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Coefficient Problems and Third Hankel Determinant for a Subclass of Analytic Functions

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ABSTRACT:

A new subclass of analytic-univalent functions is introduced and investigated, which is an extension of an existing subclass of analytic-univalent functions in literature. We establish coefficient estimates for this class and derive sharp bounds for the third Hankel determinants. The new results obtained generalize some previously known estimates for related function classes.

***Key Words**: Analytic-Univalent Functions; Subordination ; Coefficient Estimates; Hankel determinants.*

***Mathematics subject classification**: 30C45, 30C50*

1. Introduction

Let f be the class of functions $f(z)$ defined by

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \qquad 1$$

which are analytic in the unit disk $\mathbb{U} = \{z \in \mathbb{C}: |z| < 1\}$. Denote by S the subclass of A, consisting of functions which are analytic, univalent in the unit disk $\mathbb{U}$ and normalized by the conditions $f(0) = 0 = f'(0) - 1$.

A function $f(z) \in S$ of the form (1) is star-like in the unit disk $\mathbb{U} = \{z \in \mathbb{C}: |z| < 1\}$ if it maps a unit disk onto a star-like domain. A necessary and sufficient condition for a function $f(z)$to be star-like is that

$$\mathbb{R}\left(\frac{zf'(z)}{f(z)}\right) > 0, \quad z \in \mathbb{U}.$$

The class of all star-like functions can be denoted by S*.

An analytic function $f(z)$ of the form (1)is convex if it maps the unit disk $\mathbb{U} = \{z \in \mathbb{C}: |z| < 1\}$ conformally onto a convex domain. Equivalently, a function $f(z)$ is said to be convex if and only if it satisfies the following condition;

$$\mathbb{R}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, \quad z \in \mathbb{U}.$$

The class of all convex functions can be denoted by C^*.

Let $f(z)$ and $g(z)$ be analytic functions in the unit disk $\mathbb{U} = \{z \in \mathbb{C}: |z| < 1\}$, then $f(z)$ is subordinate to $g(z)$ in the unit disk $\mathbb{U}$ written as $f(z) \prec g(z)$, if there exists a function $\omega(z)$, analytic in the unit disk satisfying the conditions $w(0) = 0$, $|\omega(z)| < 1$, which is called a Schwartz function, such that $f(z) = \omega(g(z))$. If the function g is univalent in $\mathbb{U}$, the $f(z) \prec g(z)$, $z \in \mathbb{U} \Leftrightarrow f(0) = g(0)$ and $f(U) \subset g(U)$.

Bieberbach[3], gave a famous conjecture which asserts that the coefficients in the Taylor's series expansion of every function from the class S of normalized univalent functions in the unit disk $\mathbb{U} = \{z \in \mathbb{C}: |z| < 1\}$ satisfy the condition $|a_n| \leq n$, $n \geq 2$ which was later proved by de-Branges, [4].

Quite a number of researchers have defined several subclasses of normalized univalent functions by using methods like subordination and convolution among many others, their associated coefficients bounds were also derived by these authors.

Mediratta *et al* [5] introduced star-like and convex functions which is subordinate to exponential function and each of these classes are considered to be symmetric about the real axis.

For $q \geq 1$ and $n \geq 1$, the q^{th} Hankel determinant of $f(z)$ given in (1) is defined by

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & a_{n+2} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & a_{n+3} & \cdots & a_{n+q} \\ a_{n+2} & a_{n+3} & a_{n+4} & \cdots & a_{n+q+1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{n+q-1} & a_{n+q} & a_{n+q+1} & \cdots & a_{n+2q-2} \end{vmatrix} \qquad (2)$$

For $q = 3$ and $n = 1$;

$$H_3(1) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix}$$

$$H_3(1) = a_3a_5 - a_4^2 - a_2^2a_5 + 2a_2a_3a_4 - a_3^3$$

Several authors have studied Hankel determinants in order to examines it's growth rate as $n \to \infty$ and to also establish it's bound for diverse precise values of q and n. Babalola [6] was the first to study third Hankel determinant, he obtained the upper bound of $H_3(1)$ for the classes of star-like and convex functions. See [7], [8], [9], [10] further study on Hankel determinants of functions in class S.

Lei Shi et al [11], investigated the estimate of $|H_3(1)|$ for both classes $S_e^*$and C_e. They also studied this problem for m-fold symmetric star-like and convex functions associated with exponential function.

This paper uses Opoola differential operator to define a subclass $S_n^*(\mu, \beta, t)$ of univalent-analytic functions, which extends the subclass of star-like functions in [11] and the upper bound of the third Hankel determinant for this class of functions are investigated.

Lemma [1]:

If $p \in P$ with the series expansion $p(z) = 1 + p_1z + p_2z^2 + p_3z^3 + \cdots \quad z \in \mathbb{U}$, satisfying $Re(p(z)) > 0$, $p(0) = 1$. then $|p_n| \leq 2$, $n \geq 1$ where $p(z)$ is called a Caratheodory function.

This result is sharp and equality holds for the Mobius function $m(z) = \frac{1+z}{1-z}$.

Definition 1 [12]: For $t \geq 0, 0 \leq \mu \leq \beta, n \in \mathbb{N}_0, z \in \mathbb{U}$, Opoola differential operator $D^n(\mu, \beta, t)f: A \to A$ is defined as follows;

$$D^0(\mu, \beta, t)f = f(z)$$

$$D^1(\mu, \beta, t)f = tzf'(z) - z(\beta - \mu)t + (1 + (\beta - \mu + 1)t)f(z)$$

$$D^n(\mu, \beta, t)f = z + \sum_{k=2}^{\infty} [1 + (k + \beta - \mu - 1)t]^n a_k z^k$$

for $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$

Definition 2: A function $f \in A$ of the form (1) belongs to the class $S_n^*(\mu, \beta, t)$ if

$$\frac{z(D^n f(z))'}{D^n f(z)} \prec e^z \qquad (3)$$

where $t \geq 0, 0 \leq \mu \leq \beta, n \in \mathbb{N}_0, z \in \mathbb{U}$.

Remark 1:

When $n = 0$, $\frac{z(D^n f(z))'}{D^n f(z)} \prec e^z$ reduces to $\frac{zf'(z)}{f(z)} \prec e^z$. [11]

2. Main Results

Theorem 1: Let the functions $f \in A$ belong to the class $S_n^*(\mu, \beta, t)$, then

$$|H_3(1)| \leq \frac{13}{64\alpha_2^3} + \frac{7}{32\alpha_2\alpha_4} - \frac{289}{1296\alpha_3^2} - \frac{19}{72\alpha_1^2\alpha_4} + \frac{17}{24\alpha_1\alpha_2\alpha_3}.$$

Proof: Let $f \in S_n^*(\mu, \beta, t)$, then there is a Schwartz function $\omega(z)$, analytic in the unit disk $\mathbb{U}$ with $\omega(0) = 0, |\omega(z)| < 1$ such that from the definition of subordination and (3)

$$\frac{z(D^n f(z))'}{D^n f(z)} = e^{\omega(z)}$$

On expansion of $\frac{z(D^n f(z))'}{D^n f(z)}$;

$\frac{z(D^n f(z))'}{D^n f(z)} = (1 + 2[1 + (\beta - \mu + 1)t]^n a_2 z + 3[1 + (\beta - \mu + 2)t]^n a_3 z^2 + 4[1 + (\beta - \mu + 3)t]^n a_4 z^3 + 5[1 + (\beta - \mu + 4)t]^n a_5 z^4 + \cdots)(1 + [1 + (\beta - \mu + 1)t]^n a_2 z + [1 + (\beta - \mu + 2)t]^n a_3 z^2 + [1 + (\beta - \mu + 3)t]^n a_4 z^3 + [1 + (\beta - \mu + 4)t]^n a_5 z^4 + \cdots)^{-1}$

$= 1 + [1 + (\beta - \mu + 1)t]^n a_2 z + \{2[1 + (\beta - \mu + 2)t]^n a_3 - [1 + (\beta - \mu + 1)t]^{2n} a_2^2\} z^2 + \{3[1 + (\beta - \mu + 3)t]^n a_4 - 3[1 + (\beta - \mu + 1)t]^n [1 + (\beta - \mu + 2)t]^n a_2 a_3\} z^3 + 4\{[1 + (\beta - \mu + 4)t]^n a_5 - 4[1 + (\beta - \mu + 1)t]^n [1 + (\beta - \mu + 3)t]^n a_2 a_4 - 2[1 + (\beta - \mu + 2)t]^{2n} a_3^2 + 4[1 + (\beta - \mu + 1)t]^{2n} [1 + (\beta - \mu + 2)t]^n a_2^2 a_3 - [1 + (\beta - \mu + 1)t]^{4n} a_2^4\} z^4 + \cdots \qquad (4)$

and

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots$$

$$e^{\omega(z)} = 1 + \omega(z) + \frac{(\omega(z))^2}{2!} + \frac{(\omega(z))^3}{3!} + \frac{(\omega(z))^4}{4!} + \cdots$$

Since $\omega(z)$ is aschwartz function then the function $p(z)$ is defined by

$$p(z) = \frac{1 + \omega(z)}{1 - \omega(z)} = p_1 z + p_2 z^2 + p_3 z^3 + p_4 z^4 + \cdots$$

$$\omega(z) = \frac{p_1 z + p_2 z^2 + p_3 z^3 + p_4 z^4 + \cdots}{2 + p_1 z + p_2 z^2 + p_3 z^3 + p_4 z^4 + \cdots}$$

$$= (p_1 z + p_2 z^2 + p_3 z^3 + p_4 z^4 + \cdots)(2 + p_1 z + p_2 z^2 + p_3 z^3 + p_4 z^4 + \cdots)^{-1}$$

$$\omega(z) = \frac{p_1 z}{2} + \frac{1}{2}\left\{p_2 - \frac{p_1^2}{2}\right\} z^2 + \frac{1}{2}\left\{p_3 - p_1 p_2 + \frac{p_1^3}{4}\right\} z^3 + \frac{1}{2}\left\{p_4 - p_1 p_3 - \frac{p_2^2}{2} + \frac{3p_1^2 p_2}{4} - \frac{p_1^4}{8}\right\} z^4 + \cdots$$

So,

$$e^{\omega(z)} = 1 + \frac{p_1 z}{2} + \frac{1}{2}\left\{p_2 - \frac{p_1^2}{4}\right\} z^2 + \frac{1}{2}\left\{p_3 - \frac{p_1 p_2}{2} + \frac{p_1^3}{24}\right\} z^3 + \frac{1}{2}\left\{p_4 - \frac{p_1 p_3}{2} - \frac{p_2^2}{4} + \frac{p_1^2 p_2}{8} + \frac{p_1^4}{162}\right\} z^4 + \cdots \qquad (5)$$

Comparing the coefficient of like powers of z in (4) and (5), then;

$$a_2 = \frac{p_1}{2[1 + (\beta - \mu + 1)t]^n} \qquad (6)$$

$$a_3 = \frac{1}{4}[1 + (\beta - \mu + 2)t]^n \left\{p_2 + \frac{p_1^2}{4}\right\} \qquad (7)$$

$$a_4 = \frac{1}{6[1 + (\beta - \mu + 3)t]^n} \left\{p_3 + \frac{p_1 p_2}{4} - \frac{p_1^3}{48}\right\} \qquad (8)$$

$$a_5 = \frac{1}{8[1 + (\beta - \mu + 4)t]^n} \left\{p_4 + \frac{p_1 p_3}{6} - \frac{p_1^2 p_2}{12} + \frac{p_1^4}{144}\right\} \qquad (9)$$

Remarks 2:

When $n = 0$ in (6) through to (9), then (6) – (9) reduce to result of Lei Shi *et al* in [11] as follows;

$$a_2 = \frac{p_1}{2}$$

$$a_3 = \frac{1}{4}\left(p_2 + \frac{p_1^2}{4}\right)$$

$$a_4 = \frac{1}{6}\left\{p_3 + \frac{p_1 p_2}{4} - \frac{p_1^3}{48}\right\}$$

$$a_5 = \frac{1}{8}\left\{p_4 + \frac{p_1 p_3}{6} - \frac{p_1^2 p_2}{12} + \frac{p_1^4}{144}\right\}$$

Next is to discuss the upper bound of $H_3(1)$ for class $S_n^*(\mu, \beta, t)$.

From (2), the third Hankel determinant can be written as

$$H_3(1) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix}$$

$$H_3(1) = a_3 a_5 - a_4^2 - a_2^2 a_5 + 2a_2 a_3 a_4 - a_3^3, \quad a_1 = 1. \qquad (10)$$

Using (6) – (9) ;

$$a_3 a_5 = \frac{1}{32[1+(\beta-\mu+2)t]^n [1+(\beta-\mu+4)t]^n} \left\{p_2 p_4 + \frac{p_1 p_2 p_3}{6} - \frac{p_1^2 p_2^2}{12} - \frac{p_1^4 p_2}{72} + \frac{p_1^2 p_4}{4} + \frac{p_1^3 p_3}{24} + \frac{p_1^6}{576}\right\} \qquad (11)$$

$$a_4^2 = \frac{1}{36[1+(\beta-\mu+3)t]^{2n}}\left\{p_3^2 + \frac{p_1p_2p_3}{2} - \frac{p_1^3p_3}{24} + \frac{p_1^2p_2^2}{16} - \frac{p_1^4p_2}{96} + \frac{p_1^6}{2308}\right\} \quad (12)$$

$$a_2^2a_5 = \frac{1}{32[1+(\beta-\mu+1)t]^{2n}[1+(\beta-\mu+4)t]^n}\left\{p_1^2p_4 + \frac{p_1^3p_3}{6} - \frac{p_1^4p_2}{12} + \frac{p_1^6}{144}\right\} \quad (13)$$

$$a_3^3 = \frac{1}{64[1+(\beta-\mu+2)t]^{3n}}\left\{p_2^3 + \frac{p_1^6}{64} + \frac{3p_1^4p_2}{16} + \frac{3p_1^2p_2^2}{4}\right\} \quad (14)$$

$$2a_2a_3a_4 = \frac{1}{24[1+(\beta-\mu+1)t]^n[1+(\beta-\mu+2)t]^n[1+(\beta-\mu+3)t]^n}\left\{p_1p_2p_3 + \frac{p_1^2p_2^2}{4} + \frac{p_1^4p_2}{24} + \frac{p_1^3p_3}{4} - \frac{p_1^6}{192}\right\}$$

Substituting (11) – (14) into (10) and for convenience , we let

$\alpha_1 = [1+(\beta-\mu+1)t]^n$;

$\alpha_2 = [1+(\beta-\mu+2)t]^n$;

$\alpha_3 = [1+(\beta-\mu+3)t]^n$;

$\alpha_4 = [1+(\beta-\mu+4)t]^n$;

$$H_3(1) = \frac{1}{32\,\alpha_2\,\alpha_4}\left\{p_2p_4 + \frac{p_1p_2p_3}{6} - \frac{p_1^2p_2^2}{12} - \frac{p_1^4p_2}{72} + \frac{p_1^2p_4}{4} + \frac{p_1^3p_3}{24} + \frac{p_1^6}{576}\right\} - \frac{1}{36\alpha_3^2}\left\{p_3^2 + \frac{p_1p_2p_3}{2} - \frac{p_1^3p_3}{24} + \frac{p_1^2p_2^2}{16} - \frac{p_1^4p_2}{96} + \frac{p_1^6}{2304}\right\} - \frac{1}{32\alpha_1^2\alpha_4}\left\{p_1^2p_4 + \frac{p_1^3p_3}{6} - \frac{p_1^4p_2}{12} + \frac{p_1^6}{144}\right\} + \frac{1}{24\alpha_1\alpha_2\alpha_3}\left\{p_1p_2p_3 + \frac{p_1^2p_2^2}{4} + \frac{p_1^4p_2}{24} + \frac{p_1^3p_3}{4} - \frac{p_1^6}{192}\right\} - \frac{1}{64\alpha_2^3}\left\{p_2^3 + \frac{p_1^6}{64} + \frac{3p_1^4p_2}{16} + \frac{3p_1^2p_2^2}{4}\right\}.$$

$$H_3(1) = p_1^4p_2\left\{-\frac{1}{2304\alpha_2\alpha_4} - \frac{1}{3456\alpha_3^2} + \frac{1}{384\alpha_1^2\alpha_4} + \frac{3}{1024\alpha_2^3} + \frac{1}{576\alpha_1\alpha_2\alpha_3}\right\} - p_1^6\left\{\frac{1}{4096\alpha_2^3} + \frac{1}{4608\alpha_1\alpha_2\alpha_3} + \frac{1}{4608\alpha_1^2\alpha_4} + \frac{1}{82944\alpha_3^2} - \frac{1}{18432\alpha_2\alpha_4}\right\} + \frac{p_2p_4}{32\,\alpha_2\,\alpha_4} + p_1^2p_4\left\{\frac{1}{128\alpha_2\alpha_4} - \frac{1}{32\alpha_1^2\alpha_4}\right\} + p_1p_2p_3\left\{\frac{1}{192\alpha_2\alpha_4} - \frac{1}{72\alpha_3^2} + \frac{1}{24\alpha_1\alpha_2\alpha_3}\right\} + p_1^3p_3\left\{\frac{1}{768\,\alpha_2\,\alpha_4} + \frac{1}{864\alpha_3^2} - \frac{1}{192\alpha_1^2\alpha_4} + \frac{1}{96\alpha_1\alpha_2\alpha_3}\right\} - p_1^2p_2^2\left\{\frac{1}{384\,\alpha_2\,\alpha_4} + \frac{1}{576\alpha_3^2} - \frac{1}{96\alpha_1\alpha_2\alpha_3} - \frac{3}{256\alpha_2^3}\right\} - \frac{p_3^2}{36\alpha_3^2} - \frac{p_2^3}{64\alpha_2^3}. \quad (15)$$

Using triangle inequality and lemma 1 in (15)

$$|H_3(1)| \le \frac{3}{32\alpha_2^3} - \frac{1}{72\alpha_2\alpha_4} - \frac{1}{108\alpha_3^2} + \frac{1}{12\alpha_1^2\alpha_4} + \frac{1}{18\alpha_1\alpha_2\alpha_3} - \frac{1}{64\alpha_2^3} - \frac{1}{72\alpha_1\alpha_2\alpha_3} - \frac{1}{72\alpha_1^2\alpha_4} - \frac{1}{1296\alpha_3^2} + \frac{1}{288\alpha_2\alpha_4} + \frac{1}{8\,\alpha_2\,\alpha_4} + \frac{1}{16\alpha_2\alpha_4} - \frac{1}{4\alpha_1^2\alpha_4} + \frac{1}{24\alpha_2\alpha_4} - \frac{1}{9\alpha_3^2} + \frac{1}{3\alpha_1\alpha_2\alpha_3} + \frac{1}{48\,\alpha_2\,\alpha_4} + \frac{1}{54\alpha_3^2} - \frac{1}{12\alpha_1^2\alpha_4} + \frac{1}{6\alpha_1\alpha_2\alpha_3} - \frac{1}{24\,\alpha_2\,\alpha_4} - \frac{1}{36\alpha_3^2} + \frac{1}{6\alpha_1\alpha_2\alpha_3} + \frac{3}{16\alpha_2^3} - \frac{1}{9\alpha_3^2} - \frac{1}{8\alpha_2^3}.$$

$$|H_3(1)| \le \frac{13}{64\alpha_2^3} + \frac{7}{32\alpha_2\alpha_4} - \frac{289}{1296\alpha_3^2} - \frac{19}{72\alpha_1^2\alpha_4} + \frac{17}{24\alpha_1\alpha_2\alpha_3}. \quad (16)$$

Coollary:

When $n = 0$, (16) reduces to

$$|H_3(1)| \le 0.64332451$$

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