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Generalized Fractional Derivative Operators of the Multi-Index Mittag-Leffler Functions with Applications

Ram Niwas Meghwal[1] Sunil Kumar[2] and Krishna Gopal Bhadana[3]

[1] Department of Mathematics, G. H. S. Government College, Sujangarh-331507
[2, 3] Department of Mathematics, S. P. C. Government College, Ajmer
Maharshi Dayanand Saraswati University, Ajmer, Rajasthan-305009, India
E-mail(s): meghwalramniwas@gmail.com;sunilgehlot286@gmail.com; dr.bhadanakg@gmail.com

ABSTRACT

In this paper we use fractional differential operators $D^{n}_{\alpha,\beta,x}$ and D^{η}_{t} to derive a number of key formulas of multivariable H-function. We use the generalized Leibnitz's rule for fractional derivatives in order to obtain one of the aforementioned formulas, which involve a product of generalized multi-index Mittag Leffler function. It is further shown that, each of these formulas yield interesting new formulas for generalized multi-index Mittag Leffler function 2020 Mathematics Subject Classification: 26A33, 33C45, 33E20.

Keywords and Phrases: generalized fractional derivative operators, multi-index Mittag Leffler function.

Definitions

Generalized Fractional Derivative Operators

We use the fractional derivative operator defined in the following manner [7]

$$D^{n}_{\alpha,\beta,x}(x^{\lambda}) = \prod_{r=0}^{n-1}\left[\frac{\Gamma(\lambda+r\alpha+1)}{\Gamma(\lambda+r\alpha-\beta+1)}\right]x^{\lambda+n\alpha} \tag{1.1}$$

Where $\beta \neq \lambda+1$ and α and β are not necessarily integers

We use the binomial expansion in the following manner

$$(ax^{\mu}+b)^{\lambda} = b^{\lambda}\sum_{l=0}^{\infty}\binom{\lambda}{l}\left(\frac{ax^{\mu}}{b}\right)^{l} \qquad where \quad [\frac{ax^{\mu}}{b}]<1 \tag{1.2}$$

the familiar differential operator ${}_{\alpha}D^{\mu}_{x}$ is defined by [7]

$${}_{\alpha}D^{\mu}_{x} f(\mathrm{x}) = \begin{cases} \dfrac{1}{\sqrt{-\mu}}\displaystyle\int_{\alpha}^{x}(x-t)^{-\mu-1}f(t)dt\ , & [\mathrm{Re}(\mu)<0] \\ \dfrac{d^{m}}{dx^{m}}\,{}_{\alpha}D^{\mu-m}_{x} f(x), & [0\le \mathrm{Re}(\mu)<\mathrm{m}] \end{cases} \tag{1.3}$$

Where m is a positive integer

For $\alpha = 0$,(1.3) Defines the classical Riemann-Liouville fractional derivative of order μ (or-μ) when α→∞ (1.3) may be identified with the definition of the well known Weyl fraction derivative of order μ (or-μ) [1,chap.13);3] the special case of fractional calculus operator ${}_{\alpha}D_x^{\mu}$ when $\alpha = 0, \mu = \eta, x = t$ is written as D_t^{η} thus we have

$$D_t^{\eta} = {}_{\alpha}D_x^{\mu}$$

$$D_t^{\eta}(x^{\lambda}) = \frac{\Gamma(1+\lambda)}{\Gamma(1+\lambda-\eta)} x^{\lambda-\eta} \quad \{Re(\lambda) > -1\} \qquad (1.4)$$

2. Generalized Multi-Index Mittag Leffler Function

The generalized multi-index Mittag Leffler function is defined by Saxena and Nishimoto [16] in the following summation form:

$$E_{(A_j,B_j)_m}^{\lambda,\rho}(x) = \sum_{k=0}^{\infty} \frac{(\lambda)_{\rho k}}{\prod_{j=1}^{m} \Gamma(A_j\, k + B_j)} \frac{x^k}{k!}; \quad (m \in \mathbb{N}) \qquad (2.1)$$

where $A_j,\ B_j,\ \lambda,\ \rho \in \mathbb{C};\ \mathcal{R}e(B_j) > 0$ and

$$\sum_{j=1}^{m} \mathcal{R}e(A_j) > max\,\{\, \mathcal{R}e(\rho)\, - 1; 0\}.$$

For $m = 1$, the generalized multi-index Mittag Leffler function (2.1) reduce into the generalized Mittag-Leffler function given by Shukla and Prajapati [19] and defined as follows:

$$E_{A,B}^{\lambda,\rho}(x) = \sum_{k=0}^{\infty} \frac{(\lambda)_{\rho k}}{\Gamma(Ak + B)} \frac{x^k}{k!}, \qquad (2.2)$$

where $A,\ B,\ \lambda \in \mathbb{C};\ \mathcal{R}e(A)\ > 0, \mathcal{R}e(B)\ >\ 0, \mathcal{R}e(\lambda)\ >\ 0$ and $\rho \in (0,1)\ \cup\ \mathbb{N}$

For $m = 1$ and $\rho = 1$, the generalized multi-index Mittag-Leffler function (2.1) reduce into the generalized Mittag-Leffler function given by Prabhakar [12] defined as follows:

$$E_{A,B}^{\lambda}(x) = \sum_{k=0}^{\infty} \frac{(\lambda)_k}{\Gamma(Ak + B)} \frac{x^k}{k!}, \qquad (2.3)$$

where $A, B, \lambda \in \mathbb{C}\,;\ \mathcal{R}e(A)\ > 0,\ \mathcal{R}e(B)\ >\ 0,\ \mathcal{R}e(\lambda)\ >\ 0, x \in \mathbb{C}$ and $(\lambda)_k$ is the well known Pochhammer symbol.

3. Main Results

Theorem 1. Fractional derivatives operator $D_t^{\eta}(x^{\lambda})$ associated with the product of two multi-index Mittag-Leffler functions**.**

$$D_t^{\eta}\left\{t^{\delta-1}\, E_{(A_j,B_j)_m}^{\lambda,\rho}(x_1 t) \times E_{(A_j,B_j)_m}^{\lambda,\rho}(x_2 t)\right\}$$

$$= t^{\delta-\eta-1}\left\{E_{(A_j,B_j)_m}^{\lambda,\rho}(x_1) \times E_{(A_j,B_j)_m}^{\lambda,\rho}(x_2)\right\} \otimes \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{\Gamma(\delta+k+l)}{\Gamma(\delta+k+l-\eta)} t^{k+l} \qquad (3.1)$$

Where ⊗ stands for convolution product of two functions.

Proof. We refer to the left hand side of equation (3.1) by the symbol D_1.

Then making the use of equation (2.1) in equation (3.1), we have

$D_1 \equiv$

$$D_t^{\eta}\left\{t^{\delta-1} \sum_{k=0}^{\infty} \frac{(\lambda)_{\rho k}}{\prod_{j=1}^{m}\Gamma(A_j\, k + B_j)} \frac{(x_1 t)^k}{k!} \times \sum_{l=0}^{\infty} \frac{(\lambda)_{\rho l}}{\prod_{j=1}^{m}\Gamma(A_j\, l + B_j)} \frac{(x_2 t)^l}{l!}\right\}$$

After changing the order of summations and derivative operator under the conditions of theorem, we obtain the above as

$$= \sum_{k=0}^{\infty} \frac{(\lambda)_{\rho k}}{\prod_{j=1}^{m}\Gamma(A_j\, k + B_j)} \frac{(x_1)^k}{k!} \times \sum_{l=0}^{\infty} \frac{(\lambda)_{\rho l}}{\prod_{j=1}^{m}\Gamma(A_j\, l + B_j)} \frac{(x_2)^l}{l!}$$

$\times D_t^{\eta}(t^{\delta+k+l-1})$

We use the fractional derivative operator $D_t^{\eta}(x^{\lambda})$ after simplification we get

$$= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{(\lambda)_{\rho k}(\lambda)_{\rho l}}{\prod_{j=1}^{m}\Gamma(A_j\, k + B_j)\prod_{j=1}^{m}\Gamma(A_j\, l + B_j)} \frac{(x_1)^k}{k!}\frac{(x_2)^l}{l!} \times \frac{\Gamma(\delta+k+l)}{\Gamma(\delta+k+l-\eta)} t^{\delta+k+l-\eta-1}$$

Further, applying the definition (2.1) and convolution product on two series,

we obtain

$$D_1 \equiv t^{\delta-\eta-1}\left\{E_{(A_j,B_j)_m}^{\lambda,\rho}(x_1) \times E_{(A_j,B_j)_m}^{\lambda,\rho}(x_2)\right\} \otimes \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{\Gamma(\delta+k+l)}{\Gamma(\delta+k+l-\eta)} t^{k+l}$$

Where ⊗ stands for convolution product of two functions.

Theorem 2. Fractional derivatives operator $D_{\alpha,\beta,x}^{n}(x^{\lambda})$ associated with the product of two multi-index Mittag-Leffler functions.

$$D_{\alpha,\beta,t}^{n}\left\{t^{\delta-1} E_{(A_j,B_j)_m}^{\lambda,\rho}(x_1 t) \times E_{(A_j,B_j)_m}^{\lambda,\rho}(x_2 t)\right\}$$

$$= t^{\delta+n\alpha-1}\left\{E_{(A_j,B_j)_m}^{\lambda,\rho}(x_1) \times E_{(A_j,B_j)_m}^{\lambda,\rho}(x_2)\right\}$$

$$\otimes \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\left[\prod_{r=0}^{n-1}\left\{\frac{\Gamma(\delta+k+l+r\alpha)}{\Gamma(\delta+k+l+r\alpha-\beta)}\right\} t^{k+l}\right] \tag{3.2}$$

Where ⊗ stands for convolution product of two functions.

Proof. We refer to the left hand side of equation (3.2) by the symbol D_2.

Then making the use of equation (2.1) in equation (3.2), we have

$D_2 \equiv$

$$D_{\alpha,\beta,t}^{n}\left\{t^{\delta-1}\sum_{k=0}^{\infty}\frac{(\lambda)_{\rho k}}{\prod_{j=1}^{m}\Gamma(A_j\, k+B_j)}\frac{(x_1 t)^k}{k!} \times \sum_{l=0}^{\infty}\frac{(\lambda)_{\rho l}}{\prod_{j=1}^{m}\Gamma(A_j\, l+B_j)}\frac{(x_2 t)^l}{l!}\right\}$$

After changing the order of summations and derivative operator under the conditions of theorem, we obtain the above as

$$= \sum_{k=0}^{\infty}\frac{(\lambda)_{\rho k}}{\prod_{j=1}^{m}\Gamma(A_j\, k+B_j)}\frac{(x_1)^k}{k!} \times \sum_{l=0}^{\infty}\frac{(\lambda)_{\rho l}}{\prod_{j=1}^{m}\Gamma(A_j\, l+B_j)}\frac{(x_2)^l}{l!}$$

$$\times\; D_{\alpha,\beta,t}^{n}(t^{\delta+k+l-1})$$

We use the fractional derivative operator $D_{\alpha,\beta,x}^{n}(x^{\lambda})$ after simplification we get

$$= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{(\lambda)_{\rho k}(\lambda)_{\rho l}}{\prod_{j=1}^{m}\Gamma(A_j\, k + B_j)\prod_{j=1}^{m}\Gamma(A_j\, l + B_j)} \frac{(x_1)^k}{k!}\frac{(x_2)^l}{l!}$$

$$\times \prod_{r=0}^{n-1}\left[\frac{\Gamma(\delta+k+l+r\alpha)}{\Gamma(\delta+k+l+r\alpha-\beta)}\right] t^{\delta+k+l-1+n\alpha}$$

Further, applying the definition (2.1) and convolution product on two series,

we obtain

$$D_2 \equiv t^{\delta+n\alpha-1}\left\{E_{(A_j,B_j)_m}^{\lambda,\rho}(x_1) \times E_{(A_j,B_j)_m}^{\lambda,\rho}(x_2)\right\}$$

$$\otimes \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\left[\prod_{r=0}^{n-1}\left\{\frac{\Gamma(\delta+k+l+r\alpha)}{\Gamma(\delta+k+l+r\alpha-\beta)}\right\} t^{k+l}\right]$$

Where ⊗ stands for convolution product of two functions.

Theorem 3. Double fractional derivatives operators $D_t^{\eta}(x^{\lambda})$ and $D_{\alpha,\beta,x}^{n}(x^{\lambda})$

associated with the product of two multi-index Mittag-Leffler functions**.**

$$D_{\alpha,\beta,t}^{n}\left[D_t^{\eta}\left\{t^{\delta-1} E_{(A_j,B_j)_m}^{\lambda,\rho}(x_1 t) \times E_{(A_j,B_j)_m}^{\lambda,\rho}(x_2 t)\right\}\right]$$

$$= t^{\delta-\eta+n\alpha-1}\left\{E^{\lambda,\rho}_{(A_j,B_j)_m}(x_1)\times E^{\lambda,\rho}_{(A_j,B_j)_m}(x_2)\right\}$$

$$\otimes\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\left[\prod_{r=0}^{n-1}\left\{\frac{\Gamma(\delta+k+l-\eta+r\alpha)}{\Gamma(\delta+k+l-\eta+r\alpha-\beta)}\right\}t^{k+l}\right] \quad (3.3)$$

Proof. We refer to the left hand side of equation (3.3) by the symbol D_3.

Then making the use of equation (2.1) in equation (3.3), we have

$$D_3 \equiv$$

$$D^n_{\alpha,\beta,t}\left[D^\eta_t\left\{t^{\delta-1}\sum_{k=0}^{\infty}\frac{(\lambda)_{\rho k}}{\prod_{j=1}^{m}\Gamma(A_j\,k+B_j)}\frac{(x_1t)^k}{k!}\times\sum_{l=0}^{\infty}\frac{(\lambda)_{\rho l}}{\prod_{j=1}^{m}\Gamma(A_j\,l+B_j)}\frac{(x_2t)^l}{l!}\right\}\right]$$

After changing the order of summations and derivative operator under the conditions of theorem, we obtain the above as.

$$D^n_{\alpha,\beta,t}\left[\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{(\lambda)_{\rho k}(\lambda)_{\rho l}}{\prod_{j=1}^{m}\Gamma(A_j\,k+B_j)\prod_{j=1}^{m}\Gamma(A_j\,l+B_j)}\frac{(x_1)^k}{k!}\frac{(x_2)^l}{l!}\times\left\{D^\eta_t\left(t^{\delta+k+l-1}\right)\right\}\right]$$

We use the fractional derivative operator $D^\eta_t(x^\lambda)$ after simplification we get.

$$D^n_{\alpha,\beta,t}\left[\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{(\lambda)_{\rho k}(\lambda)_{\rho l}}{\prod_{j=1}^{m}\Gamma(A_j\,k+B_j)\prod_{j=1}^{m}\Gamma(A_j\,l+B_j)}\frac{(x_1)^k}{k!}\frac{(x_2)^l}{l!}\times\frac{\Gamma(\delta+k+l)}{\Gamma(\delta+k+l-\eta)}\,t^{\delta+k+l-\eta-1}\right]$$

$$=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{(\lambda)_{\rho k}(\lambda)_{\rho l}}{\prod_{j=1}^{m}\Gamma(A_j\,k+B_j)\prod_{j=1}^{m}\Gamma(A_j\,l+B_j)}\frac{(x_1)^k}{k!}\frac{(x_2)^l}{l!}\times\frac{\Gamma(\delta+k+l)}{\Gamma(\delta+k+l-\eta)}$$

$$\times\left\{D^n_{\alpha,\beta,t}\left(t^{\delta+k+l-\eta-1}\right)\right\}$$

We use the fractional derivative operator $D^n_{\alpha,\beta,x}(x^\lambda)$ after simplification we get

$$=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{(\lambda)_{\rho k}(\lambda)_{\rho l}}{\prod_{j=1}^{m}\Gamma(A_j\,k+B_j)\prod_{j=1}^{m}\Gamma(A_j\,l+B_j)}\frac{(x_1)^k}{k!}\frac{(x_2)^l}{l!}\times\frac{\Gamma(\delta+k+l)}{\Gamma(\delta+k+l-\eta)}$$

$$\times\prod_{r=0}^{n-1}\left[\frac{\Gamma(\delta+k+l-\eta+r\alpha)}{\Gamma(\delta+k+l-\eta+r\alpha-\beta)}\right]t^{\delta+k+l-\eta-1+n\alpha}$$

Further, applying the definition (2.1) and convolution product on two series,

we obtain.

$$D_3\equiv t^{\delta-\eta+n\alpha-1}\left\{E^{\lambda,\rho}_{(A_j,B_j)_m}(x_1)\times E^{\lambda,\rho}_{(A_j,B_j)_m}(x_2)\right\}$$

$$\otimes\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\left[\prod_{r=0}^{n-1}\left\{\frac{\Gamma(\delta+k+l-\eta+r\alpha)}{\Gamma(\delta+k+l-\eta+r\alpha-\beta)}\right\}t^{k+l}\right]$$

Where ⊗ stands for convolution product of two functions.

4. Special cases

Our main provides unifications and extensions of various (known or new) results fractional differential operators. For the sake of illustration, we mentior the following special cases

Corollary 1. Let the conditions of Theorem 1 be satisfied and $\eta = 1$, $m = 1$ then the theorem 1 reduced in the following form:

$$D^1_t\left\{t^{\delta-1}E^{\lambda,\rho}_{A,B}(x_1t)\times E^{\lambda,\rho}_{A,B}(x_2t)\right\}$$

$$=t^{\delta-2}\left\{E^{\lambda,\rho}_{A,B}(x_1)\times E^{\lambda,\rho}_{A,B}(x_2)\right\}\otimes\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{\Gamma(\delta+k+l)}{\Gamma(\delta+k+l-1)}t^{k+l} \quad (4.1)$$

Corollary 2. Let the conditions of Theorem 2 be satisfied and $\alpha = 0,\ \beta = 1$ then the theorem 2 reduced in the following form:

$$D_{0,1,t}^{n}\left\{t^{\delta-1}\,E_{(A_j,B_j)_m}^{\lambda,\rho}(x_1t)\times E_{(A_j,B_j)_m}^{\lambda,\rho}(x_2t)\right\}$$
$$=t^{\delta-1}\left\{E_{(A_j,B_j)_m}^{\lambda,\rho}(x_1)\times E_{(A_j,B_j)_m}^{\lambda,\rho}(x_2)\right\}$$
$$\otimes\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\left[\prod_{r=0}^{n-1}\left\{\frac{\Gamma(\delta+k+l)}{\Gamma(\delta+k+l-1)}\right\}t^{k+l}\right] \tag{4.2}$$

Corollary 3. Let the conditions of Theorem 3 be satisfied and $\alpha=1,\ \beta=0,$

$\eta=1, m=1$then the theorem 3 reduced in the following form:

$$D_{1,0,t}^{n}\left[\,D_t^1\left\{t^{\delta-1}\,E_{A,B}^{\lambda,\rho}(x_1t)\times E_{A,B}^{\lambda,\rho}(x_2t)\right\}\right]$$
$$=t^{\delta+n-2}\left\{\,E_{A,B}^{\lambda,\rho}(x_1)\times E_{A,B}^{\lambda,\rho}(x_2)\right\}$$
$$\otimes\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\left[\prod_{r=0}^{n-1}\left\{\frac{\Gamma(\delta+k+l+r-1)}{\Gamma(\delta+k+l+r-1)}\right\}t^{k+l}\right] \tag{4.3}$$

Corollary 4. Let the conditions of Theorem 3 be satisfied and $\alpha=0,\ \beta=1,$

$\eta=1, m=1, \rho=1$then the theorem 3 reduced in the following form:

$$D_{0,1,t}^{n}\left[\,D_t^1\left\{t^{\delta-1}\,E_{A,B}^{\lambda}(x_1t)\times E_{A,B}^{\lambda}(x_2t)\right\}\right]$$
$$=t^{\delta-\eta+n\alpha-1}\left\{\,E_{A,B}^{\lambda}(x_1)\times E_{A,B}^{\lambda}(x_2)\right\}$$
$$\otimes\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\left[\prod_{r=0}^{n-1}\left\{\frac{\Gamma(\delta+k+l-1)}{\Gamma(\delta+k+l-2)}\right\}t^{k+l}\right] \tag{4.4}$$

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