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On the Idempotent Representation and Singularities of Tricomplex Numbers

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A B S T R A C T

Tricomplex numbers, as a natural extension of complex and bicomplex systems [cf. 1, 2, 3, 4, 5, 6, 7, 8, 9, 10], possess a rich algebraic structure with wide-ranging applications in physics, signal processing, and multidimensional analysis. A central tool for their study is the idempotent representation, which simplifies algebraic operations by decomposing elements into orthogonal components. In this paper, we develop a comprehensive framework for the idempotent representation of tricomplex numbers, establishing its algebraic foundation and canonical forms. Special emphasis is given to the characterization of singular and non-singular elements within this framework, along with conditions for invertibility and non-invertibility in tricomplex algebra.

Keywords: Tricomplex Numbers; Idempotent Elements; Idempotent representation; Cartesian idempotent sets; Singular and non-singular elements; Algebraic decomposition.

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1. Introduction

The set of Tricomplex Numbers defined as:

$\mathbb{C}_3 = \mathbb{C}(i_1, i_2, i_3)$

$\quad = \{x_1 + i_1x_2 + i_2x_3 + i_3x_4 + i_1i_2x_5 + i_1i_3x_6 + i_2i_3x_7 + i_1i_2i_3x_8 : x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8 \in \mathbb{C}_0\}$

Where $i_1 \neq i_2 \neq i_3$; $i_1^2 = i_2^2 = i_3^2 = -1$, and $i_1i_2 = i_2i_1, i_1i_3 = i_3i_1, i_2i_3 = i_3i_2$.

We present the following sets that are frequently utilised:

$\mathbb{C}_0 =$ Set of Real Numbers

$\mathbb{C}(i_1) = \{u + i_1v : u, v \in \mathbb{C}_0\}$

$\mathbb{C}(i_2) = \{u + i_2v : u, v \in \mathbb{C}_0\}$

$\mathbb{C}(i_3) = \{u + i_3v : u, v \in \mathbb{C}_0\}$

$\mathbb{C}(i_1, i_2) = \{x_1 + i_1x_2 + i_2x_3 + i_1i_2x_4 : x_1, x_2, x_3, x_4 \in \mathbb{C}_0\}$

$\mathbb{C}(i_1, i_3) = \{x_1 + i_1x_2 + i_3x_3 + i_1i_3x_4 : x_1, x_2, x_3, x_4 \in \mathbb{C}_0\}$

$\mathbb{C}(i_2, i_3) = \{x_1 + i_2x_2 + i_3x_3 + i_2i_3x_4 : x_1, x_2, x_3, x_4 \in \mathbb{C}_0\}$

$\mathbb{C}(i_1, i_2i_3) = \{x_1 + i_1x_2 + i_2i_3x_3 + i_1i_2i_3x_4 : x_1, x_2, x_3, x_4 \in \mathbb{C}_0\}$

$\mathbb{C}(i_2, i_1i_3) = \{x_1 + i_2x_2 + i_1i_3x_3 + i_1i_2i_3x_4 : x_1, x_2, x_3, x_4 \in \mathbb{C}_0\}$

$\mathbb{C}(i_3, i_1i_2) = \{x_1 + i_3x_2 + i_1i_2x_3 + i_1i_2i_3x_4 : x_1, x_2, x_3, x_4 \in \mathbb{C}_0\}$

$\mathbb{C}(i_1i_2, i_1i_3) = \{x_1 + i_1i_2x_2 + i_1i_3x_3 + i_2i_3x_4 : x_1, x_2, x_3, x_4 \in \mathbb{C}_0\}$

$\mathbb{H}(i_1i_2) = \{u + i_1i_2v : u, v \in \mathbb{C}_0\}$

$\mathbb{H}(i_1i_3) = \{u + i_1i_3v : u, v \in \mathbb{C}_0\}$

$\mathbb{H}(i_2 i_3) = \{u + i_2 i_3 v: u, v \in \mathbb{C}_0\}$

$\mathbb{H}(i_1 i_2 i_3) = \{u + i_1 i_2 i_3 v: u, v \in \mathbb{C}_0\}$

1.1 Idempotent elements in $\mathbb{C}(i_1, i_2, i_3)$

In the tricomplex algebra $\mathbb{C}(i_1, i_2, i_3)$, there exist exactly **16 idempotent elements**. These idempotents play a fundamental role in decomposing tricomplex numbers and in the construction of Cartesian idempotent sets. The complete list of idempotent elements is given by: $\{0, 1, e_1, e_1^\dagger, e_2, e_2^\dagger, e_3, e_3^\dagger, e_4, e_4^\dagger, e_5, e_5^\dagger, e_6, e_6^\dagger, e_7, e_7^\dagger\}$.

All are listed below along with their values.

SN	Notation of idempotent element	Value of idempotent element
1	*0*	*0*
2	*1*	*1*
3	e_1	$\frac{1 + i_1 i_2}{2}$
4	$e_1^\dagger$	$\frac{1 - i_1 i_2}{2}$
5	e_2	$\frac{1 + i_1 i_3}{2}$
6	$e_2^\dagger$	$\frac{1 - i_1 i_3}{2}$
7	e_3	$\frac{1 + i_2 i_3}{2}$
8	$e_3^\dagger$	$\frac{1 - i_2 i_3}{2}$
9	$e_4 = e_1 e_2$	$\frac{1}{4}(1 + i_1 i_2 + i_1 i_3 - i_2 i_3)$
10	$e_4^\dagger = 1 - e_1 e_2$	$\frac{1}{4}(3 - i_1 i_2 - i_1 i_3 + i_2 i_3)$
11	$e_5 = e_1 e_2^\dagger$	$\frac{1}{4}(1 + i_1 i_2 - i_1 i_3 + i_2 i_3)$
12	$e_5^\dagger = 1 - e_1 e_2^\dagger$	$\frac{1}{4}(3 - i_1 i_2 + i_1 i_3 - i_2 i_3)$
13	$e_6 = e_1^\dagger e_2$	$\frac{1}{4}(1 - i_1 i_2 + i_1 i_3 + i_2 i_3)$
14	$e_6^\dagger = 1 - e_1^\dagger e_2$	$\frac{1}{4}(3 + i_1 i_2 - i_1 i_3 - i_2 i_3)$
15	$e_7 = e_1^\dagger e_2^\dagger$	$\frac{1}{4}(1 - i_1 i_2 - i_1 i_3 - i_2 i_3)$
16	$e_7^\dagger = 1 - e_1^\dagger e_2^\dagger$	$\frac{1}{4}(3 + i_1 i_2 + i_1 i_3 + i_2 i_3)$

1.2 Relation between idempotent elements:

(i) Product relations among idempotent elements

$$e_1 e_1^\dagger = e_1 e_6 = e_1 e_7 = e_1^\dagger e_4 = e_1^\dagger e_5 = e_2 e_2^\dagger = e_2 e_5 = e_2 e_7 = e_2^\dagger e_4 = e_2^\dagger e_6 = e_3 e_3^\dagger = e_3 e_4 = e_3 e_7 = e_3^\dagger e_5 = e_3^\dagger e_6$$
$$= e_4 e_4^\dagger = e_4 e_5 = e_4 e_6 = e_4 e_7 = e_5 e_5^\dagger = e_5 e_6 = e_5 e_7 = e_6 e_6^\dagger = e_6 e_7 = e_7 e_7^\dagger = e_1 e_2 e_3 = 0$$

(ii) Addition relations among idempotent elements

$$e_1 + e_1^\dagger = e_2 + e_2^\dagger = e_3 + e_3^\dagger = e_4 + e_4^\dagger = e_5 + e_5^\dagger = e_6 + e_6^\dagger = e_7 + e_7^\dagger = e_4 + e_5 + e_6 + e_7 = 1$$

2. Cartesian Idempotent Set:

The concept of the Cartesian idempotent set is central in the study of tricomplex numbers. It provides a systematic way of expressing elements of the algebra in terms of idempotent components, thereby simplifying algebraic manipulations such as inversion, factorization, and norm computations.

The set $\{0,1,e_1,e_1^\dagger,e_2,e_2^\dagger,e_3,e_3^\dagger,e_4,e_4^\dagger,e_5,e_5^\dagger,e_6,e_6^\dagger,e_7,e_7^\dagger\}$ constitutes the collection of **idempotent elements** in the algebra $\mathbb{C}(i_1,i_2,i_3)$. Any tricomplex number $\zeta \in \mathbb{C}(i_1,i_2,i_3)$ can be decomposed uniquely with respect to these idempotents, and the notion of **Cartesian idempotent sets** formalizes such decompositions.

We now define the different forms of Cartesian idempotent sets:

(i) Binary Cartesian Idempotent Set with respect to $e_1, e_1^\dagger$

Let X_1 and X_2 be two sets. The Cartesian idempotent set determined by X_1 and X_2 with respect to the idempotent pair $(e_1,e_1^\dagger)$ is denoted as

$X = X_1 \times_e X_2$

and is defined as

$X = X_1 \times_e X_2 = \{\xi \in X: \xi = \alpha e_1 + \beta e_1^\dagger: (\alpha,\beta) \in X_1 \times X_2\}.$

Here, $e_1 e_1^\dagger = 0$ and $e_1 + e_1^\dagger = 1$.

(ii) Binary Cartesian Idempotent Set with respect to $e_2, e_2^\dagger$

Similarly, the Cartesian idempotent set determined by X_1 and X_2 with respect to the idempotent pair $(e_2,e_2^\dagger)$ is given by

$X = X_1 \times_e X_2 = \{\xi \in X: \xi = \alpha e_2 + \beta e_2^\dagger: (\alpha,\beta) \in X_1 \times X_2\}.$

Here, $e_2 e_2^\dagger = 0$ and $e_2 + e_2^\dagger = 1$.

(iii) Binary Cartesian Idempotent Set with respect to $e_3, e_3^\dagger$

With respect to the idempotent pair $(e_3,e_3^\dagger)$, we define

$$X = X_1 \times_e X_2 = \{\xi \in X: \xi = \alpha e_3 + \beta e_3^\dagger: (\alpha,\beta) \in X_1 \times X_2\}$$

Here, $e_3 e_3^\dagger = 0$ and $e_3 + e_3^\dagger = 1$.

(iv) Quaternary Cartesian Idempotent Set

Finally, let X_1, X_2, X_3 and X_4 be four sets. The Cartesian idempotent set determined by X_1, X_2, X_3 and X_4 is with respect to the idempotents e_4, e_5, e_6, e_7 is denoted by $X = X_1 \times_e X_2 \times_e X_3 \times_e X_4$

and is defined as

$X = X_1 \times_e X_2 \times_e X_3 \times_e X_4 = \{\zeta \in X: \zeta = \alpha e_4 + \beta e_5 + \gamma e_6 + \delta e_7: (\alpha,\beta,\gamma,\delta) \in X_1 \times X_2 \times X_3 \times X_4\}.$

Here, $e_4e_5 = e_4e_6 = e_4e_7 = e_5e_6 = e_5e_7 = e_6e_7 = 0$ and $e_4 + e_5 + e_6 + e_7 = 1$.

3. Idempotent Representation of $\mathbb{C}(i_1,i_2), \mathbb{C}(i_1,i_3)$ and $\mathbb{C}(i_2,i_3)$:

In what follows, we present the idempotent representations of the three bicomplex subalgebras of the tricomplex algebra $\mathbb{C}(i_1,i_2,i_3)$, namely $\mathbb{C}(i_1,i_2), \mathbb{C}(i_1,i_3)$ and $\mathbb{C}(i_2,i_3)$. These representations provide a systematic decomposition of elements in each subalgebra with respect to their corresponding idempotent bases, thereby facilitating further structural and analytical investigations.

3.1 Idempotent Representation of $\mathbb{C}(i_1,i_2)$:

(i) $\mathbb{C}(i_1,i_2) = \mathbb{C}(i_1) \times_e \mathbb{C}(i_1)$

$= \mathbb{C}(i_1)e_1 + \mathbb{C}(i_1)e_1^\dagger$

$= \{\xi \in \mathbb{C}(i_1,i_2): \xi = ue_1 + ve_1^\dagger, (u,v) \in \mathbb{C}(i_1) \times \mathbb{C}(i_1)\}$

The $\mathbb{C}(i_1)$ idempotent representation of $\xi \in \mathbb{C}(i_1,i_2)$ is given by

$\xi = x_1 + i_1x_2 + i_2x_3 + i_1i_2x_4$

$= (x_1 + i_1x_2) + i_2(x_3 + i_1x_4)$

$= z_1 + i_2z_2$

$= (z_1 - i_1 z_2)e_1 + (z_1 + i_1 z_2)e_1^\dagger$

$= \{(x_1 + x_4) + i_1(x_2 - x_3)\}e_1 + \{(x_1 - x_4) + i_1(x_2 + x_3)\}e_1^\dagger$

(ii) $\mathbb{C}(i_1, i_2) = \mathbb{C}(i_2) \times_e \mathbb{C}(i_2)$

$= \mathbb{C}(i_2)e_1 + \mathbb{C}(i_2)e_1^\dagger$

$= \{\xi \in \mathbb{C}(i_1, i_2): \xi = ue_1 + ve_1^\dagger, (u, v) \in \mathbb{C}(i_2) \times \mathbb{C}(i_2)\}$

The $\mathbb{C}(i_2)$ idempotent representation of $\xi \in \mathbb{C}(i_1, i_2)$ is given by

$\xi = x_1 + i_1 x_2 + i_2 x_3 + i_1 i_2 x_4$

$= (x_1 + i_2 x_3) + i_1(x_2 + i_2 x_4)$

$= w_1 + i_1 w_2$

$= (w_1 - i_2 w_2)e_1 + (w_1 + i_2 w_2)e_1^\dagger$

$= \{(x_1 + x_4) - i_2(x_2 - x_3)\}e_1 + \{(x_1 - x_4) + i_2(x_2 + x_3)\}e_1^\dagger$

3.2 Idempotent Representation of $\mathbb{C}(\boldsymbol{i_1}, \boldsymbol{i_3})$***:***

(i) $\mathbb{C}(i_1, i_3) = \mathbb{C}(i_1) \times_e \mathbb{C}(i_1)$

$= \mathbb{C}(i_1)e_2 + \mathbb{C}(i_1)e_2^\dagger$

$= \{\xi \in \mathbb{C}(i_1, i_3): \xi = ue_2 + ve_2^\dagger, (u, v) \in \mathbb{C}(i_1) \times \mathbb{C}(i_1)\}$

The $\mathbb{C}(i_1)$ idempotent representation of $\xi \in \mathbb{C}(i_1, i_3)$ is given by

$\xi = x_1 + i_1 x_2 + i_3 x_3 + i_1 i_3 x_4$

$= (x_1 + i_1 x_2) + i_3(x_3 + i_1 x_4)$

$= z_1 + i_3 z_2$

$= (z_1 - i_1 z_2)e_2 + (z_1 + i_1 z_2)e_2^\dagger$

$= \{(x_1 + x_4) + i_1(x_2 - x_3)\}e_2 + \{(x_1 - x_4) + i_1(x_2 + x_3)\}e_2^\dagger$

(ii) $\mathbb{C}(i_1, i_3) = \mathbb{C}(i_3) \times_e \mathbb{C}(i_3)$

$= \mathbb{C}(i_3)e_2 + \mathbb{C}(i_3)e_2^\dagger$

$= \{\xi \in \mathbb{C}(i_1, i_3): \xi = ue_2 + ve_2^\dagger, (u, v) \in \mathbb{C}(i_3) \times \mathbb{C}(i_3)\}$

The $\mathbb{C}(i_3)$ idempotent representation of $\xi \in \mathbb{C}(i_1, i_3)$ is given by

$\xi = x_1 + i_1 x_2 + i_3 x_3 + i_1 i_3 x_4$

$= (x_1 + i_3 x_3) + i_1(x_2 + i_3 x_4)$

$= w_1 + i_1 w_2$

$= (w_1 - i_3 w_2)e_2 + (w_1 + i_3 w_2)e_2^\dagger$

$= \{(x_1 + x_4) - i_3(x_2 - x_3)\}e_2 + \{(x_1 - x_4) + i_3(x_2 + x_3)\}e_2^\dagger$

3.3 Idempotent Representation of $\mathbb{C}(\boldsymbol{i_2}, \boldsymbol{i_3})$***:***

(i) $\mathbb{C}(i_2, i_3) = \mathbb{C}(i_2) \times_e \mathbb{C}(i_2)$

$= \mathbb{C}(i_2)e_3 + \mathbb{C}(i_2)e_3^\dagger$

$= \{\xi \in \mathbb{C}(i_2, i_3): \xi = ue_3 + ve_3^\dagger, (u, v) \in \mathbb{C}(i_2) \times \mathbb{C}(i_2)\}$

The $\mathbb{C}(i_2)$ idempotent representation of $\xi \in \mathbb{C}(i_2, i_3)$ is given by

$\xi = x_1 + i_2x_2 + i_3x_3 + i_2i_3x_4$

$= (x_1 + i_2x_2) + i_3(x_3 + i_2x_4)$

$= z_1 + i_3z_2$

$= (z_1 - i_2z_2)e_3 + (z_1 + i_2z_2)e_3^{\dagger}$

$= \{(x_1 + x_4) + i_2(x_2 - x_3)\}e_3 + \{(x_1 - x_4) + i_2(x_2 + x_3)\}e_3^{\dagger}$

(ii) $\mathbb{C}(i_2, i_3) = \mathbb{C}(i_3)\times_e\mathbb{C}(i_3)$

$= \mathbb{C}(i_3)e_3 + \mathbb{C}(i_3)e_3^{\dagger}$

$= \{\xi \in \mathbb{C}(i_2, i_3): \xi = ue_3 + ve_3^{\dagger}, (u, v) \in \mathbb{C}(i_3) \times \mathbb{C}(i_3)\}$

The $\mathbb{C}(i_3) -$ idempotent representation of $\xi \in \mathbb{C}(i_2, i_3)$ is given by

$\xi = x_1 + i_2x_2 + i_3x_3 + i_2i_3x_4$

$= (x_1 + i_3x_3) + i_2(x_2 + i_3x_4)$

$= w_1 + i_2w_2$

$= (w_1 - i_3w_2)e_3 + (w_1 + i_3w_2)e_3^{\dagger}$

$= \{(x_1 + x_4) - i_3(x_2 - x_3)\}e_3 + \{(x_1 - x_4) + i_3(x_2 + x_3)\}e_3^{\dagger}$

4. Idempotent Representation of $\mathbb{C}(i_1, i_2, i_3)$

We present the idempotent representation of the tricomplex algebra $\mathbb{C}(i_1, i_2, i_3)$ in terms of its decompositions with respect to the subalgebras $\mathbb{C}(i_1, i_2), \mathbb{C}(i_1, i_3)$, and $\mathbb{C}(i_2, i_3)$, as well as the simpler forms involving $\mathbb{C}(i_1), \mathbb{C}(i_2)$, and $\mathbb{C}(i_3)$.

4.1 Idempotent Representation of $\mathbb{C}(i_1, i_2, i_3)$ via Its Bicomplex Subalgebras

In this subsection, we present the idempotent representation of the tricomplex algebra $\mathbb{C}(i_1, i_2, i_3)$ in terms of its decompositions with respect to the bicomplex subalgebras $\mathbb{C}(i_1, i_2), \mathbb{C}(i_1, i_3)$, and $\mathbb{C}(i_2, i_3)$.

4.1.1 Idempotent representation of $\mathbb{C}(i_1, i_2, i_3)$ in $\mathbb{C}(i_1, i_2)$

(i) $\mathbb{C}(i_1, i_2, i_3) = \mathbb{C}(i_1, i_2)\times_e\mathbb{C}(i_1, i_2)$

$= \mathbb{C}(i_1, i_2)e_2 + \mathbb{C}(i_1, i_2)e_2^{\dagger}$

$= \{\zeta \in \mathbb{C}(i_1, i_2, i_3): \zeta = \xi e_2 + \eta e_2^{\dagger}, (\xi, \eta) \in \mathbb{C}(i_1, i_2) \times \mathbb{C}(i_1, i_2)\}$

The $\mathbb{C}(i_1, i_2) -$ idempotent representation of $\zeta = \xi + \dot{i}_3\, \eta;\ \xi, \eta \in \mathbb{C}(i_1, i_2)$ is given by

$\zeta = \xi + \dot{i}_3\, \eta = (\xi - \dot{i}_1\, \eta)e_2 + (\xi + \dot{i}_1\, \eta)e_2^{\dagger}$

(ii) $\mathbb{C}(i_1, i_2, i_3) = \mathbb{C}(i_1, i_2)\times_e \mathbb{C}(i_1, i_2)$

$= \mathbb{C}(i_1, i_2)e_3 + \mathbb{C}(i_1, i_2)e_3^{\dagger}$

$= \{\zeta \in \mathbb{C}(i_1, i_2, i_3): \zeta = \xi e_2 + \eta e_2^{\dagger}, (\xi, \eta) \in \mathbb{C}(i_1, i_2) \times \mathbb{C}(i_1, i_2)\}$

The $\mathbb{C}(i_1, i_2) -$ idempotent representation of $\zeta = \xi + \dot{i}_3\, \eta;\ \xi, \eta \in \mathbb{C}(i_1, i_2)$ is given by

$\xi + \dot{i}_3\, \eta = (\xi - \dot{i}_2\, \eta)e_3 + (\xi + \dot{i}_2\, \eta)e_3^{\dagger}$

4.1.2 Idempotent representation of $\mathbb{C}(i_1, i_2, i_3)$ in $\mathbb{C}(i_1, i_3)$

(i) $\mathbb{C}(i_1, i_2, i_3) = \mathbb{C}(i_1, i_3)\times_e \mathbb{C}(i_1, i_3)$

$= \mathbb{C}(i_1, i_3)e_1 + \mathbb{C}(i_1, i_3)e_1^{\dagger}$

$= \{\zeta \in \mathbb{C}(i_1, i_2, i_3): \zeta = \xi e_1 + \eta e_1^{\dagger}, (\xi, \eta) \in \mathbb{C}(i_1, i_3) \times \mathbb{C}(i_1, i_3)\}$

The $\mathbb{C}(i_1, i_3) -$ idempotent representation of $\zeta = \xi + \dot{i}_2\,\eta;\ \xi, \eta \in \mathbb{C}(i_1, i_3)$ is given by

$\xi + \dot{i}_2\,\eta = (\xi - \dot{i}_1\,\eta)e_1 + (\xi + \dot{i}_1\,\eta)e_1^{\dagger}$

(ii) $\mathbb{C}(i_1, i_2, i_3) = \mathbb{C}(i_1, i_3) \times_e \mathbb{C}(i_1, i_3)$

$= \mathbb{C}(i_1, i_3)e_3 + \mathbb{C}(i_1, i_3)e_3^{\dagger}$

$= \{\zeta \in \mathbb{C}(i_1, i_2, i_3): \zeta = \xi e_3 + \eta e_3^{\dagger}, (\xi, \eta) \in \mathbb{C}(i_1, i_3) \times \mathbb{C}(i_1, i_3)\}$

The $\mathbb{C}(i_1, i_3) -$ idempotent representation of $\zeta = \xi + \dot{i}_2\,\eta;\ \xi, \eta \in \mathbb{C}(i_1, i_3)$ is given by

$\xi + \dot{i}_2\,\eta = (\xi - \dot{i}_3\,\eta)e_3 + (\xi + \dot{i}_3\,\eta)e_3^{\dagger}$

4.1.3 Idempotent representation of $\mathbb{C}(\boldsymbol{i_1}, \boldsymbol{i_2}, \boldsymbol{i_3})$ *in* $\mathbb{C}(\boldsymbol{i_2}, \boldsymbol{i_3})$

(i) $\mathbb{C}(i_1, i_2, i_3) = \mathbb{C}(i_2, i_3) \times_e \mathbb{C}(i_2, i_3) = \mathbb{C}(i_2, i_3)e_1 + \mathbb{C}(i_2, i_3)e_1^{\dagger}$

$= \{\zeta \in \mathbb{C}(i_1, i_2, i_3): \zeta = \xi e_1 + \eta e_1^{\dagger}, (\xi, \eta) \in \mathbb{C}(i_2, i_3) \times \mathbb{C}(i_2, i_3)\}$

The $\mathbb{C}(i_2, i_3) -$ idempotent representation of $\zeta = \xi + \dot{i}_1\,\eta;\ \xi, \eta \in \mathbb{C}(i_2, i_3)$ is given as

$\zeta = \xi + \dot{i}_1\,\eta = (\xi - \dot{i}_2\,\eta)e_1 + (\xi + \dot{i}_2\,\eta)e_1^{\dagger}$

(ii) $\mathbb{C}(i_1, i_2, i_3) = \mathbb{C}(i_2, i_3) \times_e \mathbb{C}(i_2, i_3)$

$= \mathbb{C}(i_2, i_3)e_2 + \mathbb{C}(i_2, i_3)e_2^{\dagger}$

$= \{\zeta \in \mathbb{C}(i_1, i_2, i_3): \zeta = \xi e_2 + \eta e_2^{\dagger}, (\xi, \eta) \in \mathbb{C}(i_2, i_3) \times \mathbb{C}(i_2, i_3)\}$

The $\mathbb{C}(i_2, i_3) -$ idempotent representation of $\zeta = \xi + \dot{i}_1\,\eta;\ \xi, \eta \in \mathbb{C}(i_2, i_3)$ is given as

$\zeta = \xi + \dot{i}_1\,\eta = (\xi - \dot{i}_3\,\eta)e_2 + (\xi + \dot{i}_3\,\eta)e_2^{\dagger}$

4.2 Idempotent Representation of $\mathbb{C}(\boldsymbol{i_1}, \boldsymbol{i_2}, \boldsymbol{i_3})$ *in the form of* $\mathbb{C}(\boldsymbol{i_1})$; $\mathbb{C}(\boldsymbol{i_2})$; $\mathbb{C}(\boldsymbol{i_3})$; $\mathbb{C}(\boldsymbol{i_1}), \mathbb{C}(\boldsymbol{i_2})$; $\mathbb{C}(i_1), \mathbb{C}(i_3)$; $\mathbb{C}(i_2), \mathbb{C}(i_3)$:

(i-a) $\mathbb{C}(\boldsymbol{i_1})$**-idempotent representation:**

$$\begin{aligned}
\mathbb{C}(i_1, i_2, i_3) &= \mathbb{C}(i_1, i_2) \times_e \mathbb{C}(i_1, i_2) \\
&= \mathbb{C}(i_1, i_2)e_2 + \mathbb{C}(i_1, i_2)e_2^{\dagger} \\
&= \{\mathbb{C}(i_1)e_1 + \mathbb{C}(i_1)e_1^{\dagger}\}e_2 + \{\mathbb{C}(i_1)e_1 + \mathbb{C}(i_1)e_1^{\dagger}\}e_2^{\dagger} \\
&= \mathbb{C}(i_1)e_1e_2 + \mathbb{C}(i_1)e_1^{\dagger}e_2 + \mathbb{C}(i_1)e_1e_2^{\dagger} + \mathbb{C}(i_1)e_1^{\dagger}e_2^{\dagger} \\
&= \mathbb{C}(i_1)e_4 + \mathbb{C}(i_1)e_6 + \mathbb{C}(i_1)e_5 + \mathbb{C}(i_1)e_7 \\
&= \{\zeta \in \mathbb{C}(i_1, i_2, i_3): \alpha e_4 + \beta e_5 + \gamma e_6 + \delta e_7, (\alpha, \beta, \gamma, \delta) \in \mathbb{C}(i_1) \times \mathbb{C}(i_1) \times \mathbb{C}(i_1) \times \mathbb{C}(i_1)\}
\end{aligned}$$

The $\mathbb{C}(i_1) -$ idempotent representation of $\zeta \in \mathbb{C}(i_1, i_2, i_3)$ is given as

$$\begin{aligned}
\zeta &= x_1 + i_1x_2 + i_2x_3 + i_3x_4 + i_1i_2x_5 + i_1i_3x_6 + i_2i_3x_7 + i_1i_2i_3x_8 \\
&= (x_1 + i_1x_2 + i_2x_3 + i_1i_2x_5) + i_3(x_4 + i_1x_6 + i_2x_7 + i_1i_2x_8) \\
&= \xi + i_3\eta \\
&= (\xi - i_1\eta)e_2 + (\xi + i_1\eta)e_2^{\dagger} \\
&= \{(x_1 + x_6) + i_1(x_2 - x_4) + i_2(x_3 + x_8) + i_1i_2(x_5 - x_7)\}e_2 \\
&\qquad + \{(x_1 - x_6) + i_1(x_2 + x_4) + i_2(x_3 - x_8) + i_1i_2(x_5 + x_7)\}e_2^{\dagger} \\
&= [\{(x_1 + x_6) + i_1(x_2 - x_4)\} + i_2\{(x_3 + x_8) + i_1(x_5 - x_7)\}]e_2 \\
&\qquad + [\{(x_1 - x_6) + i_1(x_2 + x_4)\} + i_2\{(x_3 - x_8) + i_1(x_5 + x_7)\}]e_2^{\dagger} \\
&= [\{(x_1 + x_5 + x_6 - x_7) + i_1(x_2 - x_3 - x_4 - x_8)\}e_1 + \{(x_1 - x_5 + x_6 + x_7) + i_1(x_2 + x_3 - x_4 + x_8)\}e_1^{\dagger}]e_2
\end{aligned}$$

$+\left[\{(x_1+x_5-x_6+x_7)+i_1(x_2-x_3+x_4+x_8)\}e_1+\{(x_1-x_5-x_6-x_7)+i_1(x_2+x_3+x_4-x_8)\}e_1^{\dagger}\right]e_2^{\dagger}$

$=\{(x_1+x_5+x_6-x_7)+i_1(x_2-x_3-x_4-x_8)\}e_1e_2+\{(x_1-x_5+x_6+x_7)+i_1(x_2+x_3-x_4+x_8)\}e_1^{\dagger}e_2$

$+\{(x_1+x_5-x_6+x_7)+i_1(x_2-x_3+x_4+x_8)\}e_1e_2^{\dagger}+\{(x_1-x_5-x_6-x_7)+i_1(x_2+x_3+x_4-x_8)\}e_1^{\dagger}e_2^{\dagger}$

$=\{(x_1+x_5+x_6-x_7)+i_1(x_2-x_3-x_4-x_8)\}e_4+\{(x_1-x_5+x_6+x_7)+i_1(x_2+x_3-x_4+x_8)\}e_6$

$+\{(x_1+x_5-x_6+x_7)+i_1(x_2-x_3+x_4+x_8)\}e_5+\{(x_1-x_5-x_6-x_7)+i_1(x_2+x_3+x_4-x_8)\}e_7$

(i-b) $\mathbb{C}(i_2)$-idempotent representation:

$$\begin{aligned}
\mathbb{C}(i_1,i_2,i_3) &= \mathbb{C}(i_1,i_2)\times_e\mathbb{C}(i_1,i_2)\\
&= \mathbb{C}(i_1,i_2)e_2+\mathbb{C}(i_1,i_2)e_2^{\dagger}\\
&= \left\{\mathbb{C}(i_2)e_1+\mathbb{C}(i_2)e_1^{\dagger}\right\}e_2+\left\{\mathbb{C}(i_2)e_1+\mathbb{C}(i_2)e_1^{\dagger}\right\}e_2^{\dagger}\\
&= \mathbb{C}(i_2)e_1e_2+\mathbb{C}(i_2)e_1^{\dagger}e_2+\mathbb{C}(i_2)e_1e_2^{\dagger}+\mathbb{C}(i_2)e_1^{\dagger}e_2^{\dagger}\\
&= \mathbb{C}(i_2)e_4+\mathbb{C}(i_2)e_6+\mathbb{C}(i_2)e_5+\mathbb{C}(i_2)e_7\\
&= \{\zeta\in\mathbb{C}(i_1,i_2,i_3):\alpha e_4+\beta e_5+\gamma e_6+\delta e_7,(\alpha,\beta,\gamma,\delta)\in\mathbb{C}(i_2)\times\mathbb{C}(i_2)\times\mathbb{C}(i_2)\times\mathbb{C}(i_2)\}
\end{aligned}$$

The $\mathbb{C}(i_2)-$ idempotent representation of $\zeta\in\mathbb{C}(i_1,i_2,i_3)$ is given as

$\zeta=x_1+i_1x_2+i_2x_3+i_3x_4+i_1i_2x_5+i_1i_3x_6+i_2i_3x_7+i_1i_2i_3x_8$

$=(x_1+i_1x_2+i_2x_3+i_1i_2x_5)+i_3(x_4+i_1x_6+i_2x_7+i_1i_2x_8)$

$=\xi+i_3\eta$

$=(\xi-i_1\eta)e_2+(\xi+i_1\eta)e_2^{\dagger}$

$=\{(x_1+x_6)+i_1(x_2-x_4)+i_2(x_3+x_8)+i_1i_2(x_5-x_7)\}e_2$

$+\{(x_1-x_6)+i_1(x_2+x_4)+i_2(x_3-x_8)+i_1i_2(x_5+x_7)\}e_2^{\dagger}$

$=[\{(x_1+x_6)+i_2(x_3+x_8)\}+i_1\{(x_2-x_4)+i_2(x_5-x_7)\}]e_2$

$+[\{(x_1-x_6)+i_2(x_3-x_8)\}+i_1\{(x_2+x_4)+i_2(x_5+x_7)\}]e_2^{\dagger}$

$=\left[\{(x_1+x_5+x_6-x_7)-i_2(x_2-x_3-x_4-x_8)\}e_1+\{(x_1-x_5+x_6+x_7)+i_2(x_2+x_3-x_4+x_8)\}e_1^{\dagger}\right]e_2$

$+\left[\{(x_1+x_5-x_6+x_7)-i_2(x_2-x_3+x_4+x_8)\}e_1+\{(x_1-x_5-x_6-x_7)+i_2(x_2+x_3+x_4-x_8)\}e_1^{\dagger}\right]e_2^{\dagger}$

$=\{(x_1+x_5+x_6-x_7)-i_2(x_2-x_3-x_4-x_8)\}e_1e_2+\{(x_1-x_5+x_6+x_7)+i_2(x_2+x_3-x_4+x_8)\}e_1^{\dagger}e_2$

$+\{(x_1+x_5-x_6+x_7)-i_2(x_2-x_3+x_4+x_8)\}e_1e_2^{\dagger}+\{(x_1-x_5-x_6-x_7)+i_2(x_2+x_3+x_4-x_8)\}e_1^{\dagger}e_2^{\dagger}$

$=\{(x_1+x_5+x_6-x_7)-i_2(x_2-x_3-x_4-x_8)\}e_4+\{(x_1-x_5+x_6+x_7)+i_2(x_2+x_3-x_4+x_8)\}e_6$

$+\{(x_1+x_5-x_6+x_7)-i_2(x_2-x_3+x_4+x_8)\}e_5+\{(x_1-x_5-x_6-x_7)+i_2(x_2+x_3+x_4-x_8)\}e_7$

(i-c) Mixed $\mathbb{C}(i_1)$ and $\mathbb{C}(i_2)$-idempotent representations:

$$\begin{aligned}
\mathbb{C}(i_1,i_2,i_3) &= \mathbb{C}(i_1,i_2)\times_e\mathbb{C}(i_1,i_2)\\
&= \mathbb{C}(i_1,i_2)e_2+\mathbb{C}(i_1,i_2)e_2^{\dagger}\\
&= \left\{\mathbb{C}(i_1)e_1+\mathbb{C}(i_1)e_1^{\dagger}\right\}e_2+\left\{\mathbb{C}(i_2)e_1+\mathbb{C}(i_2)e_1^{\dagger}\right\}e_2^{\dagger}\\
&= \mathbb{C}(i_1)e_1e_2+\mathbb{C}(i_1)e_1^{\dagger}e_2+\mathbb{C}(i_2)e_1e_2^{\dagger}+\mathbb{C}(i_2)e_1^{\dagger}e_2^{\dagger}\\
&= \mathbb{C}(i_1)e_4+\mathbb{C}(i_1)e_6+\mathbb{C}(i_2)e_5+\mathbb{C}(i_2)e_7\\
&= \{\zeta\in\mathbb{C}(i_1,i_2,i_3):\alpha e_4+\beta e_6+\gamma e_5+\delta e_7,(\alpha,\beta,\gamma,\delta)\in\mathbb{C}(i_1)\times\mathbb{C}(i_1)\times\mathbb{C}(i_2)\times\mathbb{C}(i_2)\}
\end{aligned}$$

The $\mathbb{C}(i_1)$ and $\mathbb{C}(i_2)-$ idempotent representation of $\zeta\in\mathbb{C}(i_1,i_2,i_3)$ is given as

$\zeta=x_1+i_1x_2+i_2x_3+i_3x_4+i_1i_2x_5+i_1i_3x_6+i_2i_3x_7+i_1i_2i_3x_8$

$=(x_1+i_1x_2+i_2x_3+i_1i_2x_5)+i_3(x_4+i_1x_6+i_2x_7+i_1i_2x_8)$

$= \xi + i_3\eta$

$= (\xi - i_1\eta)e_2 + (\xi + i_1\eta)e_2^{\dagger}$

$= \{(x_1 + x_6) + i_1(x_2 - x_4) + i_2(x_3 + x_8) + i_1i_2(x_5 - x_7)\}e_2$

$+ \{(x_1 - x_6) + i_1(x_2 + x_4) + i_2(x_3 - x_8) + i_1i_2(x_5 + x_7)\}e_2^{\dagger}$

$= [\{(x_1 + x_6) + i_1(x_2 - x_4)\} + i_2\{(x_3 + x_8) + i_1(x_5 - x_7)\}]e_2$

$+ [\{(x_1 - x_6) + i_2(x_3 - x_8)\} + i_1\{(x_2 + x_4) + i_2(x_5 + x_7)\}]e_2^{\dagger}$

$= \big[\{(x_1 + x_5 + x_6 - x_7) + i_1(x_2 - x_3 - x_4 - x_8)\}e_1 + \{(x_1 - x_5 + x_6 + x_7) + i_1(x_2 + x_3 - x_4 + x_8)\}e_1^{\dagger}\big]e_2$

$+ \big[\{(x_1 + x_5 - x_6 + x_7) - i_2(x_2 - x_3 + x_4 + x_8)\}e_1 + \{(x_1 - x_5 - x_6 - x_7) + i_2(x_2 + x_3 + x_4 - x_8)\}e_1^{\dagger}\big]e_2^{\dagger}$

$= \{(x_1 + x_5 + x_6 - x_7) + i_1(x_2 - x_3 - x_4 - x_8)\}e_1e_2 + \{(x_1 - x_5 + x_6 + x_7) + i_1(x_2 + x_3 - x_4 + x_8)\}e_1^{\dagger}e_2$

$+ \{(x_1 + x_5 - x_6 + x_7) - i_2(x_2 - x_3 + x_4 + x_8)\}e_1e_2^{\dagger} + \{(x_1 - x_5 - x_6 - x_7) + i_2(x_2 + x_3 + x_4 - x_8)\}e_1^{\dagger}e_2^{\dagger}$

$= \{(x_1 + x_5 + x_6 - x_7) + i_1(x_2 - x_3 - x_4 - x_8)\}e_4 + \{(x_1 - x_5 + x_6 + x_7) + i_1(x_2 + x_3 - x_4 + x_8)\}e_6$

$+ \{(x_1 + x_5 - x_6 + x_7) - i_2(x_2 - x_3 + x_4 + x_8)\}e_5 + \{(x_1 - x_5 - x_6 - x_7) + i_2(x_2 + x_3 + x_4 - x_8)\}e_7$

(i-d) Mixed $\mathbb{C}(i_1)$ and $\mathbb{C}(i_2)$-idempotent representations:

$\mathbb{C}(i_1, i_2, i_3) = \mathbb{C}(i_1, i_2)\times_e\mathbb{C}(i_1, i_2)$

$= \mathbb{C}(i_1, i_2)e_2 + \mathbb{C}(i_1, i_2)e_2^{\dagger}$

$= \{\mathbb{C}(i_2)e_1 + \mathbb{C}(i_2)e_1^{\dagger}\}e_2 + \{\mathbb{C}(i_1)e_1 + \mathbb{C}(i_1)e_1^{\dagger}\}e_2^{\dagger}$

$= \mathbb{C}(i_2)e_1e_2 + \mathbb{C}(i_2)e_1^{\dagger}e_2 + \mathbb{C}(i_1)e_1e_2^{\dagger} + \mathbb{C}(i_1)e_1^{\dagger}e_2^{\dagger}$

$= \mathbb{C}(i_2)e_4 + \mathbb{C}(i_2)e_6 + \mathbb{C}(i_1)e_5 + \mathbb{C}(i_1)e_7$

$= \{\zeta \in \mathbb{C}(i_1, i_2, i_3): \alpha e_4 + \beta e_6 + \gamma e_5 + \delta e_7, (\alpha, \beta, \gamma, \delta) \in \mathbb{C}(i_2) \times \mathbb{C}(i_2) \times \mathbb{C}(i_1) \times \mathbb{C}(i_1)\}$

The $\mathbb{C}(i_2)$ and $\mathbb{C}(i_1) -$ idempotent representation of $\zeta \in \mathbb{C}(i_1, i_2, i_3)$ is given as

$\zeta = x_1 + i_1x_2 + i_2x_3 + i_3x_4 + i_1i_2x_5 + i_1i_3x_6 + i_2i_3x_7 + i_1i_2i_3x_8$

$= (x_1 + i_1x_2 + i_2x_3 + i_1i_2x_5) + i_3(x_4 + i_1x_6 + i_2x_7 + i_1i_2x_8)$

$= \xi + i_3\eta$

$= (\xi - i_1\eta)e_2 + (\xi + i_1\eta)e_2^{\dagger}$

$= \{(x_1 + x_6) + i_1(x_2 - x_4) + i_2(x_3 + x_8) + i_1i_2(x_5 - x_7)\}e_2$

$+ \{(x_1 - x_6) + i_1(x_2 + x_4) + i_2(x_3 - x_8) + i_1i_2(x_5 + x_7)\}e_2^{\dagger}$

$= [\{(x_1 + x_6) + i_2(x_3 + x_8)\} + i_1\{(x_2 - x_4) + i_2(x_5 - x_7)\}]e_2$

$+ [\{(x_1 - x_6) + i_1(x_2 + x_4)\} + i_2\{(x_3 - x_8) + i_1(x_5 + x_7)\}]e_2^{\dagger}$

$= \big[\{(x_1 + x_5 + x_6 - x_7) - i_2(x_2 - x_3 - x_4 - x_8)\}e_1 + \{(x_1 - x_5 + x_6 + x_7) + i_2(x_2 + x_3 - x_4 + x_8)\}e_1^{\dagger}\big]e_2$

$+ \big[\{(x_1 + x_5 - x_6 + x_7) + i_1(x_2 - x_3 + x_4 + x_8)\}e_1 + \{(x_1 - x_5 - x_6 - x_7) + i_1(x_2 + x_3 + x_4 - x_8)\}e_1^{\dagger}\big]e_2^{\dagger}$

$= \{(x_1 + x_5 + x_6 - x_7) - i_2(x_2 - x_3 - x_4 - x_8)\}e_1e_2 + \{(x_1 - x_5 + x_6 + x_7) + i_2(x_2 + x_3 - x_4 + x_8)\}e_1^{\dagger}e_2$

$+ \{(x_1 + x_5 - x_6 + x_7) + i_1(x_2 - x_3 + x_4 + x_8)\}e_1e_2^{\dagger} + \{(x_1 - x_5 - x_6 - x_7) + i_1(x_2 + x_3 + x_4 - x_8)\}e_1^{\dagger}e_2^{\dagger}$

$= \{(x_1 + x_5 + x_6 - x_7) - i_2(x_2 - x_3 - x_4 - x_8)\}e_4 + \{(x_1 - x_5 + x_6 + x_7) + i_2(x_2 + x_3 - x_4 + x_8)\}e_6$

$+ \{(x_1 + x_5 - x_6 + x_7) + i_1(x_2 - x_3 + x_4 + x_8)\}e_5 + \{(x_1 - x_5 - x_6 - x_7) + i_1(x_2 + x_3 + x_4 - x_8)\}e_7$

(ii-a) $\mathbb{C}(i_1)$-idempotent representation:

$\mathbb{C}(i_1, i_2, i_3) = \mathbb{C}(i_1, i_2)\times_e \mathbb{C}(i_1, i_2)$

$= \mathbb{C}(i_1, i_2)e_3 + \mathbb{C}(i_1, i_2)e_3^{\dagger}$

$= \{\mathbb{C}(i_1)e_1 + \mathbb{C}(i_1)e_1^{\dagger}\}e_3 + \{\mathbb{C}(i_1)e_1 + \mathbb{C}(i_1)e_1^{\dagger}\}e_3^{\dagger}$

$$= \mathbb{C}(i_1)e_1e_3 + \mathbb{C}(i_1)e_1^{\dagger}e_3 + \mathbb{C}(i_1)e_1e_3^{\dagger} + \mathbb{C}(i_1)e_1^{\dagger}e_3^{\dagger}$$

$$= \mathbb{C}(i_1)e_5 + \mathbb{C}(i_1)e_6 + \mathbb{C}(i_1)e_4 + \mathbb{C}(i_1)e_7$$

$$= \{\zeta \in \mathbb{C}(i_1,i_2,i_3): \alpha e_4 + \beta e_5 + \gamma e_6 + \delta e_7, (\alpha,\beta,\gamma,\delta) \in \mathbb{C}(i_1) \times \mathbb{C}(i_1) \times \mathbb{C}(i_1) \times \mathbb{C}(i_1)\}$$

The $\mathbb{C}(i_1)$ − idempotent representation of $\zeta \in \mathbb{C}(i_1,i_2,i_3)$ is given as

$$\zeta = x_1 + i_1x_2 + i_2x_3 + i_3x_4 + i_1i_2x_5 + i_1i_3x_6 + i_2i_3x_7 + i_1i_2i_3x_8$$

$$= (x_1 + i_1x_2 + i_2x_3 + i_1i_2x_5) + i_3(x_4 + i_1x_6 + i_2x_7 + i_1i_2x_8)$$

$$= \xi + i_3\eta$$

$$= (\xi - i_2\eta)e_3 + (\xi + i_2\eta)e_3^{\dagger}$$

$$= \{(x_1 + x_7) + i_1(x_2 + x_8) + i_2(x_3 - x_4) + i_1i_2(x_5 - x_6)\}e_3 + \{(x_1 - x_7) + i_1(x_2 - x_8) + i_2(x_3 + x_4) + i_1i_2(x_5 + x_6)\}e_3^{\dagger}$$

$$= [\{(x_1 + x_7) + i_1(x_2 + x_8)\} + i_2\{(x_3 - x_4) + i_1(x_5 - x_6)\}]e_3 + [\{(x_1 - x_7) + i_1(x_2 - x_8)\} + i_2\{(x_3 + x_4) + i_1(x_5 + x_6)\}]e_3^{\dagger}$$

$$= \left[\{(x_1 + x_5 - x_6 + x_7) + i_1(x_2 - x_3 + x_4 + x_8)\}e_1 + \{(x_1 - x_5 + x_6 + x_7) + i_1(x_2 + x_3 - x_4 + x_8)\}e_1^{\dagger}\right]e_3 + \left[\{(x_1 + x_5 + x_6 - x_7) + i_1(x_2 - x_3 - x_4 - x_8)\}e_1 + \{(x_1 - x_5 - x_6 - x_7) + i_1(x_2 + x_3 + x_4 - x_8)\}e_1^{\dagger}\right]e_3^{\dagger}$$

$$= \{(x_1 + x_5 - x_6 + x_7) + i_1(x_2 - x_3 + x_4 + x_8)\}e_1e_3 + \{(x_1 - x_5 + x_6 + x_7) + i_1(x_2 + x_3 - x_4 + x_8)\}e_1^{\dagger}e_3 + \{(x_1 + x_5 + x_6 - x_7) + i_1(x_2 - x_3 - x_4 - x_8)\}e_1e_3^{\dagger} + \{(x_1 - x_5 - x_6 - x_7) + i_1(x_2 + x_3 + x_4 - x_8)\}e_1^{\dagger}e_3^{\dagger}$$

$$= \{(x_1 + x_5 - x_6 + x_7) + i_1(x_2 - x_3 + x_4 + x_8)\}e_5 + \{(x_1 - x_5 + x_6 + x_7) + i_1(x_2 + x_3 - x_4 + x_8)\}e_6 + \{(x_1 + x_5 + x_6 - x_7) + i_1(x_2 - x_3 - x_4 - x_8)\}e_4 + \{(x_1 - x_5 - x_6 - x_7) + i_1(x_2 + x_3 + x_4 - x_8)\}e_7$$

(ii-b) $\mathbb{C}(i_2)$-idempotent representation:

$$\mathbb{C}(i_1,i_2,i_3) = \mathbb{C}(i_1,i_2)\times_e \mathbb{C}(i_1,i_2)$$

$$= \mathbb{C}(i_1,i_2)e_3 + \mathbb{C}(i_1,i_2)e_3^{\dagger}$$

$$= \{\mathbb{C}(i_2)e_1 + \mathbb{C}(i_2)e_1^{\dagger}\}e_3 + \{\mathbb{C}(i_2)e_1 + \mathbb{C}(i_2)e_1^{\dagger}\}e_3^{\dagger}$$

$$= \mathbb{C}(i_2)e_1e_3 + \mathbb{C}(i_2)e_1^{\dagger}e_3 + \mathbb{C}(i_2)e_1e_3^{\dagger} + \mathbb{C}(i_2)e_1^{\dagger}e_3^{\dagger}$$

$$= \mathbb{C}(i_2)e_5 + \mathbb{C}(i_2)e_6 + \mathbb{C}(i_2)e_4 + \mathbb{C}(i_2)e_7$$

$$= \{\zeta \in \mathbb{C}(i_1,i_2,i_3): \alpha e_4 + \beta e_5 + \gamma e_6 + \delta e_7, (\alpha,\beta,\gamma,\delta) \in \mathbb{C}(i_2) \times \mathbb{C}(i_2) \times \mathbb{C}(i_2) \times \mathbb{C}(i_2)\}$$

The $\mathbb{C}(i_2)$ − idempotent representation of $\zeta \in \mathbb{C}(i_1,i_2,i_3)$ is given as

$$\zeta = x_1 + i_1x_2 + i_2x_3 + i_3x_4 + i_1i_2x_5 + i_1i_3x_6 + i_2i_3x_7 + i_1i_2i_3x_8$$

$$= (x_1 + i_1x_2 + i_2x_3 + i_1i_2x_5) + i_3(x_4 + i_1x_6 + i_2x_7 + i_1i_2x_8)$$

$$= \xi + i_3\eta$$

$$= (\xi - i_2\eta)e_3 + (\xi + i_2\eta)e_3^{\dagger}$$

$$= \{(x_1 + x_7) + i_1(x_2 + x_8) + i_2(x_3 - x_4) + i_1i_2(x_5 - x_6)\}e_3 + \{(x_1 - x_7) + i_1(x_2 - x_8) + i_2(x_3 + x_4) + i_1i_2(x_5 + x_6)\}e_3^{\dagger}$$

$$= [\{(x_1 + x_7) + i_2(x_3 - x_4)\} + i_1\{(x_2 + x_8) + i_2(x_5 - x_6)\}]e_3 + [\{(x_1 - x_7) + i_2(x_3 + x_4)\} + i_1\{(x_2 - x_8) + i_2(x_5 + x_6)\}]e_3^{\dagger}$$

$$= \left[\{(x_1 + x_5 - x_6 + x_7) - i_2(x_2 - x_3 + x_4 + x_8)\}e_1 + \{(x_1 - x_5 + x_6 + x_7) + i_2(x_2 + x_3 - x_4 + x_8)\}e_1^{\dagger}\right]e_3 + \left[\{(x_1 + x_5 + x_6 - x_7) - i_2(x_2 - x_3 - x_4 - x_8)\}e_1 + \{(x_1 - x_5 - x_6 - x_7) + i_2(x_2 + x_3 + x_4 - x_8)\}e_1^{\dagger}\right]e_3^{\dagger}$$

$$= \{(x_1 + x_5 - x_6 + x_7) - i_2(x_2 - x_3 + x_4 + x_8)\}e_1e_3 + \{(x_1 - x_5 + x_6 + x_7) + i_2(x_2 + x_3 - x_4 + x_8)\}e_1^{\dagger}e_3 + \{(x_1 + x_5 + x_6 - x_7) - i_2(x_2 - x_3 - x_4 - x_8)\}e_1e_3^{\dagger} + \{(x_1 - x_5 - x_6 - x_7) + i_2(x_2 + x_3 + x_4 - x_8)\}e_1^{\dagger}e_3^{\dagger}$$

$= \{(x_1 + x_5 - x_6 + x_7) - i_2(x_2 - x_3 + x_4 + x_8)\}e_5 + \{(x_1 - x_5 + x_6 + x_7) + i_2(x_2 + x_3 - x_4 + x_8)\}e_6$

$+ \{(x_1 + x_5 + x_6 - x_7) - i_2(x_2 - x_3 - x_4 - x_8)\}e_4 + \{(x_1 - x_5 - x_6 - x_7) + i_2(x_2 + x_3 + x_4 - x_8)\}e_7$

(ii-c) Mixed $\mathbb{C}(i_1)$ and $\mathbb{C}(i_2)$-idempotent representations:

$$\begin{aligned}
\mathbb{C}(i_1, i_2, i_3) &= \mathbb{C}(i_1, i_2) \times_e \mathbb{C}(i_1, i_2) \\
&= \mathbb{C}(i_1, i_2)e_3 + \mathbb{C}(i_1, i_2)e_3^\dagger \\
&= \{\mathbb{C}(i_1)e_1 + \mathbb{C}(i_1)e_1^\dagger\}e_3 + \{\mathbb{C}(i_2)e_1 + \mathbb{C}(i_2)e_1^\dagger\}e_3^\dagger \\
&= \mathbb{C}(i_1)e_1e_3 + \mathbb{C}(i_1)e_1^\dagger e_3 + \mathbb{C}(i_2)e_1e_3^\dagger + \mathbb{C}(i_2)e_1^\dagger e_3^\dagger \\
&= \mathbb{C}(i_1)e_5 + \mathbb{C}(i_1)e_6 + \mathbb{C}(i_2)e_4 + \mathbb{C}(i_2)e_7 \\
&= \{\zeta \in \mathbb{C}(i_1, i_2, i_3): \alpha e_5 + \beta e_6 + \gamma e_4 + \delta e_7, (\alpha, \beta, \gamma, \delta) \in \mathbb{C}(i_1) \times \mathbb{C}(i_1) \times \mathbb{C}(i_2) \times \mathbb{C}(i_2)\}
\end{aligned}$$

The $\mathbb{C}(i_1)$ and $\mathbb{C}(i_2) -$ idempotent representation of $\zeta \in \mathbb{C}(i_1, i_2, i_3)$ is given as

$$\begin{aligned}
\zeta &= x_1 + i_1x_2 + i_2x_3 + i_3x_4 + i_1i_2x_5 + i_1i_3x_6 + i_2i_3x_7 + i_1i_2i_3x_8 \\
&= (x_1 + i_1x_2 + i_2x_3 + i_1i_2x_5) + i_3(x_4 + i_1x_6 + i_2x_7 + i_1i_2x_8) \\
&= \xi + i_3\eta \\
&= (\xi - i_2\eta)e_3 + (\xi + i_2\eta)e_3^\dagger \\
&= \{(x_1 + x_7) + i_1(x_2 + x_8) + i_2(x_3 - x_4) + i_1i_2(x_5 - x_6)\}e_3 \\
&\qquad + \{(x_1 - x_7) + i_1(x_2 - x_8) + i_2(x_3 + x_4) + i_1i_2(x_5 + x_6)\}e_3^\dagger \\
&= [\{(x_1 + x_7) + i_1(x_2 + x_8)\} + i_2\{(x_3 - x_4) + i_1(x_5 - x_6)\}]e_3 \\
&\qquad + [\{(x_1 - x_7) + i_2(x_3 + x_4)\} + i_1\{(x_2 - x_8) + i_2(x_5 + x_6)\}]e_3^\dagger \\
&= [\{(x_1 + x_5 - x_6 + x_7) + i_1(x_2 - x_3 + x_4 + x_8)\}e_1 + \{(x_1 - x_5 + x_6 + x_7) + i_1(x_2 + x_3 - x_4 + x_8)\}e_1^\dagger]e_3 \\
&\qquad + [\{(x_1 + x_5 + x_6 - x_7) - i_2(x_2 - x_3 - x_4 - x_8)\}e_1 + \{(x_1 - x_5 - x_6 - x_7) + i_2(x_2 + x_3 + x_4 - x_8)\}e_1^\dagger]e_3^\dagger \\
&= \{(x_1 + x_5 - x_6 + x_7) + i_1(x_2 - x_3 + x_4 + x_8)\}e_1e_3 + \{(x_1 - x_5 + x_6 + x_7) + i_1(x_2 + x_3 - x_4 + x_8)\}e_1^\dagger e_3 \\
&\quad + \{(x_1 + x_5 + x_6 - x_7) - i_2(x_2 - x_3 - x_4 - x_8)\}e_1e_3^\dagger + \{(x_1 - x_5 - x_6 - x_7) + i_2(x_2 + x_3 + x_4 - x_8)\}e_1^\dagger e_3^\dagger \\
&= \{(x_1 + x_5 - x_6 + x_7) + i_1(x_2 - x_3 + x_4 + x_8)\}e_5 + \{(x_1 - x_5 + x_6 + x_7) + i_1(x_2 + x_3 - x_4 + x_8)\}e_6 \\
&\quad + \{(x_1 + x_5 + x_6 - x_7) - i_2(x_2 - x_3 - x_4 - x_8)\}e_4 + \{(x_1 - x_5 - x_6 - x_7) + i_2(x_2 + x_3 + x_4 - x_8)\}e_7
\end{aligned}$$

(ii-d) Mixed $\mathbb{C}(i_1)$ and $\mathbb{C}(i_2)$-idempotent representations:

$$\begin{aligned}
\mathbb{C}(i_1, i_2, i_3) &= \mathbb{C}(i_1, i_2) \times_e \mathbb{C}(i_1, i_2) \\
&= \mathbb{C}(i_1, i_2)e_3 + \mathbb{C}(i_1, i_2)e_3^\dagger \\
&= \{\mathbb{C}(i_2)e_1 + \mathbb{C}(i_2)e_1^\dagger\}e_3 + \{\mathbb{C}(i_1)e_1 + \mathbb{C}(i_1)e_1^\dagger\}e_3^\dagger \\
&= \mathbb{C}(i_2)e_1e_3 + \mathbb{C}(i_2)e_1^\dagger e_3 + \mathbb{C}(i_1)e_1e_3^\dagger + \mathbb{C}(i_1)e_1^\dagger e_3^\dagger \\
&= \mathbb{C}(i_2)e_5 + \mathbb{C}(i_2)e_6 + \mathbb{C}(i_1)e_4 + \mathbb{C}(i_1)e_7 \\
&= \{\zeta \in \mathbb{C}(i_1, i_2, i_3): \alpha e_5 + \beta e_6 + \gamma e_4 + \delta e_7, (\alpha, \beta, \gamma, \delta) \in \mathbb{C}(i_2) \times \mathbb{C}(i_2) \times \mathbb{C}(i_1) \times \mathbb{C}(i_1)\}
\end{aligned}$$

The $\mathbb{C}(i_2)$ and $\mathbb{C}(i_1) -$ idempotent representation of $\zeta \in \mathbb{C}(i_1, i_2, i_3)$ is given as

$$\begin{aligned}
\zeta &= x_1 + i_1x_2 + i_2x_3 + i_3x_4 + i_1i_2x_5 + i_1i_3x_6 + i_2i_3x_7 + i_1i_2i_3x_8 \\
&= (x_1 + i_1x_2 + i_2x_3 + i_1i_2x_5) + i_3(x_4 + i_1x_6 + i_2x_7 + i_1i_2x_8) \\
&= \xi + i_3\eta \\
&= (\xi - i_2\eta)e_3 + (\xi + i_2\eta)e_3^\dagger \\
&= \{(x_1 + x_7) + i_1(x_2 + x_8) + i_2(x_3 - x_4) + i_1i_2(x_5 - x_6)\}e_3
\end{aligned}$$

$$+\{(x_1-x_7)+i_1(x_2-x_8)+i_2(x_3+x_4)+i_1i_2(x_5+x_6)\}e_3^\dagger$$

$$=[\{(x_1+x_7)+i_2(x_3-x_4)\}+i_1\{(x_2+x_8)+i_2(x_5-x_6)\}]e_3$$

$$+[\{(x_1-x_7)+i_1(x_2-x_8)\}+i_2\{(x_3+x_4)+i_1(x_5+x_6)\}]e_3^\dagger$$

$$=\left[\{(x_1+x_5-x_6+x_7)-i_2(x_2-x_3+x_4+x_8)\}e_1+\{(x_1-x_5+x_6+x_7)+i_2(x_2+x_3-x_4+x_8)\}e_1^\dagger\right]e_3$$

$$+\left[\{(x_1+x_5+x_6-x_7)+i_1(x_2-x_3-x_4-x_8)\}e_1+\{(x_1-x_5-x_6-x_7)+i_1(x_2+x_3+x_4-x_8)\}e_1^\dagger\right]e_3^\dagger$$

$$=\{(x_1+x_5-x_6+x_7)-i_2(x_2-x_3+x_4+x_8)\}e_1e_3+\{(x_1-x_5+x_6+x_7)+i_2(x_2+x_3-x_4+x_8)\}e_1^\dagger e_3$$

$$+\{(x_1+x_5+x_6-x_7)+i_1(x_2-x_3-x_4-x_8)\}e_1e_3^\dagger+\{(x_1-x_5-x_6-x_7)+i_1(x_2+x_3+x_4-x_8)\}e_1^\dagger e_3^\dagger$$

$$=\{(x_1+x_5-x_6+x_7)-i_2(x_2-x_3+x_4+x_8)\}e_5+\{(x_1-x_5+x_6+x_7)+i_2(x_2+x_3-x_4+x_8)\}e_6$$

$$+\{(x_1+x_5+x_6-x_7)+i_1(x_2-x_3-x_4-x_8)\}e_4+\{(x_1-x_5-x_6-x_7)+i_1(x_2+x_3+x_4-x_8)\}e_7$$

(iii-a) $\mathbb{C}(i_1)$ -idempotent representation:

$$\mathbb{C}(i_1,i_2,i_3)=\mathbb{C}(i_1,i_3)\times_e\mathbb{C}(i_1,i_3)$$

$$=\mathbb{C}(i_1,i_3)e_1+\mathbb{C}(i_1,i_3)e_1^\dagger$$

$$=\{\mathbb{C}(i_1)e_2+\mathbb{C}(i_1)e_2^\dagger\}e_1+\{\mathbb{C}(i_1)e_2+\mathbb{C}(i_1)e_2^\dagger\}e_1^\dagger$$

$$=\mathbb{C}(i_1)e_1e_2+\mathbb{C}(i_1)e_1e_2^\dagger+\mathbb{C}(i_1)e_1^\dagger e_2+\mathbb{C}(i_1)e_1^\dagger e_2^\dagger$$

$$=\mathbb{C}(i_1)e_4+\mathbb{C}(i_1)e_5+\mathbb{C}(i_1)e_6+\mathbb{C}(i_1)e_7$$

$$=\{\zeta\in\mathbb{C}(i_1,i_2,i_3):\alpha e_4+\beta e_5+\gamma e_6+\delta e_7,(\alpha,\beta,\gamma,\delta)\in\mathbb{C}(i_1)\times\mathbb{C}(i_1)\times\mathbb{C}(i_1)\times\mathbb{C}(i_1)\}$$

The $\mathbb{C}(i_1)-$ idempotent representation of $\zeta\in\mathbb{C}(i_1,i_2,i_3)$ is given as

$$\zeta=x_1+i_1x_2+i_2x_3+i_3x_4+i_1i_2x_5+i_1i_3x_6+i_2i_3x_7+i_1i_2i_3x_8$$

$$=(x_1+i_1x_2+i_3x_4+i_1i_3x_6)+i_2(x_3+i_1x_5+i_3x_7+i_1i_3x_8)$$

$$=\xi+i_2\eta$$

$$=(\xi-i_1\eta)e_1+(\xi+i_1\eta)e_1^\dagger$$

$$=\{(x_1+x_5)+i_1(x_2-x_3)+i_3(x_4+x_8)+i_1i_3(x_6-x_7)\}e_1$$

$$+\{(x_1-x_5)+i_1(x_2+x_3)+i_3(x_4-x_8)+i_1i_3(x_6+x_7)\}e_1^\dagger$$

$$=[\{(x_1+x_5)+i_1(x_2-x_3)\}+i_3\{(x_4+x_8)+i_1(x_6-x_7)\}]e_1$$

$$+[\{(x_1-x_5)+i_1(x_2+x_3)\}+i_3\{(x_4-x_8)+i_1(x_6+x_7)\}]e_1^\dagger$$

$$=\left[\{(x_1+x_5+x_6-x_7)+i_1(x_2-x_3-x_4-x_8)\}e_2+\{(x_1+x_5-x_6+x_7)+i_1(x_2-x_3+x_4+x_8)\}e_2^\dagger\right]e_1$$

$$+\left[\{(x_1-x_5+x_6+x_7)+i_1(x_2+x_3-x_4+x_8)\}e_2+\{(x_1-x_5-x_6-x_7)+i_1(x_2+x_3+x_4-x_8)\}e_2^\dagger\right]e_1^\dagger$$

$$=\{(x_1+x_5+x_6-x_7)+i_1(x_2-x_3-x_4-x_8)\}e_1e_2+\{(x_1+x_5-x_6+x_7)+i_1(x_2-x_3+x_4+x_8)\}e_1e_2^\dagger$$

$$+\{(x_1-x_5+x_6+x_7)+i_1(x_2+x_3-x_4+x_8)\}e_1^\dagger e_2+\{(x_1-x_5-x_6-x_7)+i_1(x_2+x_3+x_4-x_8)\}e_1^\dagger e_2^\dagger$$

$$=\{(x_1+x_5+x_6-x_7)+i_1(x_2-x_3-x_4-x_8)\}e_4+\{(x_1+x_5-x_6+x_7)+i_1(x_2-x_3+x_4+x_8)\}e_5$$

$$+\{(x_1-x_5+x_6+x_7)+i_1(x_2+x_3-x_4+x_8)\}e_6+\{(x_1-x_5-x_6-x_7)+i_1(x_2+x_3+x_4-x_8)\}e_7$$

(iii-b) $\mathbb{C}(i_3)$-idempotent representation:

$$\mathbb{C}(i_1,i_2,i_3)=\mathbb{C}(i_1,i_3)\times_e\mathbb{C}(i_1,i_3)$$

$$=\mathbb{C}(i_1,i_3)e_1+\mathbb{C}(i_1,i_3)e_2^\dagger$$

$$=\{\mathbb{C}(i_3)e_2+\mathbb{C}(i_3)e_2^\dagger\}e_1+\{\mathbb{C}(i_3)e_2+\mathbb{C}(i_3)e_2^\dagger\}e_1^\dagger$$

$$=\mathbb{C}(i_3)e_1e_2+\mathbb{C}(i_3)e_1e_2^\dagger+\mathbb{C}(i_3)e_1^\dagger e_2+\mathbb{C}(i_3)e_1^\dagger e_2^\dagger$$

$$=\mathbb{C}(i_3)e_4+\mathbb{C}(i_3)e_5+\mathbb{C}(i_3)e_6+\mathbb{C}(i_3)e_7$$

$$= \{\zeta \in \mathbb{C}(i_1,i_2,i_3): \alpha e_4 + \beta e_5 + \gamma e_6 + \delta e_7, (\alpha,\beta,\gamma,\delta) \in \mathbb{C}(i_3)\times\mathbb{C}(i_3)\times\mathbb{C}(i_3)\times\mathbb{C}(i_3)\}$$

The $\mathbb{C}(i_3) -$ idempotent representation of $\zeta \in \mathbb{C}(i_1,i_2,i_3)$ is given as

$$\zeta = x_1 + i_1x_2 + i_2x_3 + i_3x_4 + i_1i_2x_5 + i_1i_3x_6 + i_2i_3x_7 + i_1i_2i_3x_8$$

$$= (x_1 + i_1x_2 + i_3x_4 + i_1i_3x_6) + i_2(x_3 + i_1x_5 + i_3x_7 + i_1i_3x_8)$$

$$= \xi + i_2\eta$$

$$= (\xi - i_1\eta)e_1 + (\xi + i_1\eta)e_1^\dagger$$

$$= \{(x_1+x_5) + i_1(x_2-x_3) + i_3(x_4+x_8) + i_1i_3(x_6-x_7)\}e_1$$
$$+ \{(x_1-x_5) + i_1(x_2+x_3) + i_3(x_4-x_8) + i_1i_3(x_6+x_7)\}e_1^\dagger$$

$$= [\{(x_1+x_5) + i_3(x_4+x_8)\} + i_1\{(x_2-x_3) + i_3(x_6-x_7)\}]e_1$$
$$+ [\{(x_1-x_5) + i_3(x_4-x_8)\} + i_1\{(x_2+x_3) + i_3(x_6+x_7)\}]e_1^\dagger$$

$$= \left[\{(x_1+x_5+x_6-x_7) - i_3(x_2-x_3-x_4-x_8)\}e_2 + \{(x_1+x_5-x_6+x_7) + i_3(x_2-x_3+x_4+x_8)\}e_2^\dagger\right]e_1$$
$$+ \left[\{(x_1-x_5+x_6+x_7) - i_3(x_2+x_3-x_4+x_8)\}e_2 + \{(x_1-x_5-x_6-x_7) + i_3(x_2+x_3+x_4-x_8)\}e_2^\dagger\right]e_1^\dagger$$

$$= \{(x_1+x_5+x_6-x_7) - i_3(x_2-x_3-x_4-x_8)\}e_1e_2 + \{(x_1+x_5-x_6+x_7) + i_3(x_2-x_3+x_4+x_8)\}e_1e_2^\dagger$$
$$+ \{(x_1-x_5+x_6+x_7) - i_3(x_2+x_3-x_4+x_8)\}e_1^\dagger e_2 + \{(x_1-x_5-x_6-x_7) + i_3(x_2+x_3+x_4-x_8)\}e_1^\dagger e_2^\dagger$$

$$= \{(x_1+x_5+x_6-x_7) - i_3(x_2-x_3-x_4-x_8)\}e_4 + \{(x_1+x_5-x_6+x_7) + i_3(x_2-x_3+x_4+x_8)\}e_5$$
$$+ \{(x_1-x_5+x_6+x_7) - i_3(x_2+x_3-x_4+x_8)\}e_6 + \{(x_1-x_5-x_6-x_7) + i_3(x_2+x_3+x_4-x_8)\}e_7$$

(iii-c) Mixed $\mathbb{C}(i_1)$ and $\mathbb{C}(i_3)$-idempotent representations:

$$\mathbb{C}(i_1,i_2,i_3) = \mathbb{C}(i_1,i_3)\times_e \mathbb{C}(i_1,i_3)$$
$$= \mathbb{C}(i_1,i_3)e_1 + \mathbb{C}(i_1,i_3)e_1^\dagger$$
$$= \{\mathbb{C}(i_1)e_2 + \mathbb{C}(i_1)e_2^\dagger\}e_1 + \{\mathbb{C}(i_3)e_2 + \mathbb{C}(i_3)e_2^\dagger\}e_1^\dagger$$
$$= \mathbb{C}(i_1)e_1e_2 + \mathbb{C}(i_1)e_1e_2^\dagger + \mathbb{C}(i_3)e_1^\dagger e_2 + \mathbb{C}(i_3)e_1^\dagger e_2^\dagger$$
$$= \mathbb{C}(i_1)e_4 + \mathbb{C}(i_1)e_5 + \mathbb{C}(i_3)e_6 + \mathbb{C}(i_3)e_7$$
$$= \{\zeta \in \mathbb{C}(i_1,i_2,i_3): \alpha e_4 + \beta e_5 + \gamma e_6 + \delta e_7, (\alpha,\beta,\gamma,\delta) \in \mathbb{C}(i_1)\times\mathbb{C}(i_1)\times\mathbb{C}(i_3)\times\mathbb{C}(i_3)\}$$

The $\mathbb{C}(i_1)$ and $\mathbb{C}(i_3) -$ idempotent representation of $\zeta \in \mathbb{C}(i_1,i_2,i_3)$ is given as

$$\zeta = x_1 + i_1x_2 + i_2x_3 + i_3x_4 + i_1i_2x_5 + i_1i_3x_6 + i_2i_3x_7 + i_1i_2i_3x_8$$

$$= (x_1 + i_1x_2 + i_3x_4 + i_1i_3x_6) + i_2(x_3 + i_1x_5 + i_3x_7 + i_1i_3x_8)$$

$$= \xi + i_2\eta$$

$$= (\xi - i_1\eta)e_1 + (\xi + i_1\eta)e_1^\dagger$$

$$= \{(x_1+x_5) + i_1(x_2-x_3) + i_3(x_4+x_8) + i_1i_3(x_6-x_7)\}e_1$$
$$+ \{(x_1-x_5) + i_1(x_2+x_3) + i_3(x_4-x_8) + i_1i_3(x_6+x_7)\}e_1^\dagger$$

$$= [\{(x_1+x_5) + i_1(x_2-x_3)\} + i_3\{(x_4+x_8) + i_1(x_6-x_7)\}]e_1$$
$$+ [\{(x_1-x_5) + i_3(x_4-x_8)\} + i_1\{(x_2+x_3) + i_3(x_6+x_7)\}]e_1^\dagger$$

$$= \left[\{(x_1+x_5+x_6-x_7) + i_1(x_2-x_3-x_4-x_8)\}e_2 + \{(x_1+x_5-x_6+x_7) + i_1(x_2-x_3+x_4+x_8)\}e_2^\dagger\right]e_1$$
$$+ \left[\{(x_1-x_5+x_6+x_7) - i_3(x_2+x_3-x_4+x_8)\}e_2 + \{(x_1-x_5-x_6-x_7) + i_3(x_2+x_3+x_4-x_8)\}e_2^\dagger\right]e_1^\dagger$$

$$= \{(x_1+x_5+x_6-x_7) + i_1(x_2-x_3-x_4-x_8)\}e_1e_2 + \{(x_1+x_5-x_6+x_7) + i_1(x_2-x_3+x_4+x_8)\}e_1e_2^\dagger$$
$$+ \{(x_1-x_5+x_6+x_7) - i_3(x_2+x_3-x_4+x_8)\}e_1^\dagger e_2 + \{(x_1-x_5-x_6-x_7) + i_3(x_2+x_3+x_4-x_8)\}e_1^\dagger e_2^\dagger$$

$$= \{(x_1+x_5+x_6-x_7) + i_1(x_2-x_3-x_4-x_8)\}e_4 + \{(x_1+x_5-x_6+x_7) + i_1(x_2-x_3+x_4+x_8)\}e_5$$
$$+ \{(x_1-x_5+x_6+x_7) - i_3(x_2+x_3-x_4+x_8)\}e_6 + \{(x_1-x_5-x_6-x_7) + i_3(x_2+x_3+x_4-x_8)\}e_7$$

(iii-d) Mixed $\mathbb{C}(i_1)$ and $\mathbb{C}(i_3)$-idempotent representations:

$$\mathbb{C}(i_1,i_2,i_3) = \mathbb{C}(i_1,i_3)\times_e \mathbb{C}(i_1,i_3)$$
$$= \mathbb{C}(i_1,i_3)e_1 + \mathbb{C}(i_1,i_3)e_2^{\dagger}$$
$$= \{\mathbb{C}(i_3)e_2 + \mathbb{C}(i_3)e_2^{\dagger}\}e_1 + \{\mathbb{C}(i_1)e_2 + \mathbb{C}(i_1)e_2^{\dagger}\}e_1^{\dagger}$$
$$= \mathbb{C}(i_3)e_1e_2 + \mathbb{C}(i_3)e_1e_2^{\dagger} + \mathbb{C}(i_1)e_1^{\dagger}e_2 + \mathbb{C}(i_1)e_1^{\dagger}e_2^{\dagger}$$
$$= \mathbb{C}(i_3)e_4 + \mathbb{C}(i_3)e_5 + \mathbb{C}(i_1)e_6 + \mathbb{C}(i_1)e_7$$
$$= \{\zeta \in \mathbb{C}(i_1,i_2,i_3): \alpha e_4 + \beta e_5 + \gamma e_6 + \delta e_7, (\alpha,\beta,\gamma,\delta) \in \mathbb{C}(i_3)\times\mathbb{C}(i_3)\times\mathbb{C}(i_1)\times\mathbb{C}(i_1)\}$$

The $\mathbb{C}(i_3)$ and $\mathbb{C}(i_1) -$ idempotent representation of $\zeta \in \mathbb{C}(i_1,i_2,i_3)$ is given as

$$\zeta = x_1 + i_1x_2 + i_2x_3 + i_3x_4 + i_1i_2x_5 + i_1i_3x_6 + i_2i_3x_7 + i_1i_2i_3x_8$$
$$= (x_1 + i_1x_2 + i_3x_4 + i_1i_3x_6) + i_2(x_3 + i_1x_5 + i_3x_7 + i_1i_3x_8)$$
$$= \xi + i_2\eta$$
$$= (\xi - i_1\eta)e_1 + (\xi + i_1\eta)e_1^{\dagger}$$
$$= \{(x_1+x_5) + i_1(x_2-x_3) + i_3(x_4+x_8) + i_1i_3(x_6-x_7)\}e_1 + \{(x_1-x_5) + i_1(x_2+x_3) + i_3(x_4-x_8) + i_1i_3(x_6+x_7)\}e_1^{\dagger}$$
$$= [\{(x_1+x_5) + i_3(x_4+x_8)\} + i_1\{(x_2-x_3) + i_3(x_6-x_7)\}]e_1 + [\{(x_1-x_5) + i_1(x_2+x_3)\} + i_3\{(x_4-x_8) + i_1(x_6+x_7)\}]e_1^{\dagger}$$
$$= \left[\{(x_1+x_5+x_6-x_7) - i_3(x_2-x_3-x_4-x_8)\}e_2 + \{(x_1+x_5-x_6+x_7) + i_3(x_2-x_3+x_4+x_8)\}e_2^{\dagger}\right]e_1 + \left[\{(x_1-x_5+x_6+x_7) + i_1(x_2+x_3-x_4+x_8)\}e_2 + \{(x_1-x_5-x_6-x_7) + i_1(x_2+x_3+x_4-x_8)\}e_2^{\dagger}\right]e_1^{\dagger}$$
$$= \{(x_1+x_5+x_6-x_7) - i_3(x_2-x_3-x_4-x_8)\}e_1e_2 + \{(x_1+x_5-x_6+x_7) + i_3(x_2-x_3+x_4+x_8)\}e_1e_2^{\dagger} + \{(x_1-x_5+x_6+x_7) + i_1(x_2+x_3-x_4+x_8)\}e_1^{\dagger}e_2 + \{(x_1-x_5-x_6-x_7) + i_1(x_2+x_3+x_4-x_8)\}e_1^{\dagger}e_2^{\dagger}$$
$$= \{(x_1+x_5+x_6-x_7) - i_3(x_2-x_3-x_4-x_8)\}e_4 + \{(x_1+x_5-x_6+x_7) + i_3(x_2-x_3+x_4+x_8)\}e_5 + \{(x_1-x_5+x_6+x_7) + i_1(x_2+x_3-x_4+x_8)\}e_6 + \{(x_1-x_5-x_6-x_7) + i_1(x_2+x_3+x_4-x_8)\}e_7$$

(iv-a) $\mathbb{C}(i_1)$-idempotent representation:

$$\mathbb{C}(i_1,i_2,i_3) = \mathbb{C}(i_1,i_3)\times_e \mathbb{C}(i_1,i_3)$$
$$= \mathbb{C}(i_1,i_3)e_3 + \mathbb{C}(i_1,i_3)e_3^{\dagger}$$
$$= \{\mathbb{C}(i_1)e_2 + \mathbb{C}(i_1)e_2^{\dagger}\}e_3 + \{\mathbb{C}(i_1)e_2 + \mathbb{C}(i_1)e_2^{\dagger}\}e_3^{\dagger}$$
$$= \mathbb{C}(i_1)e_2e_3 + \mathbb{C}(i_1)e_2^{\dagger}e_3 + \mathbb{C}(i_1)e_2e_3^{\dagger} + \mathbb{C}(i_1)e_2^{\dagger}e_3^{\dagger}$$
$$= \mathbb{C}(i_1)e_6 + \mathbb{C}(i_1)e_5 + \mathbb{C}(i_1)e_4 + \mathbb{C}(i_1)e_7$$
$$= \{\zeta \in \mathbb{C}(i_1,i_2,i_3): \alpha e_4 + \beta e_5 + \gamma e_6 + \delta e_7, (\alpha,\beta,\gamma,\delta) \in \mathbb{C}(i_1)\times\mathbb{C}(i_1)\times\mathbb{C}(i_1)\times\mathbb{C}(i_1)\}$$

The $\mathbb{C}(i_1) -$ idempotent representation of $\zeta \in \mathbb{C}(i_1,i_2,i_3)$ is given as

$$\zeta = x_1 + i_1x_2 + i_2x_3 + i_3x_4 + i_1i_2x_5 + i_1i_3x_6 + i_2i_3x_7 + i_1i_2i_3x_8$$
$$= (x_1 + i_1x_2 + i_3x_4 + i_1i_3x_6) + i_2(x_3 + i_1x_5 + i_3x_7 + i_1i_3x_8)$$
$$= \xi + i_2\eta$$
$$= (\xi - i_3\eta)e_3 + (\xi + i_3\eta)e_3^{\dagger}$$
$$= \{(x_1+x_7) + i_1(x_2+x_8) - i_3(x_3-x_4) - i_1i_3(x_5-x_6)\}e_3 + \{(x_1-x_7) + i_1(x_2-x_8) + i_3(x_3+x_4) + i_1i_3(x_5+x_6)\}e_3^{\dagger}$$
$$= [\{(x_1+x_7) + i_1(x_2+x_8)\} + i_3\{-(x_3-x_4) - i_1(x_5-x_6)\}]e_3 + [\{(x_1-x_7) + i_1(x_2-x_8)\} + i_3\{(x_3+x_4) + i_1(x_5+x_6)\}]e_3^{\dagger}$$
$$= \left[\{(x_1-x_5+x_6+x_7) + i_1(x_2+x_3-x_4+x_8)\}e_2 + \{(x_1+x_5-x_6+x_7) + i_1(x_2-x_3+x_4+x_8)\}e_2^{\dagger}\right]e_3$$

$+ \left[\{(x_1 + x_5 + x_6 - x_7) + i_1(x_2 - x_3 - x_4 - x_8)\}e_2 + \{(x_1 - x_5 - x_6 - x_7) + i_1(x_2 + x_3 + x_4 - x_8)\}e_2^\dagger\right]e_3^\dagger$

$= \{(x_1 - x_5 + x_6 + x_7) + i_1(x_2 + x_3 - x_4 + x_8)\}e_2e_3 + \{(x_1 + x_5 - x_6 + x_7) + i_1(x_2 - x_3 + x_4 + x_8)\}e_2^\dagger e_3$

$+ \{(x_1 + x_5 + x_6 - x_7) + i_1(x_2 - x_3 - x_4 - x_8)\}e_2e_3^\dagger + \{(x_1 - x_5 - x_6 - x_7) + i_1(x_2 + x_3 + x_4 - x_8)\}e_2^\dagger e_3^\dagger$

$= \{(x_1 - x_5 + x_6 + x_7) + i_1(x_2 + x_3 - x_4 + x_8)\}e_6 + \{(x_1 + x_5 - x_6 + x_7) + i_1(x_2 - x_3 + x_4 + x_8)\}e_5$

$+ \{(x_1 + x_5 + x_6 - x_7) + i_1(x_2 - x_3 - x_4 - x_8)\}e_4 + \{(x_1 - x_5 - x_6 - x_7) + i_1(x_2 + x_3 + x_4 - x_8)\}e_7$

(iv-b) $\mathbb{C}(i_3)$-idempotent representation:

$$\begin{aligned}
\mathbb{C}(i_1, i_2, i_3) &= \mathbb{C}(i_1, i_3) \times_e \mathbb{C}(i_1, i_3) \\
&= \mathbb{C}(i_1, i_3)e_3 + \mathbb{C}(i_1, i_3)e_3^\dagger \\
&= \{\mathbb{C}(i_3)e_2 + \mathbb{C}(i_3)e_2^\dagger\}e_3 + \{\mathbb{C}(i_3)e_2 + \mathbb{C}(i_3)e_2^\dagger\}e_3^\dagger \\
&= \mathbb{C}(i_3)e_2e_3 + \mathbb{C}(i_3)e_2^\dagger e_3 + \mathbb{C}(i_3)e_2e_3^\dagger + \mathbb{C}(i_3)e_2^\dagger e_3^\dagger \\
&= \mathbb{C}(i_3)e_6 + \mathbb{C}(i_3)e_5 + \mathbb{C}(i_3)e_4 + \mathbb{C}(i_3)e_7 \\
&= \{\zeta \in \mathbb{C}(i_1, i_2, i_3): \alpha e_4 + \beta e_5 + \gamma e_6 + \delta e_7, (\alpha, \beta, \gamma, \delta) \in \mathbb{C}(i_3) \times \mathbb{C}(i_3) \times \mathbb{C}(i_3) \times \mathbb{C}(i_3)\}
\end{aligned}$$

The $\mathbb{C}(i_3) -$ idempotent representation of $\zeta \in \mathbb{C}(i_1, i_2, i_3)$ is given as

$\zeta = x_1 + i_1x_2 + i_2x_3 + i_3x_4 + i_1i_2x_5 + i_1i_3x_6 + i_2i_3x_7 + i_1i_2i_3x_8$

$= (x_1 + i_1x_2 + i_3x_4 + i_1i_3x_6) + i_2(x_3 + i_1x_5 + i_3x_7 + i_1i_3x_8)$

$= \xi + i_2\eta$

$= (\xi - i_3\eta)e_3 + (\xi + i_3\eta)e_3^\dagger$

$= \{(x_1 + x_7) + i_1(x_2 + x_8) - i_3(x_3 - x_4) - i_1i_3(x_5 - x_6)\}e_3$

$+ \{(x_1 - x_7) + i_1(x_2 - x_8) + i_3(x_3 + x_4) + i_1i_3(x_5 + x_6)\}e_3^\dagger$

$= [\{(x_1 + x_7) - i_3(x_3 - x_4)\} + i_1\{(x_2 + x_8) - i_3(x_5 - x_6)\}]e_3$

$+ [\{(x_1 - x_7) + i_3(x_3 + x_4)\} + i_1\{(x_2 - x_8) + i_3(x_5 + x_6)\}]e_3^\dagger$

$= \left[\{(x_1 - x_5 + x_6 + x_7) - i_3(x_2 + x_3 - x_4 + x_8)\}e_2 + \{(x_1 + x_5 - x_6 + x_7) + i_3(x_2 - x_3 + x_4 + x_8)\}e_2^\dagger\right]e_3$

$+ \left[\{(x_1 + x_5 + x_6 - x_7) - i_3(x_2 - x_3 - x_4 - x_8)\}e_2 + \{(x_1 - x_5 - x_6 - x_7) + i_3(x_2 + x_3 + x_4 - x_8)\}e_2^\dagger\right]e_3^\dagger$

$= \{(x_1 - x_5 + x_6 + x_7) - i_3(x_2 + x_3 - x_4 + x_8)\}e_2e_3 + \{(x_1 + x_5 - x_6 + x_7) + i_3(x_2 - x_3 + x_4 + x_8)\}e_2^\dagger e_3$

$+ \{(x_1 + x_5 + x_6 - x_7) - i_3(x_2 - x_3 - x_4 - x_8)\}e_2e_3^\dagger + \{(x_1 - x_5 - x_6 - x_7) + i_3(x_2 + x_3 + x_4 - x_8)\}e_2^\dagger e_3^\dagger$

$= \{(x_1 - x_5 + x_6 + x_7) - i_3(x_2 + x_3 - x_4 + x_8)\}e_6 + \{(x_1 + x_5 - x_6 + x_7) + i_3(x_2 - x_3 + x_4 + x_8)\}e_5$

$+ \{(x_1 + x_5 + x_6 - x_7) - i_3(x_2 - x_3 - x_4 - x_8)\}e_4 + \{(x_1 - x_5 - x_6 - x_7) + i_3(x_2 + x_3 + x_4 - x_8)\}e_7$

(iv-c) Mixed $\mathbb{C}(i_1)$ and $\mathbb{C}(i_3)$-idempotent representations:

$$\begin{aligned}
\mathbb{C}(i_1, i_2, i_3) &= \mathbb{C}(i_1, i_3) \times_e \mathbb{C}(i_1, i_3) \\
&= \mathbb{C}(i_1, i_3)e_3 + \mathbb{C}(i_1, i_3)e_3^\dagger \\
&= \{\mathbb{C}(i_1)e_2 + \mathbb{C}(i_1)e_2^\dagger\}e_3 + \{\mathbb{C}(i_3)e_2 + \mathbb{C}(i_3)e_2^\dagger\}e_3^\dagger \\
&= \mathbb{C}(i_1)e_2e_3 + \mathbb{C}(i_1)e_2^\dagger e_3 + \mathbb{C}(i_3)e_2e_3^\dagger + \mathbb{C}(i_3)e_2^\dagger e_3^\dagger \\
&= \mathbb{C}(i_1)e_6 + \mathbb{C}(i_1)e_5 + \mathbb{C}(i_3)e_4 + \mathbb{C}(i_3)e_7 \\
&= \{\zeta \in \mathbb{C}(i_1, i_2, i_3): \alpha e_6 + \beta e_5 + \gamma e_4 + \delta e_7, (\alpha, \beta, \gamma, \delta) \in \mathbb{C}(i_1) \times \mathbb{C}(i_1) \times \mathbb{C}(i_3) \times \mathbb{C}(i_3)\}
\end{aligned}$$

The $\mathbb{C}(i_1)$ and $\mathbb{C}(i_3) -$ idempotent representation of $\zeta \in \mathbb{C}(i_1, i_2, i_3)$ is given as

$\zeta = x_1 + i_1x_2 + i_2x_3 + i_3x_4 + i_1i_2x_5 + i_1i_3x_6 + i_2i_3x_7 + i_1i_2i_3x_8$

$= (x_1 + i_1x_2 + i_3x_4 + i_1i_3x_6) + i_2(x_3 + i_1x_5 + i_3x_7 + i_1i_3x_8)$

$= \xi + i_2\eta$

$= (\xi - i_3\eta)e_3 + (\xi + i_3\eta)e_3^{\dagger}$

$= \{(x_1 + x_7) + i_1(x_2 + x_8) - i_3(x_3 - x_4) - i_1i_3(x_5 - x_6)\}e_3$

$+ \{(x_1 - x_7) + i_1(x_2 - x_8) + i_3(x_3 + x_4) + i_1i_3(x_5 + x_6)\}e_3^{\dagger}$

$= [\{(x_1 + x_7) + i_1(x_2 + x_8)\} + i_3\{-(x_3 - x_4) - i_1(x_5 - x_6)\}]e_3$

$+ [\{(x_1 - x_7) + i_3(x_3 + x_4)\} + i_1\{(x_2 - x_8) + i_3(x_5 + x_6)\}]e_3^{\dagger}$

$= \left[\{(x_1 - x_5 + x_6 + x_7) + i_1(x_2 + x_3 - x_4 + x_8)\}e_2 + \{(x_1 + x_5 - x_6 + x_7) + i_1(x_2 - x_3 + x_4 + x_8)\}e_2^{\dagger}\right]e_3$

$+ \left[\{(x_1 + x_5 + x_6 - x_7) - i_3(x_2 - x_3 - x_4 - x_8)\}e_2 + \{(x_1 - x_5 - x_6 - x_7) + i_3(x_2 + x_3 + x_4 - x_8)\}e_2^{\dagger}\right]e_3^{\dagger}$

$= \{(x_1 - x_5 + x_6 + x_7) + i_1(x_2 + x_3 - x_4 + x_8)\}e_2e_3 + \{(x_1 + x_5 - x_6 + x_7) + i_1(x_2 - x_3 + x_4 + x_8)\}e_2^{\dagger}e_3$

$+ \{(x_1 + x_5 + x_6 - x_7) - i_3(x_2 - x_3 - x_4 - x_8)\}e_2e_3^{\dagger} + \{(x_1 - x_5 - x_6 - x_7) + i_3(x_2 + x_3 + x_4 - x_8)\}e_2^{\dagger}e_3^{\dagger}$

$= \{(x_1 - x_5 + x_6 + x_7) + i_1(x_2 + x_3 - x_4 + x_8)\}e_6 + \{(x_1 + x_5 - x_6 + x_7) + i_1(x_2 - x_3 + x_4 + x_8)\}e_5$

$+ \{(x_1 + x_5 + x_6 - x_7) - i_3(x_2 - x_3 - x_4 - x_8)\}e_4 + \{(x_1 - x_5 - x_6 - x_7) + i_3(x_2 + x_3 + x_4 - x_8)\}e_7$

(iv-d) Mixed $\mathbb{C}(i_1)$ and $\mathbb{C}(i_3)$-idempotent representations:

$$\begin{aligned}
\mathbb{C}(i_1, i_2, i_3) &= \mathbb{C}(i_1, i_3) \times_e \mathbb{C}(i_1, i_3) \\
&= \mathbb{C}(i_1, i_3)e_3 + \mathbb{C}(i_1, i_3)e_3^{\dagger} \\
&= \{\mathbb{C}(i_3)e_2 + \mathbb{C}(i_3)e_2^{\dagger}\}e_3 + \{\mathbb{C}(i_1)e_2 + \mathbb{C}(i_1)e_2^{\dagger}\}e_3^{\dagger} \\
&= \mathbb{C}(i_3)e_2e_3 + \mathbb{C}(i_3)e_2^{\dagger}e_3 + \mathbb{C}(i_1)e_2e_3^{\dagger} + \mathbb{C}(i_1)e_2^{\dagger}e_3^{\dagger} \\
&= \mathbb{C}(i_3)e_6 + \mathbb{C}(i_3)e_5 + \mathbb{C}(i_1)e_4 + \mathbb{C}(i_1)e_7 \\
&= \{\zeta \in \mathbb{C}(i_1, i_2, i_3): \alpha e_6 + \beta e_5 + \gamma e_4 + \delta e_7, (\alpha, \beta, \gamma, \delta) \in \mathbb{C}(i_3) \times \mathbb{C}(i_3) \times \mathbb{C}(i_1) \times \mathbb{C}(i_1)\}
\end{aligned}$$

The $\mathbb{C}(i_3)$ and $\mathbb{C}(i_1) -$ idempotent representation of $\zeta \in \mathbb{C}(i_1, i_2, i_3)$ is given as

$\zeta = x_1 + i_1x_2 + i_2x_3 + i_3x_4 + i_1i_2x_5 + i_1i_3x_6 + i_2i_3x_7 + i_1i_2i_3x_8$

$= (x_1 + i_1x_2 + i_3x_4 + i_1i_3x_6) + i_2(x_3 + i_1x_5 + i_3x_7 + i_1i_3x_8)$

$= \xi + i_2\eta$

$= (\xi - i_3\eta)e_3 + (\xi + i_3\eta)e_3^{\dagger}$

$= \{(x_1 + x_7) + i_1(x_2 + x_8) - i_3(x_3 - x_4) - i_1i_3(x_5 - x_6)\}e_3$

$+ \{(x_1 - x_7) + i_1(x_2 - x_8) + i_3(x_3 + x_4) + i_1i_3(x_5 + x_6)\}e_3^{\dagger}$

$= [\{(x_1 + x_7) - i_3(x_3 - x_4)\} + i_1\{(x_2 + x_8) - i_3(x_5 - x_6)\}]e_3$

$+ [\{(x_1 - x_7) + i_1(x_2 - x_8)\} + i_3\{(x_3 + x_4) + i_1(x_5 + x_6)\}]e_3^{\dagger}$

$= \left[\{(x_1 - x_5 + x_6 + x_7) - i_3(x_2 + x_3 - x_4 + x_8)\}e_2 + \{(x_1 + x_5 - x_6 + x_7) + i_3(x_2 - x_3 + x_4 + x_8)\}e_2^{\dagger}\right]e_3$

$+ \left[\{(x_1 + x_5 + x_6 - x_7) + i_1(x_2 - x_3 - x_4 - x_8)\}e_2 + \{(x_1 - x_5 - x_6 - x_7) + i_1(x_2 + x_3 + x_4 - x_8)\}e_2^{\dagger}\right]e_3^{\dagger}$

$= \{(x_1 - x_5 + x_6 + x_7) - i_3(x_2 + x_3 - x_4 + x_8)\}e_2e_3 + \{(x_1 + x_5 - x_6 + x_7) + i_3(x_2 - x_3 + x_4 + x_8)\}e_2^{\dagger}e_3$

$+ \{(x_1 + x_5 + x_6 - x_7) + i_1(x_2 - x_3 - x_4 - x_8)\}e_2e_3^{\dagger} + \{(x_1 - x_5 - x_6 - x_7) + i_1(x_2 + x_3 + x_4 - x_8)\}e_2^{\dagger}e_3^{\dagger}$

$= \{(x_1 - x_5 + x_6 + x_7) - i_3(x_2 + x_3 - x_4 + x_8)\}e_6 + \{(x_1 + x_5 - x_6 + x_7) + i_3(x_2 - x_3 + x_4 + x_8)\}e_5$

$+ \{(x_1 + x_5 + x_6 - x_7) + i_1(x_2 - x_3 - x_4 - x_8)\}e_4 + \{(x_1 - x_5 - x_6 - x_7) + i_1(x_2 + x_3 + x_4 - x_8)\}e_7$

(v-a) $\mathbb{C}(i_2)$-idempotent representation:

$$\begin{aligned}
\mathbb{C}(i_1, i_2, i_3) &= \mathbb{C}(i_2, i_3) \times_e \mathbb{C}(i_2, i_3) \\
&= \mathbb{C}(i_2, i_3)e_1 + \mathbb{C}(i_2, i_3)e_1^{\dagger}
\end{aligned}$$

$$= \{\mathbb{C}(i_2)e_3 + \mathbb{C}(i_2)e_3^\dagger\}e_1 + \{\mathbb{C}(i_2)e_3 + \mathbb{C}(i_2)e_3^\dagger\}e_1^\dagger$$
$$= \mathbb{C}(i_2)e_1e_3 + \mathbb{C}(i_2)e_1e_3^\dagger + \mathbb{C}(i_2)e_1^\dagger e_3 + \mathbb{C}(i_2)e_1^\dagger e_3^\dagger$$
$$= \mathbb{C}(i_2)e_5 + \mathbb{C}(i_2)e_4 + \mathbb{C}(i_2)e_6 + \mathbb{C}(i_2)e_7$$
$$= \{\zeta \in \mathbb{C}(i_1,i_2,i_3): \alpha e_4 + \beta e_5 + \gamma e_6 + \delta e_7, (\alpha,\beta,\gamma,\delta) \in \mathbb{C}(i_2)\times\mathbb{C}(i_2)\times\mathbb{C}(i_2)\times\mathbb{C}(i_2)\}$$

The $\mathbb{C}(i_2) -$ idempotent representation of $\zeta \in \mathbb{C}(i_1,i_2,i_3)$ is given as

$$\zeta = x_1 + i_1x_2 + i_2x_3 + i_3x_4 + i_1i_2x_5 + i_1i_3x_6 + i_2i_3x_7 + i_1i_2i_3x_8$$
$$= (x_1 + i_2x_3 + i_3x_4 + i_2i_3x_7) + i_1(x_2 + i_2x_5 + i_3x_6 + i_2i_3x_8)$$
$$= \xi + i_1\eta$$
$$= (\xi - i_2\eta)e_1 + (\xi + i_2\eta)e_1^\dagger$$
$$= \{(x_1+x_5) - i_2(x_2-x_3) + i_3(x_4+x_8) - i_2i_3(x_6-x_7)\}e_1$$
$$+ \{(x_1-x_5) + i_2(x_2+x_3) + i_3(x_4-x_8) + i_2i_3(x_6+x_7)\}e_1^\dagger$$
$$= [\{(x_1+x_5) - i_2(x_2-x_3)\} + i_3\{(x_4+x_8) - i_2(x_6-x_7)\}]e_1$$
$$+ [\{(x_1-x_5) + i_2(x_2+x_3)\} + i_3\{(x_4-x_8) + i_2(x_6+x_7)\}]e_1^\dagger$$
$$= \left[\{(x_1+x_5-x_6+x_7) - i_2(x_2-x_3+x_4+x_8)\}e_3 + \{(x_1+x_5+x_6-x_7) - i_2(x_2-x_3-x_4-x_8)\}e_3^\dagger\right]e_1$$
$$+ \left[\{(x_1-x_5+x_6+x_7) + i_2(x_2+x_3-x_4+x_8)\}e_3 + \{(x_1-x_5-x_6-x_7) + i_2(x_2+x_3+x_4-x_8)\}e_3^\dagger\right]e_1^\dagger$$
$$= \{(x_1+x_5-x_6+x_7) - i_2(x_2-x_3+x_4+x_8)\}e_1e_3 + \{(x_1+x_5+x_6-x_7) - i_2(x_2-x_3-x_4-x_8)\}e_1e_3^\dagger$$
$$+ \{(x_1-x_5+x_6+x_7) + i_2(x_2+x_3-x_4+x_8)\}e_1^\dagger e_3 + \{(x_1-x_5-x_6-x_7) + i_2(x_2+x_3+x_4-x_8)\}e_1^\dagger e_3^\dagger$$
$$= \{(x_1+x_5-x_6+x_7) - i_2(x_2-x_3+x_4+x_8)\}e_5 + \{(x_1+x_5+x_6-x_7) - i_2(x_2-x_3-x_4-x_8)\}e_4$$
$$+ \{(x_1-x_5+x_6+x_7) + i_2(x_2+x_3-x_4+x_8)\}e_6 + \{(x_1-x_5-x_6-x_7) + i_2(x_2+x_3+x_4-x_8)\}e_7$$

(v-b) $\mathbb{C}(i_3)$-idempotent representation:

$$\mathbb{C}(i_1,i_2,i_3) = \mathbb{C}(i_2,i_3)\times_e \mathbb{C}(i_2,i_3)$$
$$= \mathbb{C}(i_2,i_3)e_1 + \mathbb{C}(i_2,i_3)e_1^\dagger$$
$$= \{\mathbb{C}(i_3)e_3 + \mathbb{C}(i_3)e_3^\dagger\}e_1 + \{\mathbb{C}(i_3)e_3 + \mathbb{C}(i_3)e_3^\dagger\}e_1^\dagger$$
$$= \mathbb{C}(i_3)e_1e_3 + \mathbb{C}(i_3)e_1e_3^\dagger + \mathbb{C}(i_3)e_1^\dagger e_3 + \mathbb{C}(i_3)e_1^\dagger e_3^\dagger$$
$$= \mathbb{C}(i_3)e_5 + \mathbb{C}(i_3)e_4 + \mathbb{C}(i_3)e_6 + \mathbb{C}(i_3)e_7$$
$$= \{\zeta \in \mathbb{C}(i_1,i_2,i_3): \alpha e_4 + \beta e_5 + \gamma e_6 + \delta e_7, (\alpha,\beta,\gamma,\delta) \in \mathbb{C}(i_3)\times\mathbb{C}(i_3)\times\mathbb{C}(i_3)\times\mathbb{C}(i_3)\}$$

The $\mathbb{C}(i_3) -$ idempotent representation of $\zeta \in \mathbb{C}(i_1,i_2,i_3)$ is given as

$$\zeta = x_1 + i_1x_2 + i_2x_3 + i_3x_4 + i_1i_2x_5 + i_1i_3x_6 + i_2i_3x_7 + i_1i_2i_3x_8$$
$$= (x_1 + i_2x_3 + i_3x_4 + i_2i_3x_7) + i_1(x_2 + i_2x_5 + i_3x_6 + i_2i_3x_8)$$
$$= \xi + i_1\eta$$
$$= (\xi - i_2\eta)e_1 + (\xi + i_2\eta)e_1^\dagger$$
$$= \{(x_1+x_5) - i_2(x_2-x_3) + i_3(x_4+x_8) - i_2i_3(x_6-x_7)\}e_1$$
$$+ \{(x_1-x_5) + i_2(x_2+x_3) + i_3(x_4-x_8) + i_2i_3(x_6+x_7)\}e_1^\dagger$$
$$= [\{(x_1+x_5) + i_3(x_4+x_8)\} + i_2\{-(x_2-x_3) - i_3(x_6-x_7)\}]e_1$$
$$+ [\{(x_1-x_5) + i_3(x_4-x_8)\} + i_2\{(x_2+x_3) + i_3(x_6+x_7)\}]e_1^\dagger$$
$$= \left[\{(x_1+x_5-x_6+x_7) + i_3(x_2-x_3+x_4+x_8)\}e_3 + \{(x_1+x_5+x_6-x_7) - i_3(x_2-x_3-x_4-x_8)\}e_3^\dagger\right]e_1$$
$$+ \left[\{(x_1-x_5+x_6+x_7) - i_3(x_2+x_3-x_4+x_8)\}e_3 + \{(x_1-x_5-x_6-x_7) + i_3(x_2+x_3+x_4-x_8)\}e_3^\dagger\right]e_1^\dagger$$
$$= \{(x_1+x_5-x_6+x_7) + i_3(x_2-x_3+x_4+x_8)\}e_1e_3 + \{(x_1+x_5+x_6-x_7) - i_3(x_2-x_3-x_4-x_8)\}e_1e_3^\dagger$$

$+\{(x_1-x_5+x_6+x_7)-i_3(x_2+x_3-x_4+x_8)\}e_1^{\dagger}e_3+\{(x_1-x_5-x_6-x_7)+i_3(x_2+x_3+x_4-x_8)\}e_1^{\dagger}e_3^{\dagger}$

$=\{(x_1+x_5-x_6+x_7)+i_3(x_2-x_3+x_4+x_8)\}e_5+\{(x_1+x_5+x_6-x_7)-i_3(x_2-x_3-x_4-x_8)\}e_4$

$+\{(x_1-x_5+x_6+x_7)-i_3(x_2+x_3-x_4+x_8)\}e_6+\{(x_1-x_5-x_6-x_7)+i_3(x_2+x_3+x_4-x_8)\}e_7$

(v-c) Mixed $\mathbb{C}(i_2)$ and $\mathbb{C}(i_3)$-idempotent representations:

$$\begin{aligned}\mathbb{C}(i_1,i_2,i_3) &= \mathbb{C}(i_2,i_3)\times_e \mathbb{C}(i_2,i_3)\\ &=\mathbb{C}(i_2,i_3)e_1+\mathbb{C}(i_2,i_3)e_1^{\dagger}\\ &=\{\mathbb{C}(i_2)e_3+\mathbb{C}(i_2)e_3^{\dagger}\}e_1+\{\mathbb{C}(i_3)e_3+\mathbb{C}(i_3)e_3^{\dagger}\}e_1^{\dagger}\\ &=\mathbb{C}(i_2)e_1e_3+\mathbb{C}(i_2)e_1e_3^{\dagger}+\mathbb{C}(i_3)e_1^{\dagger}e_3+\mathbb{C}(i_3)e_1^{\dagger}e_3^{\dagger}\\ &=\mathbb{C}(i_2)e_5+\mathbb{C}(i_2)e_4+\mathbb{C}(i_3)e_6+\mathbb{C}(i_3)e_7\\ &=\{\zeta\in\mathbb{C}(i_1,i_2,i_3):\alpha e_5+\beta e_4+\gamma e_6+\delta e_7,(\alpha,\beta,\gamma,\delta)\in\mathbb{C}(i_2)\times\mathbb{C}(i_2)\times\mathbb{C}(i_3)\times\mathbb{C}(i_3)\}\end{aligned}$$

The $\mathbb{C}(i_2)$ and $\mathbb{C}(i_3)-$ idempotent representation of $\zeta\in\mathbb{C}(i_1,i_2,i_3)$ is given as

$$\begin{aligned}\zeta &= x_1+i_1x_2+i_2x_3+i_3x_4+i_1i_2x_5+i_1i_3x_6+i_2i_3x_7+i_1i_2i_3x_8\\ &=(x_1+i_2x_3+i_3x_4+i_2i_3x_7)+i_1(x_2+i_2x_5+i_3x_6+i_2i_3x_8)\\ &=\xi+i_1\eta\\ &=(\xi-i_2\eta)e_1+(\xi+i_2\eta)e_1^{\dagger}\\ &=\{(x_1+x_5)-i_2(x_2-x_3)+i_3(x_4+x_8)-i_2i_3(x_6-x_7)\}e_1\\ &\qquad+\{(x_1-x_5)+i_2(x_2+x_3)+i_3(x_4-x_8)+i_2i_3(x_6+x_7)\}e_1^{\dagger}\\ &=[\{(x_1+x_5)-i_2(x_2-x_3)\}+i_3\{(x_4+x_8)-i_2(x_6-x_7)\}]e_1\\ &\qquad+[\{(x_1-x_5)+i_3(x_4-x_8)\}+i_2\{(x_2+x_3)+i_3(x_6+x_7)\}]e_1^{\dagger}\\ &=\left[\{(x_1+x_5-x_6+x_7)-i_2(x_2-x_3+x_4+x_8)\}e_3+\{(x_1+x_5+x_6-x_7)-i_2(x_2-x_3-x_4-x_8)\}e_3^{\dagger}\right]e_1\\ &+\left[\{(x_1-x_5+x_6+x_7)-i_3(x_2+x_3-x_4+x_8)\}e_3+\{(x_1-x_5-x_6-x_7)+i_3(x_2+x_3+x_4-x_8)\}e_3^{\dagger}\right]e_1^{\dagger}\\ &=\{(x_1+x_5-x_6+x_7)-i_2(x_2-x_3+x_4+x_8)\}e_1e_3+\{(x_1+x_5+x_6-x_7)-i_2(x_2-x_3-x_4-x_8)\}e_1e_3^{\dagger}\\ &\qquad+\{(x_1-x_5+x_6+x_7)-i_3(x_2+x_3-x_4+x_8)\}e_1^{\dagger}e_3+\{(x_1-x_5-x_6-x_7)+i_3(x_2+x_3+x_4-x_8)\}e_1^{\dagger}e_3^{\dagger}\\ &=\{(x_1+x_5-x_6+x_7)-i_2(x_2-x_3+x_4+x_8)\}e_5+\{(x_1+x_5+x_6-x_7)-i_2(x_2-x_3-x_4-x_8)\}e_4\\ &\qquad+\{(x_1-x_5+x_6+x_7)-i_3(x_2+x_3-x_4+x_8)\}e_6+\{(x_1-x_5-x_6-x_7)+i_3(x_2+x_3+x_4-x_8)\}e_7\end{aligned}$$

(v-d) Mixed $\mathbb{C}(i_2)$ and $\mathbb{C}(i_3)$-idempotent representations:

$$\begin{aligned}\mathbb{C}(i_1,i_2,i_3) &= \mathbb{C}(i_2,i_3)\times_e \mathbb{C}(i_2,i_3)\\ &=\mathbb{C}(i_2,i_3)e_1+\mathbb{C}(i_2,i_3)e_1^{\dagger}\\ &=\{\mathbb{C}(i_3)e_3+\mathbb{C}(i_3)e_3^{\dagger}\}e_1+\{\mathbb{C}(i_2)e_3+\mathbb{C}(i_2)e_3^{\dagger}\}e_1^{\dagger}\\ &=\mathbb{C}(i_3)e_1e_3+\mathbb{C}(i_3)e_1e_3^{\dagger}+\mathbb{C}(i_2)e_1^{\dagger}e_3+\mathbb{C}(i_2)e_1^{\dagger}e_3^{\dagger}\\ &=\mathbb{C}(i_3)e_5+\mathbb{C}(i_3)e_4+\mathbb{C}(i_2)e_6+\mathbb{C}(i_2)e_7\\ &=\{\zeta\in\mathbb{C}(i_1,i_2,i_3):\alpha e_5+\beta e_4+\gamma e_6+\delta e_7,(\alpha,\beta,\gamma,\delta)\in\mathbb{C}(i_3)\times\mathbb{C}(i_3)\times\mathbb{C}(i_2)\times\mathbb{C}(i_2)\}\end{aligned}$$

The $\mathbb{C}(i_3)$ and $\mathbb{C}(i_2)-$ idempotent representation of $\zeta\in\mathbb{C}(i_1,i_2,i_3)$ is given as

$$\begin{aligned}\zeta &= x_1+i_1x_2+i_2x_3+i_3x_4+i_1i_2x_5+i_1i_3x_6+i_2i_3x_7+i_1i_2i_3x_8\\ &=(x_1+i_2x_3+i_3x_4+i_2i_3x_7)+i_1(x_2+i_2x_5+i_3x_6+i_2i_3x_8)\\ &=\xi+i_1\eta\\ &=(\xi-i_2\eta)e_1+(\xi+i_2\eta)e_1^{\dagger}\\ &=\{(x_1+x_5)-i_2(x_2-x_3)+i_3(x_4+x_8)-i_2i_3(x_6-x_7)\}e_1\end{aligned}$$

$$+\{(x_1-x_5)+i_2(x_2+x_3)+i_3(x_4-x_8)+i_2i_3(x_6+x_7)\}e_1^{\dagger}$$

$$=[\{(x_1+x_5)+i_3(x_4+x_8)\}+i_2\{-(x_2-x_3)-i_3(x_6-x_7)\}]e_1$$

$$+[\{(x_1-x_5)+i_2(x_2+x_3)\}+i_3\{(x_4-x_8)+i_2(x_6+x_7)\}]e_1^{\dagger}$$

$$=\left[\{(x_1+x_5-x_6+x_7)+i_3(x_2-x_3+x_4+x_8)\}e_3+\{(x_1+x_5+x_6-x_7)-i_3(x_2-x_3-x_4-x_8)\}e_3^{\dagger}\right]e_1$$

$$+\left[\{(x_1-x_5+x_6+x_7)+i_2(x_2+x_3-x_4+x_8)\}e_3+\{(x_1-x_5-x_6-x_7)+i_2(x_2+x_3+x_4-x_8)\}e_3^{\dagger}\right]e_1^{\dagger}$$

$$=\{(x_1+x_5-x_6+x_7)+i_3(x_2-x_3+x_4+x_8)\}e_1e_3+\{(x_1+x_5+x_6-x_7)-i_3(x_2-x_3-x_4-x_8)\}e_1e_3^{\dagger}$$

$$+\{(x_1-x_5+x_6+x_7)+i_2(x_2+x_3-x_4+x_8)\}e_1^{\dagger}e_3+\{(x_1-x_5-x_6-x_7)+i_2(x_2+x_3+x_4-x_8)\}e_1^{\dagger}e_3^{\dagger}$$

$$=\{(x_1+x_5-x_6+x_7)+i_3(x_2-x_3+x_4+x_8)\}e_5+\{(x_1+x_5+x_6-x_7)-i_3(x_2-x_3-x_4-x_8)\}e_4$$

$$+\{(x_1-x_5+x_6+x_7)+i_2(x_2+x_3-x_4+x_8)\}e_6+\{(x_1-x_5-x_6-x_7)+i_2(x_2+x_3+x_4-x_8)\}e_7$$

(vi-a) $\mathbb{C}(\boldsymbol{i_2})$-idempotent representation:

$$\begin{aligned}\mathbb{C}(i_1,i_2,i_3)&=\mathbb{C}(i_2,i_3)\times_e\mathbb{C}(i_2,i_3)\\&=\mathbb{C}(i_2,i_3)e_2+\mathbb{C}(i_2,i_3)e_2^{\dagger}\\&=\{\mathbb{C}(i_2)e_3+\mathbb{C}(i_2)e_3^{\dagger}\}e_2+\{\mathbb{C}(i_2)e_3+\mathbb{C}(i_2)e_3^{\dagger}\}e_2^{\dagger}\\&=\mathbb{C}(i_2)e_2e_3+\mathbb{C}(i_2)e_2e_3^{\dagger}+\mathbb{C}(i_2)e_2^{\dagger}e_3+\mathbb{C}(i_2)e_2^{\dagger}e_3^{\dagger}\\&=\mathbb{C}(i_2)e_6+\mathbb{C}(i_2)e_4+\mathbb{C}(i_2)e_5+\mathbb{C}(i_2)e_7\\&=\{\zeta\in\mathbb{C}(i_1,i_2,i_3):\alpha e_4+\beta e_5+\gamma e_6+\delta e_7,(\alpha,\beta,\gamma,\delta)\in\mathbb{C}(i_2)\times\mathbb{C}(i_2)\times\mathbb{C}(i_2)\times\mathbb{C}(i_2)\}\end{aligned}$$

The $\mathbb{C}(i_2)-$ idempotent representation of $\zeta\in\mathbb{C}(i_1,i_2,i_3)$ is given as

$$\zeta=x_1+i_1x_2+i_2x_3+i_3x_4+i_1i_2x_5+i_1i_3x_6+i_2i_3x_7+i_1i_2i_3x_8$$

$$=(x_1+i_2x_3+i_3x_4+i_2i_3x_7)+i_1(x_2+i_2x_5+i_3x_6+i_2i_3x_8)$$

$$=\xi+i_1\eta$$

$$=(\xi-i_3\eta)e_2+(\xi+i_3\eta)e_2^{\dagger}$$

$$=\{(x_1+x_6)+i_2(x_3+x_8)-i_3(x_2-x_4)-i_2i_3(x_5-x_7)\}e_2$$

$$+\{(x_1-x_6)+i_2(x_3-x_8)+i_3(x_2+x_4)+i_2i_3(x_5+x_7)\}e_2^{\dagger}$$

$$=[\{(x_1+x_6)+i_2(x_3+x_8)\}+i_3\{-(x_2-x_4)-i_2(x_5-x_7)\}]e_2$$

$$+[\{(x_1-x_6)+i_2(x_3-x_8)\}+i_3\{(x_2+x_4)+i_2(x_5+x_7)\}]e_2^{\dagger}$$

$$=\left[\{(x_1-x_5+x_6+x_7)+i_2(x_2+x_3-x_4+x_8)\}e_3+\{(x_1+x_5+x_6-x_7)-i_2(x_2-x_3-x_4-x_8)\}e_3^{\dagger}\right]e_2$$

$$+\left[\{(x_1+x_5-x_6+x_7)-i_2(x_2-x_3+x_4+x_8)\}e_3+\{(x_1-x_5-x_6-x_7)+i_2(x_2+x_3+x_4-x_8)\}e_3^{\dagger}\right]e_2^{\dagger}$$

$$=\{(x_1-x_5+x_6+x_7)+i_2(x_2+x_3-x_4+x_8)\}e_2e_3+\{(x_1+x_5+x_6-x_7)-i_2(x_2-x_3-x_4-x_8)\}e_2e_3^{\dagger}$$

$$+\{(x_1+x_5-x_6+x_7)-i_2(x_2-x_3+x_4+x_8)\}e_2^{\dagger}e_3+\{(x_1-x_5-x_6-x_7)+i_2(x_2+x_3+x_4-x_8)\}e_2^{\dagger}e_3^{\dagger}$$

$$=\{(x_1-x_5+x_6+x_7)+i_2(x_2+x_3-x_4+x_8)\}e_6+\{(x_1+x_5+x_6-x_7)-i_2(x_2-x_3-x_4-x_8)\}e_4$$

$$+\{(x_1+x_5-x_6+x_7)-i_2(x_2-x_3+x_4+x_8)\}e_5+\{(x_1-x_5-x_6-x_7)+i_2(x_2+x_3+x_4-x_8)\}e_7$$

(vi-b) $\mathbb{C}(\boldsymbol{i_3})$-idempotent representation:

$$\begin{aligned}\mathbb{C}(i_1,i_2,i_3)&=\mathbb{C}(i_2,i_3)\times_e\mathbb{C}(i_2,i_3)\\&=\mathbb{C}(i_2,i_3)e_2+\mathbb{C}(i_2,i_3)e_2^{\dagger}\\&=\{\mathbb{C}(i_3)e_3+\mathbb{C}(i_3)e_3^{\dagger}\}e_2+\{\mathbb{C}(i_3)e_3+\mathbb{C}(i_3)e_3^{\dagger}\}e_2^{\dagger}\\&=\mathbb{C}(i_3)e_2e_3+\mathbb{C}(i_3)e_2e_3^{\dagger}+\mathbb{C}(i_3)e_2^{\dagger}e_3+\mathbb{C}(i_3)e_2^{\dagger}e_3^{\dagger}\\&=\mathbb{C}(i_3)e_6+\mathbb{C}(i_3)e_4+\mathbb{C}(i_3)e_5+\mathbb{C}(i_3)e_7\\&=\{\zeta\in\mathbb{C}(i_1,i_2,i_3):\alpha e_4+\beta e_5+\gamma e_6+\delta e_7,(\alpha,\beta,\gamma,\delta)\in\mathbb{C}(i_3)\times\mathbb{C}(i_3)\times\mathbb{C}(i_3)\times\mathbb{C}(i_3)\}\end{aligned}$$

The $\mathbb{C}(i_3)-$ idempotent representation of $\zeta \in \mathbb{C}(i_1, i_2, i_3)$ is given as

$$\zeta = x_1 + i_1x_2 + i_2x_3 + i_3x_4 + i_1i_2x_5 + i_1i_3x_6 + i_2i_3x_7 + i_1i_2i_3x_8$$

$$= (x_1 + i_2x_3 + i_3x_4 + i_2i_3x_7) + i_1(x_2 + i_2x_5 + i_3x_6 + i_2i_3x_8)$$

$$= \xi + i_1\eta$$

$$= (\xi - i_3\eta)e_2 + (\xi + i_3\eta)e_2^\dagger$$

$$= \{(x_1 + x_6) + i_2(x_3 + x_8) - i_3(x_2 - x_4) - i_2i_3(x_5 - x_7)\}e_2 + \{(x_1 - x_6) + i_2(x_3 - x_8) + i_3(x_2 + x_4) + i_2i_3(x_5 + x_7)\}e_2^\dagger$$

$$= [\{(x_1 + x_6) - i_3(x_2 - x_4)\} + i_2\{(x_3 + x_8) - i_3(x_5 - x_7)\}]e_2 + [\{(x_1 - x_6) + i_3(x_2 + x_4)\} + i_2\{(x_3 - x_8) + i_3(x_5 + x_7)\}]e_2^\dagger$$

$$= \left[\{(x_1 - x_5 + x_6 + x_7) - i_3(x_2 + x_3 - x_4 + x_8)\}e_3 + \{(x_1 + x_5 + x_6 - x_7) - i_3(x_2 - x_3 - x_4 - x_8)\}e_3^\dagger\right]e_2$$
$$+ \left[\{(x_1 + x_5 - x_6 + x_7) + i_3(x_2 - x_3 + x_4 + x_8)\}e_3 + \{(x_1 - x_5 - x_6 - x_7) + i_3(x_2 + x_3 + x_4 - x_8)\}e_3^\dagger\right]e_2^\dagger$$

$$= \{(x_1 - x_5 + x_6 + x_7) - i_3(x_2 + x_3 - x_4 + x_8)\}e_2e_3 + \{(x_1 + x_5 + x_6 - x_7) - i_3(x_2 - x_3 - x_4 - x_8)\}e_2e_3^\dagger$$
$$+ \{(x_1 + x_5 - x_6 + x_7) + i_3(x_2 - x_3 + x_4 + x_8)\}e_2^\dagger e_3 + \{(x_1 - x_5 - x_6 - x_7) + i_3(x_2 - x_3 - x_4 - x_8)\}e_2^\dagger e_3^\dagger$$

$$= \{(x_1 - x_5 + x_6 + x_7) - i_3(x_2 + x_3 - x_4 + x_8)\}e_6 + \{(x_1 + x_5 + x_6 - x_7) - i_3(x_2 - x_3 - x_4 - x_8)\}e_4$$
$$+ \{(x_1 + x_5 - x_6 + x_7) + i_3(x_2 - x_3 + x_4 + x_8)\}e_5 + \{(x_1 - x_5 - x_6 - x_7) + i_3(x_2 + x_3 + x_4 - x_8)\}\, e_7$$

(vi-c) Mixed $\mathbb{C}(i_2)$ and $\mathbb{C}(i_3)$-idempotent representations:

$$\mathbb{C}(i_1, i_2, i_3) = \mathbb{C}(i_2, i_3) \times_e \mathbb{C}(i_2, i_3)$$
$$= \mathbb{C}(i_2, i_3)e_2 + \mathbb{C}(i_2, i_3)e_2^\dagger$$
$$= \{\mathbb{C}(i_2)e_3 + \mathbb{C}(i_2)e_3^\dagger\}e_2 + \{\mathbb{C}(i_3)e_3 + \mathbb{C}(i_3)e_3^\dagger\}e_2^\dagger$$
$$= \mathbb{C}(i_2)e_2e_3 + \mathbb{C}(i_2)e_2e_3^\dagger + \mathbb{C}(i_3)e_2^\dagger e_3 + \mathbb{C}(i_3)e_2^\dagger e_3^\dagger$$
$$= \mathbb{C}(i_2)e_6 + \mathbb{C}(i_2)e_4 + \mathbb{C}(i_3)e_5 + \mathbb{C}(i_3)e_7$$
$$= \{\zeta \in \mathbb{C}(i_1, i_2, i_3): \alpha e_6 + \beta e_4 + \gamma e_5 + \delta e_7, (\alpha, \beta, \gamma, \delta) \in \mathbb{C}(i_2) \times \mathbb{C}(i_2) \times \mathbb{C}(i_3) \times \mathbb{C}(i_3)\}$$

The $\mathbb{C}(i_2)$ and $\mathbb{C}(i_3)-$ idempotent representation of $\zeta \in \mathbb{C}(i_1, i_2, i_3)$ is given as

$$\zeta = x_1 + i_1x_2 + i_2x_3 + i_3x_4 + i_1i_2x_5 + i_1i_3x_6 + i_2i_3x_7 + i_1i_2i_3x_8$$

$$= (x_1 + i_2x_3 + i_3x_4 + i_2i_3x_7) + i_1(x_2 + i_2x_5 + i_3x_6 + i_2i_3x_8)$$

$$= \xi + i_1\eta$$

$$= (\xi - i_3\eta)e_2 + (\xi + i_3\eta)e_2^\dagger$$

$$= \{(x_1 + x_6) + i_2(x_3 + x_8) - i_3(x_2 - x_4) - i_2i_3(x_5 - x_7)\}e_2 + \{(x_1 - x_6) + i_2(x_3 - x_8) + i_3(x_2 + x_4) + i_2i_3(x_5 + x_7)\}e_2^\dagger$$

$$= [\{(x_1 + x_6) + i_2(x_3 + x_8)\} + i_3\{-(x_2 - x_4) - i_2(x_5 - x_7)\}]e_2 + [\{(x_1 - x_6) + i_3(x_2 + x_4)\} + i_2\{(x_3 - x_8) + i_3(x_5 + x_7)\}]e_2^\dagger$$

$$= \left[\{(x_1 - x_5 + x_6 + x_7) + i_2(x_2 + x_3 - x_4 + x_8)\}e_3 + \{(x_1 + x_5 + x_6 - x_7) - i_2(x_2 - x_3 - x_4 - x_8)\}e_3^\dagger\right]e_2$$
$$+ \left[\{(x_1 + x_5 - x_6 + x_7) + i_3(x_2 - x_3 + x_4 + x_8)\}e_3 + \{(x_1 - x_5 - x_6 - x_7) + i_3(x_2 + x_3 + x_4 - x_8)\}e_3^\dagger\right]e_2^\dagger$$

$$= \{(x_1 - x_5 + x_6 + x_7) + i_2(x_2 + x_3 - x_4 + x_8)\}e_2e_3 + \{(x_1 + x_5 + x_6 - x_7) - i_2(x_2 - x_3 - x_4 - x_8)\}e_2e_3^\dagger$$
$$+ \{(x_1 + x_5 - x_6 + x_7) + i_3(x_2 - x_3 + x_4 + x_8)\}e_2^\dagger e_3 + \{(x_1 - x_5 - x_6 - x_7) + i_3(x_2 + x_3 + x_4 - x_8)\}e_2^\dagger e_3^\dagger$$

$$= \{(x_1 - x_5 + x_6 + x_7) + i_2(x_2 + x_3 - x_4 + x_8)\}e_6 + \{(x_1 + x_5 + x_6 - x_7) - i_2(x_2 - x_3 - x_4 - x_8)\}e_4$$
$$+ \{(x_1 + x_5 - x_6 + x_7) + i_3(x_2 - x_3 + x_4 + x_8)\}e_5 + \{(x_1 - x_5 - x_6 - x_7) + i_3(x_2 + x_3 + x_4 - x_8)\}e_7$$

(vi-d) Mixed $\mathbb{C}(i_2)$ and $\mathbb{C}(i_3)$-idempotent representations:

$$\begin{aligned}\mathbb{C}(i_1,i_2,i_3) &= \mathbb{C}(i_2,i_3)\times_e \mathbb{C}(i_2,i_3)\\ &= \mathbb{C}(i_2,i_3)e_2 + \mathbb{C}(i_2,i_3)e_2^{\dagger}\\ &= \{\mathbb{C}(i_3)e_3 + \mathbb{C}(i_3)e_3^{\dagger}\}e_2 + \{\mathbb{C}(i_2)e_3 + \mathbb{C}(i_2)e_3^{\dagger}\}e_2^{\dagger}\\ &= \mathbb{C}(i_3)e_2e_3 + \mathbb{C}(i_3)e_2e_3^{\dagger} + \mathbb{C}(i_2)e_2^{\dagger}e_3 + \mathbb{C}(i_2)e_2^{\dagger}e_3^{\dagger}\\ &= \mathbb{C}(i_3)e_6 + \mathbb{C}(i_3)e_4 + \mathbb{C}(i_2)e_5 + \mathbb{C}(i_2)e_7\\ &= \{\zeta \in \mathbb{C}(i_1,i_2,i_3): \alpha e_6 + \beta e_4 + \gamma e_5 + \delta e_7, (\alpha,\beta,\gamma,\delta) \in \mathbb{C}(i_3)\times\mathbb{C}(i_3)\times\mathbb{C}(i_2)\times\mathbb{C}(i_2)\}\end{aligned}$$

The $\mathbb{C}(i_3)$ and $\mathbb{C}(i_2) -$ idempotent representation of $\zeta \in \mathbb{C}(i_1,i_2,i_3)$ is given as

$$\begin{aligned}\zeta &= x_1 + i_1x_2 + i_2x_3 + i_3x_4 + i_1i_2x_5 + i_1i_3x_6 + i_2i_3x_7 + i_1i_2i_3x_8\\ &= (x_1 + i_2x_3 + i_3x_4 + i_2i_3x_7) + i_1(x_2 + i_2x_5 + i_3x_6 + i_2i_3x_8)\\ &= \xi + i_1\eta\\ &= (\xi - i_3\eta)e_2 + (\xi + i_3\eta)e_2^{\dagger}\\ &= \{(x_1+x_6) + i_2(x_3+x_8) - i_3(x_2-x_4) - i_2i_3(x_5-x_7)\}e_2\\ &\quad + \{(x_1-x_6) + i_2(x_3-x_8) + i_3(x_2+x_4) + i_2i_3(x_5+x_7)\}e_2^{\dagger}\\ &= [\{(x_1+x_6) - i_3(x_2-x_4)\} + i_2\{(x_3+x_8) - i_3(x_5-x_7)\}]e_2\\ &\quad + [\{(x_1-x_6) + i_2(x_3-x_8)\} + i_3\{(x_2+x_4) + i_2(x_5+x_7)\}]e_2^{\dagger}\\ &= \left[\{(x_1-x_5+x_6+x_7) - i_3(x_2+x_3-x_4+x_8)\}e_3 + \{(x_1+x_5+x_6-x_7) - i_3(x_2-x_3-x_4-x_8)\}e_3^{\dagger}\right]e_2\\ &\quad + \left[\{(x_1+x_5-x_6+x_7) - i_2(x_2-x_3+x_4+x_8)\}e_3 + \{(x_1-x_5-x_6-x_7) + i_2(x_2+x_3+x_4-x_8)\}e_3^{\dagger}\right]e_2^{\dagger}\\ &= \{(x_1-x_5+x_6+x_7) - i_3(x_2+x_3-x_4+x_8)\}e_2e_3 + \{(x_1+x_5+x_6-x_7) - i_3(x_2-x_3-x_4-x_8)\}e_2e_3^{\dagger}\\ &\quad + \{(x_1+x_5-x_6+x_7) - i_2(x_2-x_3+x_4+x_8)\}e_2^{\dagger}e_3 + \{(x_1-x_5-x_6-x_7) + i_2(x_2+x_3+x_4-x_8)\}e_2^{\dagger}e_3^{\dagger}\\ &= \{(x_1-x_5+x_6+x_7) - i_3(x_2+x_3-x_4+x_8)\}e_6 + \{(x_1+x_5+x_6-x_7) - i_3(x_2-x_3-x_4-x_8)\}e_4\\ &\quad + \{(x_1+x_5-x_6+x_7) - i_2(x_2-x_3+x_4+x_8)\}e_5 + \{(x_1-x_5-x_6-x_7) + i_2(x_2+x_3+x_4-x_8)\}e_7\end{aligned}$$

Note 4.1: We observe that $\zeta \in \mathbb{C}(i_1,i_2,i_3)$ has **6** idempotent representations of types $\mathbb{C}(i_1,i_2)$, $\mathbb{C}(i_1,i_3)$ and $\mathbb{C}(\boldsymbol{i_2},\boldsymbol{i_3})$, consisting of two of type $\mathbb{C}(i_1,i_2)$, two of type $\mathbb{C}(i_1,\boldsymbol{i_3})$, and two of type $\mathbb{C}(\boldsymbol{i_2},\boldsymbol{i_3})$.

They are given as follows:

Form of $\boldsymbol{\zeta} \in \mathbb{C}(\boldsymbol{i_1},\boldsymbol{i_2},\boldsymbol{i_3})$	Idempotent Representation-1	Idempotent Representation-2
$\boldsymbol{\zeta = \xi + i_3\,\eta;\ \xi,\eta} \in \mathbb{C}(\boldsymbol{i_1},\boldsymbol{i_2})$	$\boldsymbol{\zeta = (\xi - i_1\,\eta)e_2 + (\xi + i_1\,\eta)e_2^{\dagger}}$	$\boldsymbol{\zeta = (\xi - i_2\,\eta)e_3 + (\xi + i_2\,\eta)e_3^{\dagger}}$
$\boldsymbol{\zeta = \xi + i_2\,\eta;\ \xi,\eta} \in \mathbb{C}(\boldsymbol{i_1},\boldsymbol{i_3})$	$\boldsymbol{\zeta = (\xi - i_1\,\eta)e_1 + (\xi + i_1\,\eta)e_1^{\dagger}}$	$\boldsymbol{\zeta = (\xi - i_3\,\eta)e_3 + (\xi + i_3\,\eta)e_3^{\dagger}}$
$\boldsymbol{\zeta = \xi + i_1\,\eta;\ \xi,\eta} \in \mathbb{C}(\boldsymbol{i_2},\boldsymbol{i_3})$	$\boldsymbol{\zeta = (\xi - i_2\,\eta)e_1 + (\xi + i_2\,\eta)e_1^{\dagger}}$	$\boldsymbol{\zeta = (\xi - i_3\,\eta)e_2 + (\xi + i_3\,\eta)e_2^{\dagger}}$

Note 4.2: We observe that $\zeta = x_1 + i_1x_2 + i_2x_3 + i_3x_4 + i_1i_2x_5 + i_1i_3x_6 + i_2i_3x_7 + i_1i_2i_3x_8 \in \mathbb{C}(i_1,i_2,i_3)$ has 24 idempotent representations in $\mathbb{C}(i_1)$, $\mathbb{C}(i_2)$ and $\mathbb{C}(i_3)$.

- The four $\mathbb{C}(i_1)$-idempotent representations,
- the four $\mathbb{C}(i_2)$-idempotent representations, and
- the four $\mathbb{C}(i_3)$-idempotent representations

are the same.

The remaining **12 idempotent representations** are different.

Further Remark on Structure: After splitting

$\zeta = x_1 + i_1x_2 + i_2x_3 + i_3x_4 + i_1i_2x_5 + i_1i_3x_6 + i_2i_3x_7 + i_1i_2i_3x_8 \in \mathbb{C}(i_1,i_2,i_3)$

into idempotent parts, each idempotent component is a complex-like number with a real part and an imaginary part. We set the real parts r_k and the imaginary parts s_k as follows:

$r_1 = x_1 + x_5 + x_6 - x_7,\quad s_1 = x_2 - x_3 - x_4 - x_8$

$r_2 = x_1 + x_5 - x_6 + x_7,\quad s_2 = x_2 - x_3 + x_4 + x_8$

$r_3 = x_1 - x_5 + x_6 + x_7,\quad s_3 = x_2 + x_3 - x_4 + x_8$

$r_4 = x_1 - x_5 - x_6 - x_7,\quad s_4 = x_2 + x_3 + x_4 - x_8$

The simplest forms of the idempotent representations are listed below:

(I) $\mathbb{C}(i_1)$-idempotent representation:

$$\zeta = (r_1 + i_1 s_1)e_4 + (r_2 + i_1 s_2)e_5 + (r_3 + i_1 s_3)e_6 + (r_4 + i_1 s_4)e_7$$

(II) $\mathbb{C}(i_2)$-idempotent representation:

$$\zeta = (r_1 - i_2 s_1)e_4 + (r_2 - i_2 s_2)e_5 + (r_3 + i_2 s_3)e_6 + (r_4 + i_2 s_4)e_7$$

(III) $\mathbb{C}(i_3)$-idempotent representation:

$$\zeta = (r_1 - i_3 s_1)e_4 + (r_2 + i_3 s_2)e_5 + (r_3 - i_3 s_3)e_6 + (r_4 + i_3 s_4)e_7$$

(IV) Mixed $\mathbb{C}(i_1)$ and $\mathbb{C}(i_2)$-idempotent representations:

(i) $\zeta = (r_1 + i_1 s_1)e_4 + (r_2 - i_2 s_2)e_5 + (r_3 + i_1 s_3)e_6 + (r_4 + i_2 s_4)e_7$

(ii) $\zeta = (r_1 - i_2 s_1)e_4 + (r_2 + i_1 s_2)e_5 + (r_3 + i_1 s_3)e_6 + (r_4 + i_2 s_4)e_7$

(iii) $\zeta = (r_1 - i_2 s_1)e_4 + (r_2 + i_1 s_2)e_5 + (r_3 + i_2 s_3)e_6 + (r_4 + i_1 s_4)e_7$

(iv) $\zeta = (r_1 + i_1 s_1)e_4 + (r_2 - i_2 s_2)e_5 + (r_3 + i_2 s_3)e_6 + (r_4 + i_1 s_4)e_7$

(V) Mixed $\mathbb{C}(i_1)$ and $\mathbb{C}(i_3)$-idempotent representations:

(i) $\zeta = (r_1 + i_1 s_1)e_4 + (r_2 + i_1 s_2)e_5 + (r_3 - i_3 s_3)e_6 + (r_4 + i_3 s_4)e_7$

(ii) $\zeta = (r_1 - i_3 s_1)e_4 + (r_2 + i_1 s_2)e_5 + (r_3 + i_1 s_3)e_6 + (r_4 + i_3 s_4)e_7$

(iii) $\zeta = (r_1 - i_3 s_1)e_4 + (r_2 + i_3 s_2)e_5 + (r_3 + i_1 s_3)e_6 + (r_4 + i_1 s_4)e_7$

(iv) $\zeta = (r_1 + i_1 s_1)e_4 + (r_2 + i_3 s_2)e_5 + (r_3 - i_3 s_3)e_6 + (r_4 + i_1 s_4)e_7$

(VI) Mixed $\mathbb{C}(i_2)$ and $\mathbb{C}(i_3)$-idempotent representations:

(i) $\zeta = (r_1 - i_2 s_1)e_4 + (r_2 - i_2 s_2)e_5 + (r_3 - i_3 s_3)e_6 + (r_4 + i_3 s_4)e_7$

(ii) $\zeta = (r_1 - i_2 s_1)e_4 + (r_2 + i_3 s_2)e_5 + (r_3 + i_2 s_3)e_6 + (r_4 + i_3 s_4)e_7$

(iii) $\zeta = (r_1 - i_3 s_1)e_4 + (r_2 + i_3 s_2)e_5 + (r_3 + i_2 s_3)e_6 + (r_4 + i_2 s_4)e_7$

(iv) $\zeta = (r_1 - i_3 s_1)e_4 + (r_2 - i_2 s_2)e_5 + (r_3 - i_3 s_3)e_6 + (r_4 + i_2 s_4)e_7$

5. Singular Elements in $\mathbb{C}(i_1, i_2)$:

Let $\xi, \eta \in \mathbb{C}(i_1, i_2)$ such that $\xi\eta = \eta\xi = 1$, then η is said to be a multiplicative inverse of ξ. The invertible elements are also called non-singular elements.The set of all singular elements in $\mathbb{C}(i_1, i_2)$ is denoted as $\mathbb{O}(i_1, i_2)$ and $\mathbb{C}(i_1, i_2) \setminus \mathbb{O}(i_1, i_2)$ is the set of all non-singular elements in $\mathbb{C}(i_1, i_2)$.

Similarly, the set of all singular elements in $\mathbb{C}(i_1, i_3)$ is denoted by $\mathbb{O}(i_1, i_3)$, while $\mathbb{C}(i_1, i_3) \setminus \mathbb{O}(i_1, i_3)$ denotes the set of all non-singular elements in $\mathbb{C}(i_1, i_3)$. Likewise, the set of all singular elements in $\mathbb{C}(i_2, i_3)$ is denoted by $\mathbb{O}(i_2, i_3)$, and $\mathbb{C}(i_2, i_3) \setminus \mathbb{O}(i_2, i_3)$ denotes the set of all non-singular elements in $\mathbb{C}(i_2, i_3)$.

5.1 Singular and Non-singular Elements in $\mathbb{C}(i_1, i_2, i_3)$

The set of all singular elements in $\mathbb{C}(i_1, i_2, i_3)$ is denoted by $\mathbb{O}(i_1, i_2, i_3)$, while $\mathbb{C}(i_1, i_2, i_3) \setminus \mathbb{O}(i_1, i_2, i_3)$ denotes the set of all non-singular elements in $\mathbb{C}(i_1, i_2, i_3)$.

5.2 Set of Non-singular elements in $\mathbb{C}(i_1, i_2), \mathbb{C}(i_1, i_3)$ *and* $\mathbb{C}(i_2, i_3)$

(i) For $\mathbb{C}(i_1, i_2)$:

$$\mathbb{C}(i_1, i_2) \setminus \mathbb{O}(i_1, i_2) = \mathbb{C}(i_1) \setminus \{0\}e_1 + \mathbb{C}(i_1) \setminus \{0\}e_1^\dagger = \mathbb{C}(i_2) \setminus \{0\}e_1 + \mathbb{C}(i_2) \setminus \{0\}e_1^\dagger$$

(ii) For $\mathbb{C}(i_1, i_3)$:

$$\mathbb{C}(i_1, i_3) \setminus \mathbb{O}(i_1, i_3) = \mathbb{C}(i_1) \setminus \{0\}e_2 + \mathbb{C}(i_1) \setminus \{0\}e_2^\dagger = \mathbb{C}(i_3) \setminus \{0\}e_2 + \mathbb{C}(i_3) \setminus \{0\}e_2^\dagger$$

(iii) For $\mathbb{C}(i_2, i_3)$:

$$\mathbb{C}(i_2, i_3) \setminus \mathbb{O}(i_2, i_3) = \mathbb{C}(i_2) \setminus \{0\}e_3 + \mathbb{C}(i_2) \setminus \{0\}e_3^\dagger = \mathbb{C}(i_3) \setminus \{0\}e_3 + \mathbb{C}(i_3) \setminus \{0\}e_3^\dagger$$

5.3. Set of Non-singular elements in $\mathbb{C}(i_1, i_2, i_3)$

(i) In terms of $\mathbb{C}(i_1, i_2)$:

$$\begin{aligned}\mathbb{C}(i_1, i_2, i_3) \setminus \mathbb{O}(i_1, i_2, i_3) &= \mathbb{C}(i_1, i_2) \setminus \mathbb{O}(i_1, i_2)e_2 + \mathbb{C}(i_1, i_2) \setminus \mathbb{O}(i_1, i_2)e_2^\dagger \\ &= \mathbb{C}(i_1, i_2) \setminus \mathbb{O}(i_1, i_2)e_3 + \mathbb{C}(i_1, i_2) \setminus \mathbb{O}(i_1, i_2)e_3^\dagger\end{aligned}$$

(ii) In terms of $\mathbb{C}(i_1, i_3)$:

$$\begin{aligned}\mathbb{C}(i_1, i_2, i_3) \setminus \mathbb{O}(i_1, i_2, i_3) &= \mathbb{C}(i_1, i_3) \setminus \mathbb{O}(i_1, i_3)e_1 + \mathbb{C}(i_1, i_3) \setminus \mathbb{O}(i_1, i_3)e_1^\dagger \\ &= \mathbb{C}(i_1, i_3) \setminus \mathbb{O}(i_1, i_3)e_3 + \mathbb{C}(i_1, i_3) \setminus \mathbb{O}(i_1, i_3)e_3^\dagger\end{aligned}$$

(iii) In terms of $\mathbb{C}(i_2, i_3)$:

$$\begin{aligned}\mathbb{C}(i_1, i_2, i_3) \setminus \mathbb{O}(i_1, i_2, i_3) &= \mathbb{C}(i_2, i_3) \setminus \mathbb{O}(i_2, i_3)e_1 + \mathbb{C}(i_2, i_3) \setminus \mathbb{O}(i_2, i_3)e_1^\dagger \\ &= \mathbb{C}(i_2, i_3) \setminus \mathbb{O}(i_2, i_3)e_2 + \mathbb{C}(i_2, i_3) \setminus \mathbb{O}(i_2, i_3)e_2^\dagger\end{aligned}$$

(iv)

- Using $\mathbb{C}(i_1)$: $\mathbb{C}(i_1, i_2, i_3) \setminus \mathbb{O}(i_1, i_2, i_3) = \mathbb{C}(i_1) \setminus \{0\}e_4 + \mathbb{C}(i_1) \setminus \{0\}e_5 + \mathbb{C}(i_1) \setminus \{0\}e_6 + \mathbb{C}(i_1) \setminus \{0\}e_7$
- Using $\mathbb{C}(i_2)$: $\mathbb{C}(i_1, i_2, i_3) \setminus \mathbb{O}(i_1, i_2, i_3) = \mathbb{C}(i_2) \setminus \{0\}e_4 + \mathbb{C}(i_2) \setminus \{0\}e_5 + \mathbb{C}(i_2) \setminus \{0\}e_6 + \mathbb{C}(i_2) \setminus \{0\}e_7$
- Using $\mathbb{C}(i_3)$: $\mathbb{C}(i_1, i_2, i_3) \setminus \mathbb{O}(i_1, i_2, i_3) = \mathbb{C}(i_3) \setminus \{0\}e_4 + \mathbb{C}(i_3) \setminus \{0\}e_5 + \mathbb{C}(i_3) \setminus \{0\}e_6 + \mathbb{C}(i_3) \setminus \{0\}e_7$

Theorem (Division in Idempotent Basis) 1:

Let $\zeta_1 = z_1e_4 + z_2e_5 + z_3e_6 + z_4e_7$ and $\zeta_2 = w_1e_4 + w_2e_5 + w_3e_6 + w_4e_7$ be elements of $\mathbb{C}(i_1, i_2, i_3)$, where $z_k, w_k \in \mathbb{C}(i_j)$ for $k = 1,2,3,4$ and $j = 1,2,3$. If $\zeta_2$is invertible, then the quotient $\frac{\zeta_1}{\zeta_2}$ is given by :

$$\frac{\zeta_1}{\zeta_2} = \frac{z_1}{w_1}e_4 + \frac{z_2}{w_2}e_5 + \frac{z_3}{w_3}e_6 + \frac{z_4}{w_4}e_7$$

Proof:

Given that $\zeta_2 = w_1e_4 + w_2e_5 + w_3e_6 + w_4e_7$ is invertible, there exist $\eta = \alpha e_4 + \beta e_5 + \gamma e_6 + \delta e_7 \in \mathbb{C}(i_1, i_2, i_3)$ such that

$\frac{\zeta_1}{\zeta_2} = \eta \Rightarrow \zeta_1 = \eta\zeta_2$

Now, compute the product

$$\zeta_1 = (\alpha e_4 + \beta e_5 + \gamma e_6 + \delta e_7)(w_1e_4 + w_2e_5 + w_3e_6 + w_4e_7)$$

Using the idempotent properties:

$e_i e_j = 0$ for $i \neq j$ and $e_k^2 = e_k$ for each $k = 4,5,6,7$

We get:

$$\zeta_1 = (\alpha e_4 + \beta e_5 + \gamma e_6 + \delta e_7)(w_1 e_4 + w_2 e_5 + w_3 e_6 + w_4 e_7)$$

$$\Rightarrow z_1 e_4 + z_2 e_5 + z_3 e_6 + z_4 e_7 = \alpha w_1 e_4 + \beta w_2 e_5 + \gamma w_3 e_6 + \delta w_4 e_7$$

Comparing both sides:

$$z_1 = \alpha w_1, z_2 = \beta w_2, z_3 = \gamma w_3, z_4 = \delta w_4$$

Solving for $\alpha, \beta, \gamma, \delta$, we find

$$\alpha = \frac{z_1}{w_1}, \beta = \frac{z_2}{w_2}, \gamma = \frac{z_3}{w_3}, \delta = \frac{z_4}{w_4}$$

Thus,

$$\frac{\zeta_1}{\zeta_2} = \frac{z_1}{w_1} e_4 + \frac{z_2}{w_2} e_5 + \frac{z_3}{w_3} e_6 + \frac{z_4}{w_4} e_7 .$$

□

6. Algebraic operations in idempotent form

Let $\zeta_1 = z_1 e_4 + z_2 e_5 + z_3 e_6 + z_4 e_7$ and $\zeta_2 = w_1 e_4 + w_2 e_5 + w_3 e_6 + w_4 e_7$ be elements of $\mathbb{C}(i_1, i_2, i_3)$. Then the basic operations are given by

(i) Addition: $\zeta_1 + \zeta_2 = (z_1 + w_1)e_4 + (z_2 + w_2)e_5 + (z_3 + w_3)e_6 + (z_4 + w_4)e_7$

(ii) Subtraction: $\zeta_1 - \zeta_2 = (z_1 - w_1)e_4 + (z_2 - w_2)e_5 + (z_3 - w_3)e_6 + (z_4 - w_4)e_7$

(iii) Multiplication: $\zeta_1 \zeta_2 = (z_1 w_1)e_4 + (z_2 w_2)e_5 + (z_3 w_3)e_6 + (z_4 w_4)e_7$

(iv) Division (when $\boldsymbol{\zeta_2}$ is invertible i.e. each $\boldsymbol{w_j \neq 0}$):

$$\frac{\zeta_1}{\zeta_2} = \frac{z_1}{w_1} e_4 + \frac{z_2}{w_2} e_5 + \frac{z_3}{w_3} e_6 + \frac{z_4}{w_4} e_7$$

(v) Powers (Integer Exponentiation)

For any positive integer n:

$\zeta_1^n = z_1^n e_4 + z_2^n e_5 + z_3^n e_6 + z_4^n e_7.$

Theorem 2: $\zeta = \xi + i_3\, \eta;\ \xi, \eta \in \mathbb{C}(i_1, i_2)$ is non-singular (invertible) if one of the following hold

(i) $\zeta \notin \mathbb{O}(i_1, i_2, i_3)$

(ii) $\xi^2 + \eta^2 \notin \mathbb{O}(i_1, i_2)$

(iii) $\xi - i_1\, \eta \notin \mathbb{O}(i_1, i_2)$ and $\xi + i_1\, \eta \notin \mathbb{O}(i_1, i_2)$

(iv) $\xi - i_2\, \eta \notin \mathbb{O}(i_1, i_2)$ and $\xi + i_2\, \eta \notin \mathbb{O}(i_1, i_2)$

Proof: Let $\boldsymbol{\zeta = \xi + i_3\, \eta;\ \xi, \eta \in \mathbb{C}(i_1, i_2)}$ is non-singular (invertible)

Then the inverse of ζ is given by :

$$\zeta^{-1} = \frac{1}{\xi + i_3\, \eta} = \frac{\xi - i_3\, \eta}{\xi^2 + \eta^2}$$

Since $\xi, \eta \in \mathbb{C}(i_1, i_2)$, it follows that $\xi^2 + \eta^2 \in \mathbb{C}(i_1, i_2)$.

To ensure ζ^{-1}exist (i.e. ζ is invertible), the denominator must be non-singular:

$$\xi^2 + \eta^2 \notin \mathbb{O}(i_1, i_2)$$

Observe that:

$$\xi^2 + \eta^2 = (\xi - i_1\, \eta)(\xi + i_1\, \eta) = (\xi - i_2\, \eta)(\xi + i_2\, \eta)$$

Therefore, if $\xi^2 + \eta^2 \notin \mathbb{O}(i_1, i_2)$, then the following two conditions must each hold separately:

Condition -1 $\xi - i_1\, \eta \notin \mathbb{O}(i_1, i_2)$ and $\xi + i_1\, \eta \notin \mathbb{O}(i_1, i_2)$

Condition -2 $\xi - i_2\, \eta \notin \mathbb{O}(i_1, i_2)$ and $\xi + i_2\, \eta \notin \mathbb{O}(i_1, i_2)$

Hence, any of the conditions (ii), (iii), or (iv) implies the invertibility of ζ. Also, condition (i) is a general condition asserting that ζ does not lie in the set $\mathbb{O}(i_1, i_2, i_3)$, which directly implies invertibility. □

Theorem 3: $\zeta = \xi + \dot{i}_2\,\eta;\ \xi, \eta \in \mathbb{C}(i_1, i_3)$ is non-singular (invertible) if one of the following hold

(i) $\zeta \notin \mathbb{O}(i_1, i_2, i_3)$

(ii) $\xi^2 + \eta^2 \notin \mathbb{O}(i_1, i_3)$

(iii) $\xi - \dot{i}_1\,\eta \notin \mathbb{O}(i_1, i_3)$ and $\xi + \dot{i}_1\,\eta \notin \mathbb{O}(i_1, i_3)$

(iv) $\xi - \dot{i}_2\,\eta \notin \mathbb{O}(i_1, i_3)$ and $\xi + \dot{i}_2\,\eta \notin \mathbb{O}(i_1, i_3)$

Proof: Same as **Theorem 2**, using $\mathbb{C}(\boldsymbol{i_1}, \boldsymbol{i_3})$ subalgebra. □

Theorem 4: $\zeta = \xi + \dot{i}_3\,\eta;\ \xi, \eta \in \mathbb{C}(i_2, i_3)$ is non-singular (invertible) if one of the following hold

(i) $\zeta \notin \mathbb{O}(i_1, i_2, i_3)$

(ii) $\xi^2 + \eta^2 \notin \mathbb{O}(i_2, i_3)$

(iii) $\xi - \dot{i}_2\,\eta \notin \mathbb{O}(i_2, i_3)$ and $\xi + \dot{i}_2\,\eta \notin \mathbb{O}(i_2, i_3)$

(iv) $\xi - \dot{i}_3\,\eta \notin \mathbb{O}(i_2, i_3)$ and $\xi + \dot{i}_3\,\eta \notin \mathbb{O}(i_2, i_3)$

Proof: Follows the same structure and logic as **Theorem 2** □

Theorem 5: Let $\zeta = x_1 + i_1x_2 + i_2x_3 + i_3x_4 + i_1i_2x_5 + i_1i_3x_6 + i_2i_3x_7 + i_1i_2i_3x_8 \in \mathbb{C}(i_1, i_2, i_3)$.

Then ζ is non-singular (invertible) if any one of the following conditions holds:

(i) $x_1 + x_5 \neq |x_6 - x_7|$ and $x_1 - x_5 \neq |x_6 + x_7|$.

(ii) $x_1 + x_5 \neq |x_6 - x_7|, x_1 - x_5 + x_6 + x_7 \neq 0, x_2 + x_3 + x_4 - x_8 \neq 0.$

(iii) $x_1 + x_5 \neq |x_6 - x_7|, x_2 + x_3 - x_4 + x_8 \neq 0, x_1 - x_5 - x_6 - x_7 \neq 0.$

(iv) $x_1 + x_5 \neq |x_6 - x_7|$ and $x_2 + x_3 \neq |x_4 - x_8|$.

(v) $x_1 + x_5 + x_6 - x_7 \neq 0, x_2 - x_3 + x_4 + x_8 \neq 0, x_1 - x_5 \neq |x_6 + x_7|.$

(vi) $x_1 + x_6 \neq |x_5 - x_7|$ and $x_2 + x_4 \neq |x_3 - x_8|$.

(vii) $x_1 - x_7 \neq |x_5 + x_6|$ and $x_2 + x_8 \neq |x_3 - x_4|$.

(viii) $x_1 + x_5 + x_6 - x_7 \neq 0, x_2 - x_3 + x_4 + x_8 \neq 0, x_2 + x_3 \neq |x_4 - x_8|.$

(ix) $x_2 - x_3 - x_4 - x_8 \neq 0, x_1 + x_5 - x_6 + x_7 \neq 0, x_1 - x_5 \neq |x_6 + x_7|.$

(x) $x_2 - x_8 \neq |x_3 + x_4|$ and $x_1 + x_7 \neq |x_5 - x_6|$.

(xi) $x_2 - x_4 \neq |x_3 + x_8|$ and $x_1 - x_6 \neq |x_5 + x_7|$.

(xii) $x_2 - x_3 - x_4 - x_8 \neq 0, x_1 + x_5 - x_6 + x_7 \neq 0, x_2 + x_3 \neq |x_4 - x_8|.$

(xiii) $x_2 - x_3 \neq |x_4 + x_8|$ and $x_1 - x_5 \neq |x_6 + x_7|$.

(xiv) $x_2 - x_3 \neq |x_4 + x_8|, x_1 - x_5 + x_6 + x_7 \neq 0, x_2 + x_3 + x_4 - x_8 \neq 0.$

(xv) $x_2 - x_3 \neq |x_4 + x_8|,\ x_2 + x_3 - x_4 + x_8 \neq 0, x_1 - x_5 - x_6 - x_7 \neq 0.$

(xvi) $x_2 - x_3 \neq |x_4 + x_8|$ and $x_2 + x_3 \neq |x_4 - x_8|$.

Proof: By decomposing $\zeta = x_1 + i_1x_2 + i_2x_3 + i_3x_4 + i_1i_2x_5 + i_1i_3x_6 + i_2i_3x_7 + i_1i_2i_3x_8$

into one of its idempotent form

$$\zeta = (r_1 + i_1s_1)e_4 + (r_2 + i_1s_2)e_5 + (r_3 + i_1s_3)e_6 + (r_4 + i_1s_4)e_7,$$

We see that ζ is non-singular if and only if all its idempotent coefficients are nonzero, i.e.

$r_1 + i_1s_1 \neq 0, \quad r_2 + i_1s_2 \neq 0, \quad r_3 + i_1s_3 \neq 0, \quad r_4 + i_1s_4 \neq 0.$

By examining all possible vanishing cases of these coefficients, we obtain precisely the conditions (i)–(xvi) stated in the theorem.

Therorem 6: Let $\zeta \in \mathbb{C}(i_1, i_2, i_3)$

Write different forms of ζ

$\zeta = \xi + i_3\,\eta;\ \xi, \eta \in \mathbb{C}(i_1, i_2)$

$\zeta = \chi + i_2\psi;\ \chi, \psi \in \mathbb{C}(i_1, i_3)$

$\zeta = \mu + i_1\nu;\ \mu, \nu \in \mathbb{C}(i_2, i_3)$

$\zeta = \mu + i_1\nu;\ \mu, \nu \in \mathbb{C}(i_2, i_3)$

$\zeta = (r_1 + i_1 s_1)e_4 + (r_2 + i_1 s_2)e_5 + (r_3 + i_1 s_3)e_6 + (r_4 + i_1 s_4)e_7$

Then ζ is singular if one of the following condition hold:

(i) $\zeta \in \mathbb{O}(i_1, i_2, i_3)$

(ii) $\xi^2 + \eta^2 \in \mathbb{O}(i_1, i_2)$

(iii) $\xi + i_1\eta \in \mathbb{O}(i_1, i_2)$ or $\xi - i_1\eta \in \mathbb{O}(i_1, i_2)$

(iv) $\xi + i_2\eta \in \mathbb{O}(i_1, i_2)$ or $\xi - i_2\eta \in \mathbb{O}(i_1, i_2)$

(v) $\chi^2 + \psi^2 \in \mathbb{O}(i_1, i_3)$

(vi) $\chi + i_1\eta \in \mathbb{O}(i_1, i_3)$ or $\chi - i_1\eta \in \mathbb{O}(i_1, i_3)$

(vii) $\chi + i_3\eta \in \mathbb{O}(i_1, i_3)$ or $\chi - i_3\eta \in \mathbb{O}(i_1, i_3)$

(viii) $\mu^2 + \nu^2 \in \mathbb{O}(i_2, i_3)$

(ix) $\mu + i_2\nu \in \mathbb{O}(i_1, i_3)$ or $\mu - i_2\nu \in \mathbb{O}(i_1, i_3)$

(x) $\mu + i_3\nu \in \mathbb{O}(i_1, i_3)$ or $\mu - i_3\nu \in \mathbb{O}(i_1, i_3)$

(xi) $r_1 = s_1 = 0$ or $r_2 = s_2 = 0$ or $r_3 = s_3 = 0$ or $r_4 = s_4 = 0$

Proof: From **Theorems 2, 3, 4,** and **5**, we know that an element $\zeta \in \mathbb{C}(i_1, i_2, i_3)$ is **non-singular (invertible)** if and only if:

- It does **not** lie in the null cone $\mathbb{O}(i_1, i_2, i_3)$,
- All idempotent components $r_k + s_k \neq 0$ for $k = 1,2,3,4$,
- Or equivalently, certain expressions such as $\xi^2 + \eta^2$, $\chi^2 + \psi^2$, $\mu^2 + \nu^2$, and their linear factors are not in the respective null cones.

Therefore, if **any** of these expressions **lie in** the corresponding null cone or **vanish**, then ζ is **not invertible**, i.e., **singular.**

Thus, the listed conditions **(i)–(xi)** are necessary and sufficient for ζ to be singular. □

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