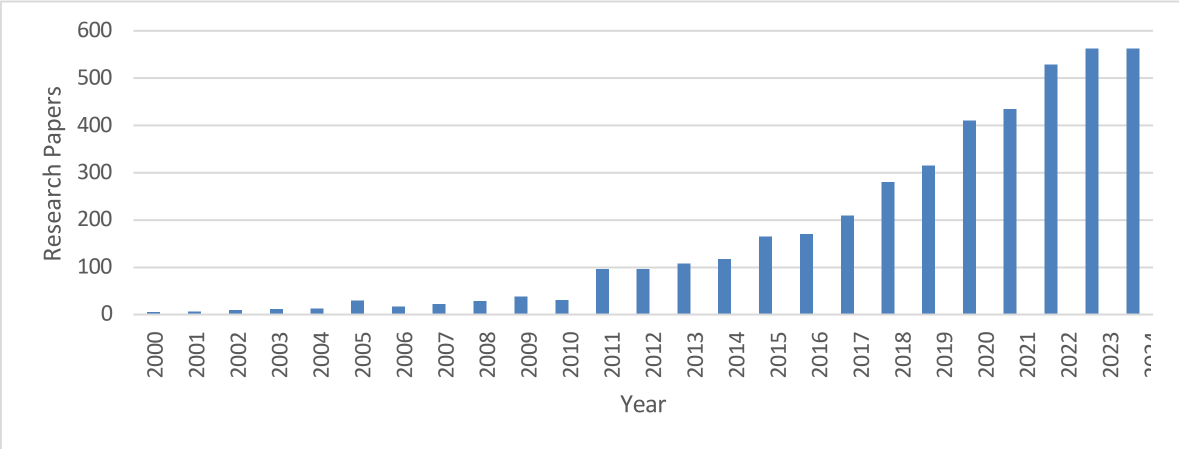

Figure 1. Yearly distribution of published graph energy papers.

The information provided demonstrates a distinct upward trend in graph energy research activities throughout time as shown in figure 1. The field has grown significantly and attracted significant attention since its modest beginnings in the early 2000s, as shown by the marked increase in research publications over time. The quantity of research articles increased steadily between 2000 and 2024, with certain years experiencing dips but an overall rising direction. The research output increased considerably in 2000 and 2024, suggesting a spike in interest or advancements during that time. A more notable increase in research activity occurred between 2000 and 2024, with nearly twice as many papers published than in the previous ten years. This might be ascribed to more financing, developments in technology, or newly discovered uses that sparked the field's research. There is a notable increase in the number of research papers from 2015 to 2024— approximately more than doubling every few years. This exponential increase is a reflection of graph energy's expanding significance across a range of scientific fields and commercial sectors. The tendency is still present in 2023, as evidenced by the large number of research articles that have been published so far this year, and continuing the trend as seen in the year 2024 (Till now). Demonstrating the field's continued interest and progress in graph energy research. All things considered, the data shows how graph energy research has developed from a specialized topic to a dynamic and quickly growing one that draws interest from scholars all over the world and makes a substantial contribution to the body of knowledge and applications in graph theory and related fields.

3.1 Expanding Studies into Graph Energies

In recent years, a multitude of new graph energies have arisen in the literature. We just provide their names here; further information and references can be obtained in the [4]. This list now includes roughly 25 different types of graph energies, with more to come. Thus, in addition to extended, distance, Laplacian, and Randi´c energies, we have: color energy, color Laplacian energy, color signless Laplacian energy, Randi´c energy, complementary distance energy, e- energy eccentric, Laplacian energy, edge energy, Hermitian energy, Hermitian–Randi´c energy, Laplacian distance energy, Laplacian incidence energy, minimum-covering Gutman energy, path energy, path Laplacian energy, Randi´c incidence energy, Seidel energy, skew energy, skew Randi´c energy, skew Laplacian energy, so-energy, vertex energy, vertex degree energy etc.

(a) **The extended energy** is calculated by adding the absolute values of the eigenvalues of the extended adjacency matrix.

(b) **The Laplacian energy** of a graph with order n and size m is equal to the sum of the absolute values of the eigenvalues of $\boldsymbol{L}(G) - \frac{2m}{n}\boldsymbol{I}_n$, where $\boldsymbol{I}_n$ is the order n unit matrix.

(c) **The distance energy** of a linked graph is calculated by summing the absolute values of the distance matrix's eigenvalues.

(d) **The Randic energy** sums the absolute values of the Randic matrix's eigenvalues.

4. Conclusion

The graph energy research has grown at an impressive rate between 2000 and 2023, as evidenced by the statistics. After a slow start in the early 2000s, the field had a sharp spike in publications starting in 2015, which suggested more attention and progress. The exponential increase from 2015 to 2024 (Till now) highlights how graph energy is becoming more and more relevant in a variety of fields. This pattern illustrates how the discipline has developed from a specialized area to a dynamic and important area of scientific study. The large volume of research articles in 2023 is indicative of continued innovation and momentum, underscoring the long-term importance and promise of graph energy in solving difficult problems and expanding our understanding of graph theory and related subjects.

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