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SOME RESULTS RELATED TO THE CONTINUOUS WEINSTEIN WAVELET TRANSFORM

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ABSTRACT.

In this paper, we investigate uncertainty principle and some results for the Weinstein wavelet transform. As a variant of Heisenberg-Type uncertainty principle, Pitt's inequality, Benkner-Type uncertainty principle, logarithmic Sobolev inequality.

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1. Introduction

The Weinstein operator is the elliptic partial differential operator Δ_W considered in the upper half space $\mathbb{R}_+^d = \mathbb{R}^{d-1} \times [0, \infty[$

$$\Delta_W = \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2} + \frac{2\alpha+1}{x_d}\frac{\partial}{\partial x_d}, \alpha > -\frac{1}{2}$$

$$\Delta_W = \Delta_{d-1} + \ell_\alpha$$

where Δ_{d-1} is the Laplacian operator on $\mathbb{R}^{d-1}$ and ℓ_α is the Bessel operator with respect to the variable x_d defined by

$$\ell_\alpha = \frac{\partial^2}{\partial x_d^2} + \frac{2\alpha+1}{x_d}\frac{\partial}{\partial x_d}.$$

The harmonic analysis associated with the Weinstein operator is studied by Nahia and Ben Salem [3],[4]. In particular the authors have introduced and studied the generalized Fourier transform associated with the Weinstein operator. The theory of wavelet and continuous wavelet transform has been extended to hypergroups, in particular, to Chébli-Trimche hypergroup see [23]. Recently, there many studies about the wavelet transforms see [13], [18], [14], the authors have studied the uncertainty inequalities for the continuous Weinstein wavelet transform, and deformed wavelet transform. In this paper, we introduce some new uncertain inequalities for the continuous Weinstein wavelet transform.

Let us now to be more precise and describe our results. To do so, we need to introduce some notation. For $1 \leq p < \infty$, we denote by $L_\alpha^p(\mathbb{R}_+^d)$ the Lebesgue space consisting of measurable functions f on $\mathbb{R}_+^d = \mathbb{R}^{d-1} \times \mathbb{R}_+$ equipped with the norm

$$\|f\|_{L_\alpha^p(\mathbb{R}_+^d)} = \left(\int_{\mathbb{R}_+^d} |f(x', x_d)|^p \, \mathrm{d}\mu_\alpha(x', x_d) \right)^{1/p}$$

where

$$\mathrm{d}\mu_\alpha(x) = \mathrm{d}\mu_\alpha(x', x_d) = \frac{x_d^{2\alpha+1}}{\pi^{\frac{d-1}{2}} 2^{\alpha+\frac{d-1}{2}} \Gamma(\alpha+1)} \mathrm{d}x' \mathrm{d}x_d = \frac{x_d^{2\alpha+1}}{\pi^{\frac{d-1}{2}} 2^{\alpha+\frac{d-1}{2}} \Gamma(\alpha+1)} \mathrm{d}x_1 \ldots \mathrm{d}x_d$$

For $f \in L_\alpha^1(\mathbb{R}_+^d)$, the Weinstein (or Laplace-Bessel) transform is defined by

$$\mathcal{F}_W(f)(\xi', \xi_d) = \int_{\mathbb{R}_+^d} f(x', x_d) e^{-i\langle x', \xi' \rangle} j_\alpha(x_d \xi_d) \mathrm{d}\mu_\alpha(x', x_d).$$

The Weinstein wavelet on $\mathbb{R}_+^d$ is a measurable function h on $\mathbb{R}_+^d$ satisfying, for almost all $\xi \in \mathbb{R}_+^d$

$$0 < C_h = \int_0^\infty |\mathcal{F}_W(h)(t)|^2 \frac{dt}{t} < \infty.$$

We denote by $L_{\omega_\alpha}^p, 1 \leq p \leq \infty$ the space of measurable functions f on $\mathbb{R}_+^d \times [0, \infty] = \mathbb{R}_{++}^{d+1}$, $\mathrm{d}\omega_\alpha(x, b) = \mathrm{d}\mu_\alpha(x) \frac{1}{b^{2\alpha+d+2}} \mathrm{d}b$ such that

$$\|f\|_{L^p_{\omega_\alpha}(\mathbb{R}^{d+1}_{++})} = \left(\int_{\mathbb{R}^{d+1}_{++}} |f(x,b)|^p \, d\omega_\alpha(x,b)\right)^{\frac{1}{p}} < \infty, 1 \le p < \infty.$$

Let h be a Weinstein wavelet on $\mathbb{R}^d_+ \in L^2_\alpha(\mathbb{R}^d_+)$, we define the Weinstein continuous wavelet transform as follows

$$\mathcal{S}^h_W(f)(b,y) = \int_{\mathbb{R}^d_+} f(y)\overline{h_{b,y}(x)}d\mu_\alpha(x), (b,y) \in \mathbb{R}^{d+1}_{++}.$$

We define for $s \in \mathbb{R}^d, 1 < p \le 2$ the $\mathrm{H}^{2,s}_\alpha(\mathbb{R}^d_+)$ space by

$$\mathrm{H}^{2,s}_\alpha(\mathbb{R}^d_+) = \{f \in L^2_\alpha(\mathbb{R}^d_+) : |\xi|^s \mathcal{F}_W(f) \in L^2_\alpha(\mathbb{R}^d_+) \cdot\}$$

Recently, there are many results for the wavelet transform see [18], [13] and [14]. We will here concentrate on some uncertainties principles and some results for the Weinstein wavelet transform.

Our first result is the Heisenberg-type uncertainty principle for the wavelet transform:

Theorem 2.1. Let $s, t > 1$. Then, for all $f \in L^2_\alpha(\mathbb{R}^d_+)$, we have

$$\||x|^s \mathcal{S}^h_W(f)(b,x)\|^{\frac{t}{s+t}}_{L^2_{\omega_\alpha}\mathbb{R}^{d+1}_{++}} \||\xi|^t \mathcal{F}_W(f)(\xi)\|^{\frac{s}{s+t}}_{L^2_\alpha \mathbb{R}^d_+} \ge C_{\alpha,d}(s,t) \mathcal{C}_h^{\frac{t}{2(s+t)}} \|f\|_{L^2_\alpha(\mathbb{R}^d_+)}$$

where $C_{\alpha,d}(s,t) = \left(\alpha + \frac{d+1}{2}\right)^{\frac{ts}{s+t}}$.

The second result is the Pitt's inequality for the Weinstein wavelet transform:

Theorem 2.2. For $0 \le s < \alpha + \frac{d+1}{2}$ and $f \in S(\mathbb{R}^d_+)$, the Pitt's-type inequality for the wavelet transform is given by

$$\||\xi|^s \mathcal{F}_W(f)\|_{L^2_\alpha(\mathbb{R}^d_+)} \le \mathcal{C}_h^{-1} C(\alpha, s, d) \||x|^s \mathcal{S}^h_W(f)(b,x)\|_{L^2_\alpha(\mathbb{R}^{d+1}_{++})}.$$

The third result is the Benkner-Type uncertainty principle for the Weintein wavelet transform:

Theorem 2.3. Let $f \in S(\mathbb{R}^d_+)$, the following logarithmic uncertainty principle inequality for the Weinstein wavelet transform holds

$$\int_{\mathbb{R}^{d+1}_{++}} \ln(|x|)|\mathcal{S}^h_W(f)(b,x)|^2 d\omega_\alpha(b,x) + \mathcal{C}_h \int_{\mathbb{R}^d_+} \ln(|\xi|)|\mathcal{F}_W(f)(\xi)|^2 d\mu_\alpha(\xi)$$
$$\ge \left(\frac{\Gamma'\left(\frac{2\alpha+d+1}{4}\right)}{\Gamma\left(\frac{2\alpha+d+1}{4}\right)} + \ln(2)\right)\mathcal{C}_h \|f\|^2_{L^2_\alpha(\mathbb{R}^d_+)}$$

Finally, the logarithmic Sobolev inequalities for the Weinstein wavelet transform:

Theorem 2.4. Let $s > \frac{2\alpha+d+1}{2q}, 1 < p \le 2$ and for all $f \in \mathrm{H}^{p,s}_\alpha(\mathbb{R}^d_+)$ there exists a positive constant $\varepsilon(\alpha, d, s, h, p)$, such that

$$\|\mathcal{S}^h_W(f)\|^2_{L^2_{\omega_\alpha}(\mathbb{R}^d_+)} \le \varepsilon(\alpha, d, s, h, p, q)\left(\|f\|^{2p}_{L^{2p}_\alpha(\mathbb{R}^d_+)} + \||\xi|^s \mathcal{F}_W\|^{2p}_{L^{2p}_\alpha(\mathbb{R}^d_+)}\right).$$

Theorem 2.5. Let $h \in L^2_\alpha(\mathbb{R}^d_+)$ be a Weinstein wavelet and for all $f \in \mathrm{H}^{2,1}_\alpha$, there exists a positive constant $\mathcal{C}(\alpha, d)$, such that

$$\int_{\mathbb{R}^{d++}_{++}} |\mathcal{S}^h_W(f)(b,x)|^2 \ln\left(\frac{|\mathcal{S}^h_W(f)(b,x)|}{\|\mathcal{S}^h_W(f)(b,.)\|_{L^2_\alpha(\mathbb{R}^d_+)}}\right) d\omega_\alpha(x)$$
$$\le \left(\alpha + \frac{d+1}{2}\right) \mathrm{C}_h \int_{\mathbb{R}^d_+} |\mathcal{F}_W(f)(\xi)|^2 \ln|\xi| d\mu_\alpha(x) - \mathcal{C}(\alpha,d)\mathrm{C}_h \|f\|^2_{L^2_\alpha(\mathbb{R}^d_+)}$$

Theorem 2.6. For all $f \in L^2_\alpha(\mathbb{R}^d_+) \cap \mathrm{H}^{2,1}_\alpha$, there exists a positive constant such that

$$\int_{\mathbb{R}^{d+1}_{++}} |\mathcal{S}^h_W(f)(b,x)|^2 \ln(1+|x|^2) d\omega_\alpha(b,x) + \mathrm{C}_h \int_{\mathbb{R}^d_+} |\mathcal{F}_W(f)(x)|^2 \ln(1+|x|^2) d\mu_\alpha(x)$$
$$\ge \mathcal{C}(\alpha,d)\mathcal{C}_h \|f\|^2_{L^2_\alpha(\mathbb{R}^d_+)}$$

The structure of the paper is as follows. In the next section we introduce some further notations as well as some preliminary results. the Section 3, is devoted to prove some result for the Weinstein wavelet transform.

1.1. Harmonic analysis associated with the Weinstein operator

In order to describe our paper, we first need to introduce some notations.

The unit sphere of $\mathbb{R}^d$ is denoted by $\mathbb{S}^{d-1}$, if we denote by $\mathbb{S}^{d-1}_+ = \mathbb{S}^{d-1} \cap \mathbb{R}^d_+$, then

$$w_{d,\alpha} := \int_{\mathbb{S}_+^{d-1}} x_d^{2\alpha+1}\, \mathrm{d}\sigma_d(x) = \frac{\pi^{\frac{d-1}{2}}\Gamma(\alpha+1)}{\Gamma\left(\alpha+\frac{d+1}{2}\right)}$$

where σ_d is the normalized surface measure on $\mathbb{S}_+^{d-1}$.

For a radial function $f \in L_\alpha^1(\mathbb{R}_+^d)$ the function $\tilde{f}$ defined on $\mathbb{R}_+$such that $f(x) = \tilde{f}(|x|)$, for all $x \in \mathbb{R}_+^d$ is integrable with respect to the measure $r^{2\alpha+d}\,\mathrm{d}r$. More precisely, we have

$$\int_{\mathbb{R}_+^d} f(x)\mathrm{d}\mu_\alpha(x) = a_\alpha \int_0^\infty \tilde{f}(r) r^{2\alpha+d}\,\mathrm{d}r$$

where

$$a_\alpha = \frac{w_{d,\alpha}}{\pi^{\frac{d-1}{2}} 2^{\alpha+\frac{d-1}{2}}\Gamma(\alpha+1)} = \frac{1}{2^{\alpha+\frac{d-1}{2}}\Gamma\left(\alpha+\frac{d+1}{2}\right)}$$

For $r > 0$ we will denote by $B_r = \{x \in \mathbb{R}_+^d : |x| < r\}$ the "ball" in $\mathbb{R}_+^d$ of center 0 and radius r and the characteristic function of a set A will be denoted by χ_A, so that $\chi_A(x) = \begin{cases} 1, & \text{if } x \in A; \\ 0, & \text{otherwise} \end{cases}$.

We consider the Weinstein operator (also called Laplace-Bessel operator), (see [3, 4]), defined on $\mathbb{R}^{d-1} \times (0,\infty)$ by

$$\Delta_W = \sum_{i=1}^{d} \frac{\partial}{\partial x_i^2} + \frac{2\alpha+1}{x_d}\frac{\partial}{\partial x_{d-1}};\ d \geq 2, \alpha > -1/2$$

For $d > 2$, the operator Δ_W is the Laplace-Beltrami operator on the Riemanian space $\mathbb{R}^{d-1} \times (0,\infty)$ equipped with the metric [3]

$$\mathrm{d}s^2 = x_d^{4\alpha+2/(d-2)} \sum_{i=1}^{d} \mathrm{d}x_i^2$$

The Weinstein operator has several applications in pure and applied Mathematics especially in Fluid Mechanics (see e.g. [7, 24]). For $1 \leq p \leq \infty$, we denote by $L_\alpha^p(\mathbb{R}_+^d)$ the Lebesgue space consisting of measurable functions f on $\mathbb{R}_+^d = \mathbb{R}^{d-1} \times \mathbb{R}_+$equipped with the norm,

(1) $$\|f\|_{L_\alpha^p(\mathbb{R}_+^d)} = \left(\int_{\mathbb{R}_+^d} |f(x',x_d)|^p\, \mathrm{d}\mu_\alpha(x',x_d)\right)^{1/p}, 1 \leq p < \infty,$$

$$\|f\|_{L_\alpha^\infty(\mathbb{R}_+^d)} = \operatorname*{ess\,sup}_{x\in\mathbb{R}_+^d} |f(x)| < \infty$$

where for $x = (x_1, \dots, x_{d-1}, x_d) = (x', x_d)$ and

$$\mathrm{d}\mu_\alpha(x) = \frac{x_d^{2\alpha+1}}{\pi^{\frac{d-1}{2}} 2^{\alpha+\frac{d-1}{2}}\Gamma(\alpha+1)}\mathrm{d}x'\mathrm{d}x_d = \frac{x_d^{2\alpha+1}}{\pi^{\frac{d-1}{2}} 2^{\alpha+\frac{d-1}{2}}\Gamma(\alpha+1)}\mathrm{d}x_1 \dots \mathrm{d}x_d$$

For $f \in L_\alpha^1(\mathbb{R}_+^d)$, the Weinstein (or Laplace-Bessel) transform is defined by

$$\mathcal{F}_W(f)(\xi',\xi_d) = \int_{\mathbb{R}_+^d} f(x',x_d) e^{-i\langle x',\xi'\rangle} j_\alpha(x_d\xi_d)\mathrm{d}\mu_\alpha(x',x_d),$$

where j_α is the spherical Bessel function :

$$j_\alpha(z) = \Gamma(\alpha+1)\sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(\alpha+k+1)}\left(\frac{z}{2}\right)^{2k}, z \in \mathbb{C}$$

extends uniquely to an isometric isomorphism on $L_\alpha^2(\mathbb{R}_+^d)$ i.e.

(2) $$\|\mathcal{F}_W(f)\|_{L_\alpha^2(\mathbb{R}_+^d)} = \|f\|_{L_\alpha^2(\mathbb{R}_+^d)},$$

and we have

$$\mathcal{F}_W^{-1}(f)(\xi) = \mathcal{F}_W(f)(-\xi',\xi_d), \xi = (\xi',\xi_d) \in \mathbb{R}_+^d$$

Moreover if $f \in L_\alpha^1(\mathbb{R}_+^d)$, then

(3) $$\|\mathcal{F}_W(f)\|_{L_\alpha^\infty(\mathbb{R}_+^d)} \leq \|f\|_{L_\alpha^1(\mathbb{R}_+^d)}.$$

We recall the generalized translation operator $\mathcal{T}_x, x \in \mathbb{R}_+^d$ associated with the Weinstein operator Δ_W is defined for a continuous function f on $\mathbb{R}_+^d$, even with respect to the last variable by

$$\mathcal{T}_x f(y) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+1/2)}\int_0^\pi f\left(x'+y'; \sqrt{x_d^2 + y_d^2 + 2x_d^2 y_d^2 \cos\theta}\right)(\sin\theta)^{2\alpha}\,\mathrm{d}\theta, y \in \mathbb{R}_+^d$$

where $x' + y' = (x_1 + y_1, \dots, x_{d-1} + y_{d-1})$.

For any function $g \in L_\alpha^2(\mathbb{R}_+^d)$ and any $y \in \mathbb{R}_+^d$, here $*_W$ denotes the convolution product associated with the Weinstein operator given by

$$f *_W g(x) = \int_{\mathbb{R}_+^d} f(y)\mathcal{T}_{-x}(g)(y)d\mu_\alpha(y).$$

1.2. The Weinstein wavelet transform

A Weinstein wavelet on $\mathbb{R}_+^d$ is a measurable function h on $\mathbb{R}_+^d$ satisfying, for almost all $\xi \in \mathbb{R}_+^d$

(4) $$0 < C_h = \int_0^\infty |\mathcal{F}_W(h)(t)|^2 \frac{dt}{t} < \infty.$$

Let $b > 0$, and let $h \in L_\alpha^2(\mathbb{R}_+^d)$, we define the dilation of h as follows:

$$\forall y \in \mathbb{R}_+^d, h_b(y) = \frac{1}{b^{2\alpha+d+1}} h\left(\frac{y}{b}\right).$$

It easy to see that $h_b \in L_\alpha^2(\mathbb{R}_+^d)$ and

$$\forall \xi \in \mathbb{R}_+^d, \mathcal{F}_W(h_b)(\xi) = \mathcal{F}_W(h)(b\xi)$$

(5) $$0 < C_h = \int_0^\infty |\mathcal{F}_W(h)(b\xi)|^2 \frac{db}{b} < \infty.$$

We introduce $h_{b,y}, b > 0$, and $y \in R_+^d, y \in \mathbb{R}_+^d$, of Weinstein wavelet on $\mathbb{R}_+^d$ in in $L_\alpha^2(\mathbb{R}_+^d)$, defined by

(6) $$\forall x \in \mathbb{R}_+^d, h_{b,y}(x) = b^{\alpha+\frac{d+1}{2}} \mathcal{T}_x h_b(-y', y_d).$$

We note that

$$\forall b > 0, \forall y \in \mathbb{R}_+^d, \|h_{b,y}\|_{L_\alpha^2(\mathbb{R}_+^d)} \leq \|h\|_{L_\alpha^2(\mathbb{R}_+^d)}$$

Also, we denote by $L_{\omega_\alpha}^p, 1 \leq p \leq \infty$ the space of measurable functions f on $\mathbb{R}_+^d \times [0,\infty] = \mathbb{R}_{++}^{d+1}$, $d\omega_\alpha(x,b) = d\mu_\alpha(x) \frac{1}{b^{2\alpha+d+2}} db$ such that

$$\|f\|_{L_{\omega_\alpha}^p(\mathbb{R}_{++}^{d+1})} = \left(\int_{\mathbb{R}_{++}^{d+1}} |f(x,b)|^p \, d\omega_\alpha(x,b)\right)^{\frac{1}{p}} < \infty, 1 \leq p < \infty$$

$$\|f\|_{L_{\omega_\alpha}^\infty(\mathbb{R}_{++}^{d+1})} = \operatorname{ess}\sup_{x,y \in \mathbb{R}_{++}^{d+1}} |f(x,b)| < \infty$$

Let h be a Weinstein wavelet on $\mathbb{R}_+^d \in L_\alpha^2(\mathbb{R}_+^d)$. we defined the Weinstein continuous wavelet

(7) $$\mathcal{S}_W^h(f)(b,y) = \int_{\mathbb{R}_+^d} f(y)\overline{h_{b,y}(x)} d\mu_\alpha(x), (b,y) \in \mathbb{R}_{++}^{d+1}.$$

And

(8) $$\mathcal{S}_W^h(f)(b,y) = b^{\alpha+\frac{d+1}{2}} \langle f, \mathcal{T}_x h_b \rangle_{L_\alpha^2(\mathbb{R}_+^d)} = b^{\alpha+\frac{d+1}{2}} f *_W \overline{h_b}.$$

Then

(9) $$\mathcal{S}_W^h(f)(b,y) = b^{\alpha+\frac{d+1}{2}} \mathcal{F}_W^{-1}[\mathcal{F}_W(f)(\xi)\mathcal{F}_W(h)(\xi b)](y),$$

and

(10) $$\|\mathcal{S}_W^h(f)\|_{L_{\omega_\alpha}^2(\mathbb{R}_{++}^{d+1})} \leq \|f\|_{L_\alpha^2(\mathbb{R}_+^d)} \| h\|_{L_\alpha^2(\mathbb{R}_+^d)}.$$

Theorem 1.1. (Plancherel's formula for $\mathcal{S}_W^h$)Let h be a Weinstein wavelet. For all $f \in L_\alpha^2(\mathbb{R}_+^d)$

(11) $$\int_{\mathbb{R}_+^d} |f(x)|^2 d\mu_\alpha(x) = C_h^{-1} \int_{\mathbb{R}_{++}^{d+1}} |\mathcal{S}_W^h(f)(b,y)|^2 d\omega_\alpha(b,y).$$

Corollary 1.1. (Parseval's formula for $\mathcal{S}_W^h$) Let h be a Weinstein wavelet. For all $f_1, f_2 \in L_\alpha^2(\mathbb{R}_+^d)$

(12) $$\int_{\mathbb{R}_+^d} f_1(x)\overline{f_2(x)} d\mu_\alpha(x) = C_h^{-1} \int_{\mathbb{R}_{++}^{d+1}} \mathcal{S}_W^h(f_1)(b,y)\overline{\mathcal{S}_W^h(f_2)(b,y)} d\omega_\alpha(b,y).$$

Proposition 1.1. for all $s \geq 0$ and $f \in L_\alpha^2(\mathbb{R}_+^d)$ we have

(13) $$\int_{\mathbb{R}_+^d} |\xi|^s |\mathcal{F}_W(f)(\xi)|^2 d\mu_\alpha(\xi) = C_h^{-1} \int_{\mathbb{R}_{++}^{d+1}} |\xi|^s |\mathcal{S}_W^h(f)(\xi)|^2 d\omega_\alpha(b,y).$$

Proof. By (5) and (9) we can easy proof it.

2. New results for the Weinstein wavelet transform

In this section, we will analogue of Heisenberg-Type Uncertainty Principle for the Weinstein Wavelet Transform, Pitt's Inequality for the Weinstein Wavelet Transform, the Benkner-Type uncertainty principle for the Weintein wavelet transform and the logarithmic Sobolev inequalities for the Weinstein
wavelet transform our proof is inspired from [14], whose proved some results for deformed wavelet transform and related uncertainty principle.

2.1. Heisenberg-type uncertainty principle for the Weinstein wavelet transform

Extension to our studies in [5] and there are many studies the Heisenberg uncertainty principle inequality for wavelet transforms [[18], [14]]. In this section, we introduce the uncertainty inequality for the Weinstein wavelet transform. Firstly from our study [Corollary 3.5, [5]], we present the following theorem.

Theorem 2.1. Let $s,t>1$. Then, for all $f\in L^2_\alpha(\mathbb{R}^d_+)$, we have

(14) $$\||x|^s\mathcal{S}^h_W(f)(b,x)\|^{\frac{t}{s+t}}_{L^2_{\omega_\alpha}\mathbb{R}^{d+1}_{++}}\||\xi|^t\mathcal{F}_W(f)(\xi)\|^{\frac{s}{s+t}}_{L^2_\alpha\mathbb{R}^d_+}\geq C_{\alpha,d}(s,t)\mathcal{C}_h^{\frac{t}{2(s+t)}}\|f\|_{L^2_\alpha(\mathbb{R}^d_+)},$$

where $C_{\alpha,d}(s,t)=\left(\alpha+\frac{d+1}{2}\right)^{\frac{ts}{s+t}}$.

Proof. From [(3.10),[5]], implies that for all $b>0$

$$\left(\int_{\mathbb{R}^d_+}|\xi|^{2t}|\mathcal{F}_W[\mathcal{S}^h_W(f)(b,.)](\xi)|^2d\mu_\alpha(\xi)\right)^{\frac{s}{t+s}}\left(\int_{\mathbb{R}^d_+}|x|^{2s}|\mathcal{S}^h_W(f)(b,x)|^2d\mu_\alpha(x)\right)^{\frac{t}{t+s}}$$
$$\geq\left(C_{\alpha,d}(s,t)\right)^2\int_{\mathbb{R}^d_+}|\mathcal{S}^h_W(f)(b,x)|^2d\mu_\alpha(x).$$

Integrating both sides with respect to the measure $\frac{db}{b^{2\alpha+d+2}}$, we obtain, by Hölder's inequality and Plancherel's formula,

$$\int_0^\infty\left(\int_{\mathbb{R}^d_+}|\xi|^{2t}|\mathcal{F}_W[\mathcal{S}^h_W(f)(b,.)](\xi)|^2d\mu_\alpha(\xi)\right)^{\frac{s}{t+s}}\left(\int_{\mathbb{R}^d_+}|x|^{2s}|\mathcal{S}^h_W(f)(b,x)|^2d\mu_\alpha(x)\right)^{\frac{t}{t+s}}\frac{db}{b^{2\alpha+d+2}}$$
$$\geq\left(C_{\alpha,d}(s,t)\right)^2\int_0^\infty\int_{\mathbb{R}^d_+}|\mathcal{S}^h_W(f)(b,x)|^2d\mu_\alpha(x)\frac{db}{b^{2\alpha+d+2}}.$$

Thus, form (13), we deduce

$$\mathcal{C}_h^{\frac{s}{t+s}}\left(\int_{\mathbb{R}^d_+}|\xi|^{2t}|\mathcal{F}_W(f)(\xi)|^2d\omega_\alpha(b,\xi)\right)^{\frac{s}{t+s}}\left(\int_{\mathbb{R}^{d+1}_{++}}|x|^{2s}|\mathcal{S}^h_W(f)(b,x)|^2d\omega_\alpha(b,x)\right)^{\frac{t}{t+s}}$$
$$\geq\left(C_{\alpha,d}(s,t)\right)^2\int_{\mathbb{R}^{d+1}_{++}}|\mathcal{S}^h_W(f)(b,x)|^2d\omega_\alpha(x)$$
$$\geq\left(C_{\alpha,d}(s,t)\right)^2\mathcal{C}_h\int_{\mathbb{R}^d_+}|f(x)|^2d\mu_\alpha(x)$$
$$\geq\left(C_{\alpha,d}(s,t)\right)^2\mathcal{C}_h\|f(x)\|^2_{\mathbb{R}^d_+}.$$

2.1. Pitt's inequality for the Weinstein wavelet transform

The Pitt's inequality for the Weinstein transform is studied in [1], for all $\in S(\mathbb{R}^d_+)$, (the Schwartz space of rapidly decreasing functions on $\mathbb{R}^d_+$, even with respect to the last variable) and $0\leq s<\alpha+\frac{d+1}{2}$

(15) $$\||\xi|^{-s}\mathcal{F}_W(f)(\xi)\|_{L^2_\alpha(\mathbb{R}^d_+)}\leq C(\alpha,s,d)\||x|^sf\|_{L^2_\alpha(\mathbb{R}^d_+)},$$

where

$$C(\alpha,s,d)=2^{-s}\frac{\Gamma\left(\frac{\alpha+\frac{d+1}{2}-s}{2}\right)}{\Gamma\left(\frac{\alpha+\frac{d+1}{2}+s}{2}\right)}.$$

The main aim of this section is to formulate an analogue of Pitt's inequality (15) for the Weinstein wavelet transform.

Theorem 2.2. For $0\leq s<\alpha+\frac{d+1}{2}$ and $\in S(\mathbb{R}^d_+)$, the Pitt's-type inequality for the wavelet transform is given by

(16) $$\||\xi|^s\mathcal{F}_W(f)\|_{L^2_\alpha(\mathbb{R}^d_+)}\leq\mathcal{C}_h^{-1}C(\alpha,s,d)\||x|^s\mathcal{S}^h_W(f)(b,x)\|_{L^2_\alpha(\mathbb{R}^{d+1}_{++})}.$$

Proof. From (15), replace f with $\mathcal{S}^h_W(f)$ and for $b>0$, we have

$$\int_{\mathbb{R}^d_+}|\xi|^s\mathcal{F}_W\left[\mathcal{S}^h_W(f)((b,.)](\xi)\mathrm{d}\mu_\alpha(x)\leq C(\alpha,s,d)\int_{\mathbb{R}^{d+1}_{++}}|x|^s\mathcal{S}^h_W(f)\big((b,x)d\omega_\alpha(x).$$

By integrating the both sides with respect to the measure $\frac{db}{b^{2\alpha+d+2}}$, and by Fubini's theorem, we have

$$\int_{\mathbb{R}^d_+}\int_0^\infty|\xi|^{2s}\mid\mathcal{F}_W[\mathcal{S}^h_W(f)((b,.)]|^2\frac{db}{b^{2\alpha+d+2}}\,\mathrm{d}\mu_\alpha(\xi)$$
$$\leq C(\alpha,s,d)\int_{\mathbb{R}^d_+}\int_0^\infty|x|^{2s}\left|\mathcal{S}^h_W(f)\left((b,x)\right|\frac{db}{b^{2\alpha+d+2}}d\mu_\alpha(x)\right.$$

Using (9), the last inequality has introduced as follows:

$$\int_{\mathbb{R}_+^d}\int_0^\infty |\xi|^{2s}|\mathcal{F}_W(h_b)(\xi)|^2|\mathcal{F}_W(f)(\xi)|^2\frac{db}{b^{2\alpha+d+2}}\ d\mu_\alpha(\xi)$$
$$\leq C(\alpha,s,d)\int_{\mathbb{R}_{++}^{d+1}}|x|^{2s}\left|\mathcal{S}_W^h(f)\left((b,x)\right|^2\frac{db}{b^{2\alpha+d+2}}d\omega_\alpha(x).\right.$$

Thus

$$\int_{\mathbb{R}_+^d}|\xi|^{2s}\left(\int_0^\infty |\mathcal{F}_W(h_b)(\xi)|^2\frac{db}{b}\right)|\mathcal{F}_W(f)(\xi)|^2\ d\mu_\alpha(\xi)$$
$$\leq C(\alpha,s,d)\int_{\mathbb{R}_+^d}\int_0^\infty |x|^{2s}\left|\mathcal{S}_W^h(f)\left((b,x)\right|^2\frac{db}{b^{2\alpha+d+2}}d\omega_\alpha(x).\right.$$

From (5), we obtain

$$\mathcal{C}_h\int_{\mathbb{R}_+^d}|\xi|^{2s}|\mathcal{F}_W(f)(\xi)|^2\ d\mu_\alpha(\xi)\leq C(\alpha,s,d)\int_{\mathbb{R}_{++}^{d+1}}|x|^{2s}\left|\mathcal{S}_W^h(f)\left((b,x)\right|^2\frac{db}{b^{2\alpha+d+2}}d\omega_\alpha(x).\right.$$

This completes the proof.

2.3. Benkner-Type uncertainty principle for the Weinstein wavelet transform

In this section study the Benkner-Type uncertainty principle for the Weinstein wavelet transform as the following theorem.

Theorem 2.3. Let $\in$ $S(\mathbb{R}_+^d)$, the following logarithmic uncertainty principle inequality for the Weinstein wavelet transform holds

$$\int_{\mathbb{R}_{++}^{d+1}}\ln(|x|)\,|\mathcal{S}_W^h(f)(b,x)|^2 d\omega_\alpha(b,x)+\mathcal{C}_h\int_{\mathbb{R}_+^d}\ln(|\xi|)|\mathcal{F}_W(f)(\xi)|^2 d\mu_\alpha(\xi)$$

$$\geq\left(\frac{\Gamma'\left(\frac{2\alpha+d+1}{4}\right)}{\Gamma\left(\frac{2\alpha+d+1}{4}\right)}+\ln(2)\right)\mathcal{C}_h\|f\|^2_{L^2_\alpha(\mathbb{R}_+^d)} \tag{17}$$

Proof. From [Theorem4.5,[1]], we observe

$$\int_{\mathbb{R}_+^d}\ln(|x|)|f(x)|^2 d\mu_\alpha(x)+\int_{\mathbb{R}_+^d}\ln(|\xi|)\,|\mathcal{F}_W(f)(\xi)|^2 d\mu_\alpha(\xi)$$
$$\geq\left(\frac{\Gamma'\left(\frac{2\alpha+d+1}{4}\right)}{\Gamma\left(\frac{2\alpha+d+1}{4}\right)}+\ln(2)\right)\|f\|^2_{L^2_\alpha(\mathbb{R}_+^d)}.$$

Here we replace f by $\mathcal{S}_W^h(b,.), weget$

$$\int_{\mathbb{R}_+^d}\ln(|x|)|\mathcal{S}_W^h(f)(b,x)|^2 d\mu_\alpha(x)+\int_{\mathbb{R}_+^d}\ln(|\xi|)|\mathcal{F}_W[\mathcal{S}_W^h(f)(b,.)](\xi)|^2 d\mu_\alpha(\xi)$$
$$\geq\left(\frac{\Gamma'\left(\frac{2\alpha+d+1}{4}\right)}{\Gamma\left(\frac{2\alpha+d+1}{4}\right)}+\ln(2)\right)\int_{\mathbb{R}_+^d}|\mathcal{S}_W^h(f)(b,x)|^2 d\mu_\alpha(x)$$

Integrating the last inequality, the both sides with respect to the measure $\frac{db}{b^{2\alpha+d+2}}$, and by Fubini's theorem, we have

$$\int_{\mathbb{R}_{++}^{d+1}}\ln(|x|)|\mathcal{S}_W^h(f)(b,x)|^2 d\omega_\alpha(b,x)+\int_{\mathbb{R}_{++}^{d+1}}\ln(|\xi|)|\mathcal{F}_W[\mathcal{S}_W^h(f)(b,.)](\xi)|^2 d\omega_\alpha(b,\xi)$$
$$\geq\left(\frac{\Gamma'\left(\frac{2\alpha+d+1}{4}\right)}{\Gamma\left(\frac{2\alpha+d+1}{4}\right)}+\ln(2)\right)\int_{\mathbb{R}_{++}^{d+1}}|\mathcal{S}_W^h(f)(b,x)|^2 d\mu_\alpha(x)$$

Using (9), and Plancherel's formula, we obtain

$$\int_{\mathbb{R}_{++}^{d+1}}\ln(|x|)|\mathcal{S}_W^h(f)(b,x)|^2 d\omega_\alpha(b,x)+\int_{\mathbb{R}_{++}^{d+1}}\ln(|\xi|)|\mathcal{F}_W(h)(b,\xi)|^2|\mathcal{F}_W(f)(\xi)|^2 d\omega_\alpha(b,\xi)$$
$$\geq\left(\frac{\Gamma'\left(\frac{2\alpha+d+1}{4}\right)}{\Gamma\left(\frac{2\alpha+d+1}{4}\right)}+\ln(2)\right)\int_{\mathbb{R}_{++}^{d+1}}|\mathcal{F}_W(h)(b,\xi)|^2|\mathcal{F}_W(f)(\xi)|^2 d\omega_\alpha(b,x).$$

Consequently

$$\int_{\mathbb{R}_{++}^{d+1}}\ln(|x|)\,|\mathcal{S}_W^h(f)(b,x)|^2 d\omega_\alpha(b,x)+\mathcal{C}_h\int_{\mathbb{R}_+^d}\ln(|\xi|)\,|\mathcal{F}_W(f)(\xi)|^2 d\mu_\alpha(\xi).$$

Which proves the desired result.

2.4. Logarithmic Sobolev inequalities for the Weinstein wavelet transform

Definition. Let N $s \in \mathbb{R}_+^d$, $1 \leq p < \infty$ and $T \in S'(\mathbb{R}_+^d)$ be a distribution we define:

(18) $H_\alpha{}^{p,s}(\mathbb{R}_+^d) = \{ T \in S(\mathbb{R}_+^d) : (1 + |\xi|^2)^{\frac{ps}{2}} \mathcal{F}_W(T) \in L_\alpha^p(\mathbb{R}_+^d)\}$.

Means that for $s > 0$, $T \in H_\alpha{}^{p,s}(\mathbb{R}_+^d)$ and $\mathcal{F}_W(T)$ is given by a function in $L_\alpha^p(\mathbb{R}_+^d)$, the norm on $H_\alpha{}^{p,s}$ is defined by

$$\|T\|_{H_\alpha{}^{p,s}} = \left\|(1 + |\xi|^2)^{\frac{ps}{2}}\right\|_{L_\alpha^p(\mathbb{R}_+^d)}$$

In case $p = 2$, for $T \in H_\alpha{}^{p,s}(\mathbb{R}_+^d)$, it be can seen that T is necessary given by a function $f \in L^2_\alpha(\mathbb{R}_+^d)$ the space $H_\alpha{}^{p,s}(\mathbb{R}_+^d)$ can be defined as

(19) $$H_\alpha^{2,s}(\mathbb{R}_+^d) = \{f \in L_\alpha^2(\mathbb{R}_+^d) : |\xi|^s \mathcal{F}_W(f) \in L_\alpha^2(\mathbb{R}_+^d)\}.$$

In the following, we give a Sobolev embedding theorem.

Theorem 2.4 Let $s > \frac{2\alpha+d+1}{2q}$, $1 < p \leq 2$ and for all $f \in H_\alpha{}^{p,s}(\mathbb{R}_+^d)$ there exists a positive constant $\varepsilon(\alpha, d, s, h, p)$, such that

(20) $\|S_w^h(f)\|^2_{L^2_{W_\alpha}(\mathbb{R}_+^d)} \leq \varepsilon(\alpha, d, s, h, p, q)(\|f\|^{2p}{}_{L_\alpha^{2p}(\mathbb{R}_+^d)} + \||\xi|^s \mathcal{F}_w(f)\|^{2p}_{L_\alpha^p(\mathbb{R}_+^d)}$

Proof. From Plancherel's formula, we obtain

$$\|S_w^h(f)\|^2_{L^2_{W_\alpha}(\mathbb{R}_+^d)} \leq C_h \int_{\mathbb{R}_+^d} |\mathcal{F}_W(f)(\xi)|^2 (1 + |\xi|^2)^s (1 + |\xi|^2)^{-s} \, d\mu_\alpha(\xi)$$

Using Hölder inequality, we get

$$\|S_w^h(f)\|^2_{L^2_{W_\alpha}(\mathbb{R}_+^d)} \leq (\int_{\mathbb{R}_+^d} |\mathcal{F}_W(f)(\xi)|^{2p} (1 + |\xi|^2)^{sp} d\mu_\alpha(\xi))^{\frac{1}{p}} \left(\int_{\mathbb{R}_+^d} (1 + |\xi|^2)^{-sq} d\mu_\alpha(\xi)\right)^{1/q}$$

By (1), we have

$$\int_{\mathbb{R}_+^d} (1 + |\xi|^2)^{-sq} d\mu_\alpha(\xi) = \frac{\Gamma\left(sq - \left(\alpha + \frac{d+1}{2}\right)\right)\Gamma\left(\alpha + \frac{d+1}{2}\right)}{2^{\alpha + \frac{d+1}{2}}\Gamma\left(\alpha + \frac{d+1}{2}\right)\Gamma(sq)}.$$

Using the fact that $(a + b)^s \leq 2^s(a^s + b^s)$, we get

$$\|S_w^h(f)\|^2_{L^2_{W_\alpha}(\mathbb{R}_+^d)} \leq C_h{}^p \left[\frac{\Gamma\left(sq-\left(\alpha+\frac{d+1}{2}\right)\right)\Gamma\left(\alpha+\frac{d+1}{2}\right)}{2^{\alpha+\frac{d+1}{2}}\Gamma\left(\alpha+\frac{d+1}{2}\right)\Gamma(sq)}\right]^{\frac{p}{q}} 2^{sp} \left(\|f\|^{2p}{}_{L_\alpha^{2p}(\mathbb{R}_+^d)} + \||\xi|^s \mathcal{F}_w(f)\|^{2p}_{L_\alpha^p(\mathbb{R}_+^d)}\right).$$

Thus

$$\varepsilon(\alpha, d, s, h, p, q) = C_h{}^p \left[\frac{\Gamma\left(sq - \left(\alpha + \frac{d+1}{2}\right)\right)\Gamma\left(\alpha + \frac{d+1}{2}\right)}{2^{\alpha+\frac{d+1}{2}}\Gamma\left(\alpha + \frac{d+1}{2}\right)\Gamma(sq)}\right]^{\frac{p}{q}} 2^{sp}.$$

Corollary 2.1. Let $s > \frac{2\alpha+d+1}{2q}$, there exists a positive constant $\varepsilon(\alpha, d, s, h, p)$, such that

(21) $\|S_w^h(f)\|^2_{L^2_{W_\alpha}(\mathbb{R}_+^d)} \leq \varepsilon(\alpha, d, s, h, p, q)(\|f\|^{2p}{}_{L_\alpha^{2p}(\mathbb{R}_+^d)} + \||\xi|^s \mathcal{F}_w(f)\|^{2p}_{L_\alpha^p(\mathbb{R}_+^d)})$.

Proof. Using the dilated $D_\lambda(f)(x) = \lambda^{\alpha+\frac{d+1}{2}} f(\lambda x)$ to (20) , we get

$$\|S_w^h(f)\|^2_{L^2_{W_\alpha}(\mathbb{R}_+^d)} \leq \varepsilon(\alpha, d, s, h, p, q)(\lambda^{2\alpha+d+1}\|f\|^{2p}{}_{L_\alpha^{2p}(\mathbb{R}_+^d)} + \lambda^{2sp-(2\alpha+d+1)}\||\xi|^s \mathcal{F}_w(f)\|^{2p}_{L_\alpha^p(\mathbb{R}_+^d)})$$

By minimizing the right-hand side last inequality, we obtain

$$\|S_w^h(f)\|^2_{L^2_{W_\alpha}(\mathbb{R}_+^d)} \leq \varepsilon(\alpha, d, s, h, p, q)^{\frac{1}{2}} \left(\frac{\mathrm{sp}\left(\mathrm{ps} - \left(\alpha + \frac{d+1}{2}\right)\right)^{\frac{2\alpha+d+1-2sp}{2sp}}}{\left(\alpha + \frac{d+1}{2}\right)^{\frac{2\alpha+d+1}{2sp}}}\right)$$

$$\times \left(\|f\|^{p-\frac{2\alpha+d+1}{2s}}_{L_\alpha^{2p}(\mathbb{R}_+^d)} \||\xi|^s \mathcal{F}_w(f)\|^{\frac{2\alpha+d+1}{2s}}_{L_\alpha^p(\mathbb{R}_+^d)}\right)$$

This is the desired result.

Now, to describe next our result we need to recall the Theorem 5.6 in [2], there exists a positive constant *C(α, d)* such that for al $f \in H_\alpha^{2,s}(\mathbb{R}_+^d)$

$$\int_{\mathbb{R}_+^d} |f(x)|^2 n\left(\frac{f(x)}{\|f\|_{L_\alpha^2(\mathbb{R}_+^d)}}\right) d\mu_\alpha(x) \leq \int_{\mathbb{R}_+^d} |\mathcal{F}_W(h)(\xi)|^2 \ln(|\xi|)\, d\mu_\alpha(\xi)$$

(22) $$-C(\alpha,d)\|f\|^2_{L_\alpha^2(\mathbb{R}_+^d)}.$$

Now, with the same constant C(α, d), we show the following theorem

Theorem 2.5. Let $h \in L_\alpha^2(\mathbb{R}_+^d)$ be a Weinstein wavelet and for all $f \in H_\alpha^{2,s}(\mathbb{R}_+^d)$, there exist a positive constant *C(α, d)*, Such that

$$\int_{\mathbb{R}_+^d} |\mathcal{S}_W^h(f)(b,x)|^2 \ln\left(\frac{|\mathcal{S}_W^h(f)(b,x)|}{\|\mathcal{S}_W^h(f)(b,.)\|_{L_\alpha^2(\mathbb{R}_+^d)}}\right) dw_\alpha(x)$$

(23) $$\leq (\alpha + \tfrac{d+1}{2})\mathcal{C}_h \int_{\mathbb{R}_+^d} \ln(|\xi|)\, |\mathcal{F}_W(f)(\xi)|^2 d\mu_\alpha(\xi) - C(\alpha,d)\mathcal{C}_h \|f\|^2_{L_\alpha^2(\mathbb{R}_+^d)}.$$

Proof. Replacing $\mathcal{S}_W^h(f)\ (b,x)$ and integrate with respect to the measure $\frac{db}{b^{2\alpha+d+2}}$ both sides in (22), we get

$$\int_0^\infty \int_{\mathbb{R}_+^d} |\mathcal{S}_W^h(f)(b,x)|^2 \ln\left(\frac{|\mathcal{S}_W^h(f)(b,x)|}{\|\mathcal{S}_W^h(f)(b,.)\|_{L_\alpha^2(\mathbb{R}_+^d)}}\right) d\mu_\alpha(x) \frac{db}{b^{2\alpha+d+2}}$$

$$\leq (\alpha + \frac{d+1}{2}) \int_0^\infty \int_{\mathbb{R}_+^d} |\mathcal{F}_W(\mathcal{S}_W^h(f)(b,.))(\xi)|^2 \ln(|\xi|) d\mu_\alpha(\xi) \frac{db}{b^{2\alpha+d+2}} - C(\alpha,d) \int_0^\infty \|\mathcal{S}_W^h(f)(b,.)\|^2_{L_\alpha^2(\mathbb{R}_+^d)}.$$

Now, by Fubini's theorem and Plancherel's formula, we get the required result.

Finally, we give another version of logarithmic uncertainty for the Weinstein wavelet transform.

Theorem 2.6. For all $f \in L_\alpha^2(\mathbb{R}_+^d)$, there exists a positive constant such that

$$\int_{\mathbb{R}_{++}^{d+1}} |\mathcal{S}_W^h(f)(b,x)|^2 \ln(1+|x|^2) d\omega_\alpha(b,x) + \mathcal{C}_h \int_{\mathbb{R}_+^d} |\mathcal{F}_W(f)(\xi)|^2 \ln(1+|x|^2) d\mu_\alpha(\xi)$$

(24) $$\geq C(\alpha,d)\mathcal{C}_h \|f\|^2_{L_\alpha^2(\mathbb{R}_+^d)}.$$

Proof. In the same manner. Replacing f with $\mathcal{S}_W^h(f)$ and by integrating both sides to inequality of [Theorem 5.8, [2]], we obtain

$$\int_{\mathbb{R}_{++}^{d+1}} |\mathcal{S}_W^h(f)(b,x)|^2 \ln(1+|x|^2) d\omega_\alpha(b,x) + \int_{\mathbb{R}_{++}^{d+1}} |\mathcal{F}_W[\mathcal{S}_W^h(f)(b,.)](\xi)|^2 \ln(1+|x|^2) d\omega_\alpha(b,\xi)$$

$$\geq C(\alpha,d)\|\mathcal{S}_W^h(f)(b,.)\|^2_{L_\alpha^2(\mathbb{R}_+^d)}.$$

Applying Proposition 2.3 Plancherel's formula, we obtain

(25) $$\int_{\mathbb{R}_{++}^{d+1}} |\mathcal{S}_W^h(f)(b,x)|^2 \ln(1+|x|^2) d\omega_\alpha(b,x) + \mathcal{C}_h \int_{\mathbb{R}_+^d} |\mathcal{F}_W(f)(\xi)|^2 \ln(1+|x|^2) d\mu_\alpha(\xi)$$

$$\geq C(\alpha,d)\mathcal{C}_h \|f\|^2_{L_\alpha^2(\mathbb{R}_+^d)}.$$

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