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Spectral Analysis of Economic Growth of India and Uttar Pradesh Using Locally Stationary Wavelet Process

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ABSTRACT:

Wavelet transform is a powerful analytical technique increasingly utilized for analysing non-stationary and transient signals. It decomposes a signal into two key components: approximations (low-frequency parts reflecting the average behaviour) and details (high-frequency parts capturing differential features). The Multiresolution Analysis (MRA) framework allows for adaptive resolution levels depending on the characteristics of the signal. A variant known as the Stationary Wavelet Transform (SWT) enhances the Discrete Wavelet Transform (DWT) by eliminating down-sampling; instead, it up-samples the filter by a factor of two at each decomposition level, thereby preserving translation invariance. The Locally Stationary Wavelet (LSW) process is particularly suited for examining non-stationary datasets, such as financial time series, by breaking them down into distinct frequency bands using wavelets. In this study, the economic trends of India and Uttar Pradesh are forecasted using both general linear models and wavelet-based prediction methods. Historical economic data spanning 2004–05 to 2023–24 serves as the input, and future projections are extended up to 2035–36 through analysis using the LSW process. The study highlights a notable and consistent rise in the economic growth of India and Uttar Pradesh, observed in the present and projected for the near future.

Keywords: wavelet, India, Uttar Pradesh, Income, LSW, MRA

1. Introduction

Fourier transform is a widely used technique in spectral analysis, where functions are represented using trigonometric components of varying periods across different scales. It is a powerful analytical method with applications spanning mathematics, physics, and engineering. From a statistical perspective, computing the Fourier spectrum of a function is equivalent to fitting sine and cosine functions of different frequencies using the least squares method. This type of multiple regression with trigonometric functions is both elegant and straightforward. Since sine and cosine functions are orthonormal, the Fourier coefficients can be easily computed either as a summation in the discrete case or an integral in the continuous case. This entire procedure is referred to as Fourier transformation [1]. Fourier transform plays a fundamental role in several scientific disciplines. In fields such as optics, acoustics, and electronics, both the waveform and its spectral representation are physically observable and measurable. Instruments like oscilloscopes visually display optical or electrical waveforms, while the human ear directly perceives acoustical spectra. The waveform and its spectrum are mathematically linked through the Fourier transform, underlining its deep physical significance. Additionally, the theory behind phase contrast microscopes and frequency modulation detection circuits can both be explained through transformation concepts. A linear and time-invariant system responds harmonically to harmonic inputs at the same frequency, which further illustrates the practical utility of Fourier analysis [2].

Wavelet theory emerged in the early 1980s through interdisciplinary collaboration among mathematicians, engineers, and physicists. Unlike the relatively simpler domain of stationary signals, transient signals are more complex and varied. Wavelets provide a framework for analysing these signals by decomposing them into basic, localized elements at various positions and scales. This has proven especially effective in applications like image edge detection. In the discrete wavelet transform (DWT), filters (both high-pass and low-pass) are used to achieve variable time and frequency resolutions, with sub-sampling used to control scaling [3-4]. The input signal is repeatedly processed through these filters to extract information at multiple levels.

2. Wavelet Transforms

While the Fourier transform is effective for analysing time-invariant signals with finite energy, it does not offer information about how frequency components evolve over time in non-stationary signals. To address this, a windowing technique was introduced, where a time-localized window function is applied before performing the Fourier transform. This concept, introduced by Gabor in 1946, allows for the analysis of localized frequency content by sliding the window across the time domain and computing the transform at each position (Antoine, 2004). This approach is known as the Short-Time Fourier Transform (STFT) or Windowed Fourier Transform (WFT). However, according to the Heisenberg uncertainty principle, one cannot simultaneously achieve high precision in both time and frequency. Therefore, a trade-off must be made: high-frequency components require finer time resolution, while low-frequency components benefit from better frequency resolution. This leads to the idea of analysing the signal in segments using short windows for high-frequency content and longer windows for low-frequency trends. Wavelets address this challenge by using functions that oscillate briefly and then decay, making them ideal for capturing both transient and long-duration features [5].

In essence, a wavelet is a localized, oscillatory function that captures signal characteristics over short intervals and then fades out, making it well-suited for analysing nonstationary and time-varying signals. For any two real numbers a and b, a wavelet function is defined as [6]: -

$$\psi_{a,b}(t) = \frac{1}{\sqrt{a}}\psi\left(\frac{t-b}{a}\right) = T_b D_a \psi \tag{1}$$

Putting $a = 2^{-j}$ and $\frac{b}{a} = k$, we get discrete wavelets as following: -

$$\psi_{j,k}(t) = 2^{j/2}\psi(2^j t - k) \tag{2}$$

where a and b are the dilation and translation parameter respectively. In this case, ψ(t) is a real-valued function, and the set of wavelets forms an orthonormal basis

The continuous wavelet transform is the modified WFT and defined as: -

$$W_{a,b} = \int f(t)\frac{1}{\sqrt{a}}\psi\left(\frac{t-b}{a}\right)dt \tag{3}$$

The discrete wavelet transform is defined as: -

$$W_{j,k} = \int f(t)2^{j/2}\psi(2^j t - k)\,dt \tag{4}$$

2.1 Multiresolution Analysis (MRA):

An MRA is a new recursive method to perform the discrete wavelet transforms [7-9]. It consists of a sequence $V_j : j \in \mathbb{Z}$ of closed subspaces of Lebesgue space $L^2(\mathbb{R})$, a space of square integrable functions, satisfying the following properties: -

1) $V_{j+1} \subset V_j : j \in \mathbb{Z}$

2) $\cap_{j\in\mathbb{Z}} V_j = \{0\}$, $\cup_{j\in\mathbb{Z}} = L^2(\mathbb{R})$

3) For every $L^2(\mathbb{R})$, $f(t) \in V_j \Rightarrow f(2t) \in V_{j+1}$,

4) There exists a function $\phi(t) \in V_0$ such that $\{\phi(t-k): k \in \mathbb{Z}\}$ is orthonormal basis of V_0.

The function $\phi(t)$ is called scaling function of given MRA and property 3 implies a dilation equation as follows: -

$$\phi(t) = \sum_{k\in Z} h_k \sqrt{2}\,\phi(2t-k) \tag{5}$$

Where h_k is low pass filter and is defined as:

$$h_k = \left(\frac{1}{\sqrt{2}}\right)\int_{-\infty}^{\infty} \phi(t)\phi(2t-k)\,dt \tag{6}$$

Now we consider W_1 be orthogonal compliment of V_1 in V_0 i.e.

$$V_0 = V_1 \oplus W_1$$

If $\psi \in W_1$ be any wavelet function then,

$$\psi(t) = \sum_{k\in Z} g_k \sqrt{2}\,\phi(2t-k) \tag{7}$$

where $g_k = (-1)^{k+1} h_{1-k}$ are high pass filters. In general, we can write,

$$V_j = V_{j+1} \oplus W_{j+1} \tag{8}$$

But, $$V_{j+1} = V_{j+2} \oplus W_{j+2}$$

Therefore,

$$V_j = W_{j+1} \oplus W_{j+2} \oplus V_{j+2}$$

.. ..

$$V_j = W_{j+1} \oplus W_{j+2} \oplus W_{j+3} \oplus \ldots\ldots\ldots\ldots W_{j+p} \oplus V_{j+p} \tag{9}$$

where p is any desired number representing the order of level of decomposition.

2.2 One dimensional (1D) wavelet transform:

A function f (that is for which $\int_{\mathbb{R}}|f(x)|\, dx < \infty$) has a wavelet series expansion in vector space V_j,

$$f(x) = \sum_{k\in\mathbb{Z}} a_{j+p,k}\ \phi(x-k) + \sum_{p=1}^{\infty}\sum_{k\in\mathbb{Z}} d_{j+p,k}\ \psi_{j+p,k}(x) \quad (10)$$

It also follows that the sum $\sum_{k\in\mathbb{Z}} a_{j+p,k}\phi(x-k)$ is the orthogonal projection of f on the space V_{j+p} of square integrable functions that are constant on integer end point intervals $[k, k+1)$. For $j = 0$, the sum $\sum_{p=1}^{\infty}\sum_{k\in\mathbb{Z}} d_{j+p,k}\ \psi_{j+p,k}(x)$ adds the details required to obtain an approximation in the space V_p of square integrable functions that are constant on all intervals [10].

If all such functions u and v are orthogonal ($\langle u, v\rangle = 0$), then W_j is the orthogonal complement of V_j in V_{j-1} ($V_j \perp W_j$) and the construction below will give the scaling function and mother wavelet of an orthonormal wavelet basis for $L^2((\mathbb{R})$. By MRA, the orthogonal decomposition of pth level of space V_j is as following: -

$$V_j = V_{j+p} \oplus \sum_{p=1}^{\infty} W_{j+p}$$

A discrete signal is approximated in space of square summable sequences $\ell^2(\mathbb{Z})$ as follows: -

$$f[n] = \frac{1}{\sqrt{M}}\sum_k a[j+p,k]\phi_{j+p,k}[n] + \frac{1}{\sqrt{M}}\sum_{p=1}^{\infty}\sum_k d[j+p,k]\psi_{j+p,k}[n] \quad (11)$$

Here $f[n]$, $\phi_{j+p,k}[n]$ and $\psi_{j+p,k}[n]$ are discrete functions defined in $[0, M-1]$, totally M points. Because the sets $\{\phi_{j+p,k}[n]\}_{k\epsilon\mathbb{Z}}$ and $\{\psi_{j+p,k}[n]\}_{k\epsilon\mathbb{Z},p\epsilon\mathbb{Z}^+}$ are orthogonal to each other. The We wavelet coefficients can be obtained by taking the inner product as follows: -

$$a[j+p,k] = \frac{1}{\sqrt{M}}\sum_n f[n]\ \phi_{j+p,k}[n] \quad (12)$$

$$d[j+p,k] = \frac{1}{\sqrt{M}}\sum_n f[n]\ \psi_{j+p,k}[n] \quad (13)$$

where $a[j+p,k]$ and $d[j+p,k]$ are called approximation and detailed coefficients respectively. From property of scaling function,

$$\phi_{j,k}[n] = 2^{j/2}\phi[2^j n - k]$$
$$= 2^{j/2}\sum_{n'} h[n']\sqrt{2}\phi[2(2^j n - k) - n']$$
$$= 2^{(j+1)/2}\sum_{n'} h[n']\ \phi\left[2^{(j+1)}n - 2k - n'\right]$$

Let $n' = m - 2k$, we have $\phi_{j,k}[n] = 2^{(j+1)/2}\sum_m h[m-2k]\phi\left[2^{(j+1)}n - m\right]$. Now the approximation coefficient,

$$a[j,k] = \frac{1}{\sqrt{M}}\sum_n f[n]\ \phi_{j,k}[n]$$
$$= \frac{1}{\sqrt{M}}\sum_n f[n]\ 2^{j/2}\phi[2^j n - k]$$
$$= \frac{1}{\sqrt{M}}\sum_n f[n]\ 2^{j/2}\sum_m h[m-2k]\ \sqrt{2}\phi[2^{j+1}n - m]$$
$$= \sum_m\ h[m-2k]\left(\frac{1}{\sqrt{M}}\sum_n f[n]\ 2^{j+1/2}\ \phi[2^{j+1}n - m]\right)$$
$$= \sum_m\ h[m-2k]\ a[j+1,n]$$
$$= \sum_{n'}\ h[n'] * a[j+1,n], \text{ where } k \geq 0.$$

Similarly, for the detail coefficients, it is,

$$a[j,k] = g[n'] * d[j+1,n], \text{ where } k \geq 0.$$

By taking $j = 0$, we get,

$$a[0,k] = \sum_{n'}\ h[n'] * a[1,n]$$

$$a[0,k] = g[n'] * d[1,n] \text{ where } k \geq 0.$$

2.3 Stationary Wavelet Transforms (SWT):

In SWT, the same number of samples as the input is maintained at every decomposition level at decomposition of levels redundancy of in the wavelet coefficients exists. The SWT reconstructions result lower and error values and faster convergence compared to DWT [11]. This is achieved by SWT thresholding, which provides a translation-invariant basis. By SWT thresholding, a redundant decomposition can be obtained as follows: -

$$\tilde{a}_{2^j}^{2^j k+p} = \langle f(t), 2^{j/2}\phi(2^j(t-p)-k)\rangle$$
$$\tilde{d}_{2^j}^{2^j k+p} = \langle f(t), 2^{j/2}\psi(2^j(t-p)-k)\rangle$$

Where $p \in \{0, \ldots. 2^j - 1\}$. For decomposition level $j_m, 2j_m$ different orthogonal bases are generated. Each path from the root of the tree to a leaf corresponds to the set of functions as follows: -

$$\{2^{j/2}\psi(2^j(t-p_j)-k)\} \cup \{2^{j_m/2}\psi(2^j(t-p_{j_m})-k)\}$$

Where $1 \le j \le j_m$, $k \in \mathbb{Z}$. form an orthogonal wavelet basis, resulting in a standard wavelet reconstruction.

3. Locally stationary wavelet (LSW) prediction

Every covariance-stationary process X_t has a Cramer representation as follows: -

$$X_t = \int_{-\pi}^{\pi} A(\omega) e^{i\omega t} dz(\omega) \tag{14}$$

Where $dz(\omega)$ represents a stochastic process having orthonormal increments. Non-stationary processes represent a slow change over time of the amplitude $A(\omega)$. In LSW process the amplitude $A(\omega)$ in the Cramer representation is replaced by a time varying quantity and the Fourier harmonics $e^{i\omega t}$ by non-decimated discrete wavelets $\psi_{j,k}(t)$: $j, k \in \mathbb{Z}$; . Here j and k are the scale and location parameter respectively. Time-modulated (TM) process $X_{t,T}$ is defined as follows: -

$$X_{t,T} = \sigma\left(\frac{t}{T}\right) Y_t \tag{15}$$

Where Y_t represents a zero-mean stationary process with variance one and the local standard deviation function $\sigma(z)$ is Lipschitz continuous on (0, 1). Process $X_{t,T}$ is locally stationary wavelet (LSW) if,

i) The auto covariance function of Y_t is absolutely summable so that Y_t is an LSW with a time-invariant spectrum S_j^Y

ii) The Lipschitz constants $L_j^X = D.\left(S_j^Y\right)^{1/2}$ satisfy the Cramer representation, where D is the Lipschitz constant.

If above two conditions are satisfied, the spectrum $S_j(z)$ of $X_{t,T}$ is expressed as follows: -

$$S_j(z) = \sigma^2(z)\, S_j^Y$$

The general LSW processes are applicable to model processes whose variance and autocorrelation function both vary with time. The prediction operator can be expressed as in wavelet domain [12],

$$X_{t,T} = \sum_{j=1}^{J} \sum_{k \in \mathbb{Z}} w_{j,k;T}\, \psi_{j,k}(t)\, \xi_{j,k} \tag{16}$$

Where $T = 2^J$ and $\{\psi_{j,k}(t)\}_{j,k}$ is a discrete non-decimated family of wavelets for $j = 1,2, \ldots \ldots J = log_2(T)$ based on a mother wavelet $\psi(t)$ of compact support. Here $\xi_{j,k}$ is a random orthogonal increment sequence with $E\, \xi_{j,k} = 0$ and $Cov\,(\xi_{j,k}, \xi_{\ell,m}) = \xi_{j\ell}\, \xi_{km}$. LSW processes are not uniquely determined by the sequence $\{w_{j,k;T}\}$. For t observations of non-stationary data $X_{0,T}, X_{1,T}, X_{2,T}, \ldots \ldots . X_{T-1,T}$ of an LSW process, the general linear predictor $X_{t+h,T}$ corresponding to h-step ahead, is expressed as follows: -

$$\hat{X}_{t+h,T} = \sum_{s=0}^{t-1} b_{t+s;T}^{(t)} X_{s,T} \tag{17}$$

Where the coefficients $b_{t+s;T}^{(t)}$ minimise the Mean Square Prediction Error (MSPE) defined as,

$$MSPE(\hat{X}_{t,T}, X_{t,T}) = E(\hat{X}_{t,T} - X_{t,T})^2$$

That is, as $T \to \infty$, allows us to fit coefficients $b_{t-s;T}^{(t)}$ with more accuracy. Here h is the prediction horizon, we set $T = t + h$. Let us consider the forecasting horizon $h = 1$, so that T= $t + 1$. The empirical wavelet coefficients in the wavelet domain in terms of prediction operator are defined as follows: -

$$d_{j,k;T} = \sum_{t=0}^{T-1} X_{t,T}, \psi_{j,k}(t) \tag{18}$$

for all $j = 1,2, \ldots \ldots \ldots J$ and $k \in \mathbb{Z}$. The one-step ahead predictor in terms of wavelet coefficients is defined as: -

$$\hat{X}_{t+h,T} = \sum_{j=1}^{J} \sum_{k \in \mathbb{Z}} d_{j,k;T}\, a_{j,k;T}^{(t)}\, \psi_{j,k}(t) \tag{19}$$

Where the estimated coefficients $c_{j,k;T}^{(t)}$ are such that they minimise the MSPE. This predictor is defined as a projection of $X_{t,T}$ on the space having random variables and spanned by $d_{j,k;T}$; $j = 1,2, \ldots \ldots \ldots J$ and $k \in \mathbb{Z}$. Because of the redundancy of the non-orthogonal wavelet system $\psi_{j,k}(t)$ the predictor has more than one solution $\{c_{j,k}^{(t)}\}$ (and every solution corresponds to the same predictor in terms of the different linear combination of redundant functions $\{\psi_{j,k}(t)\}$. Therefore, the wavelet predictor and the linear predictor can be expressed as follows: -

$$b_{t+s;T}^{(t)} = \sum_{j=1}^{J} \sum_{k \in \mathbb{Z}} c_{j,k;T}^{(t)}\, \psi_{j,k}(t)\, \psi_{j,k}(s) \tag{20}$$

Due to the redundancy of the non-decimated wavelet system, a fixed sequence $b_{t+s;T}^{(t)}$ is expressed as the linear combination of more than one sequence $c_{j,k;T}^{(t+1)}$. Therefore, the prediction work is carried out directly with the general linear predictor, and wavelet predictor is determined from above equation because linear predictor is a non-unique projection onto the wavelet domain [13].

4. Results and discussion

The data of Indian and Uttar Pradesh economy from financial year 2004-05 to 2023-24 (Total 20 years) have been taken from website of Indian Climate and Energy Dashboard (Website- https://iced.niti.gov.in).

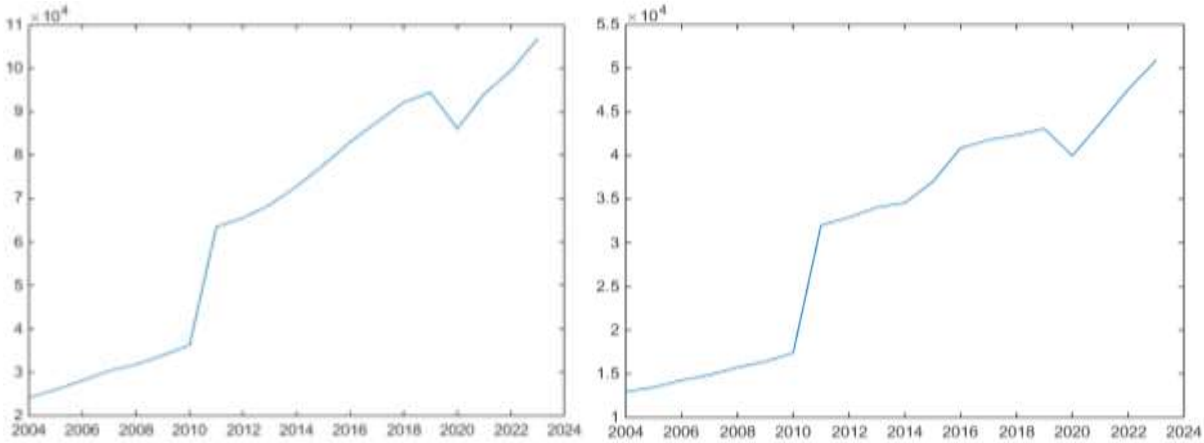

Figure 1: Per capita income of India and U.P. from year 2004-05 to 2023-24

This data is decomposed up to level-1 using Haar wavelet transform.

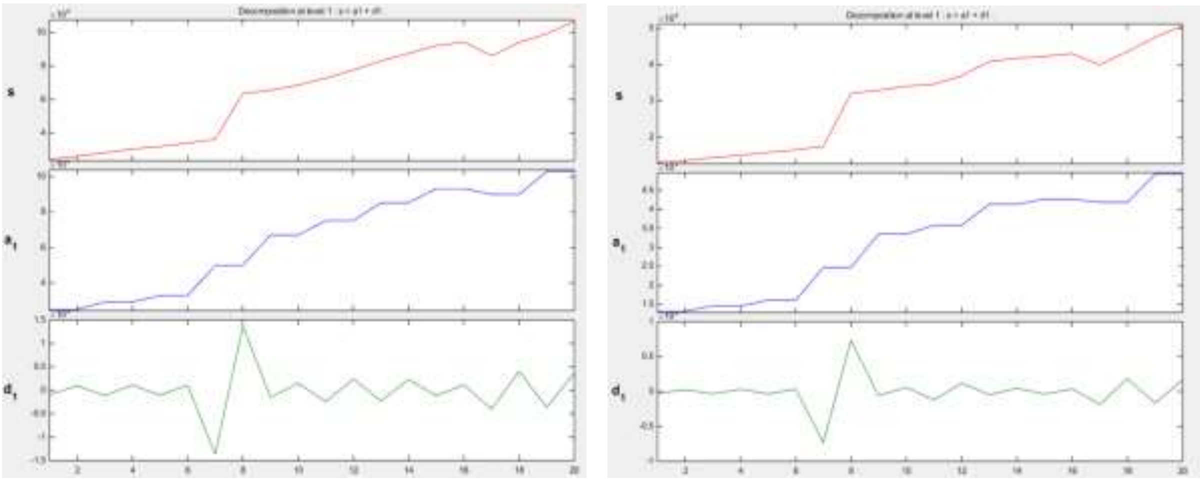

Figure 2: Wavelet decomposition of per capita income of India and U.P.

The given data is extended up to year 2035-36 (Total 32 years) using locally stationary wavelet process [14-15].

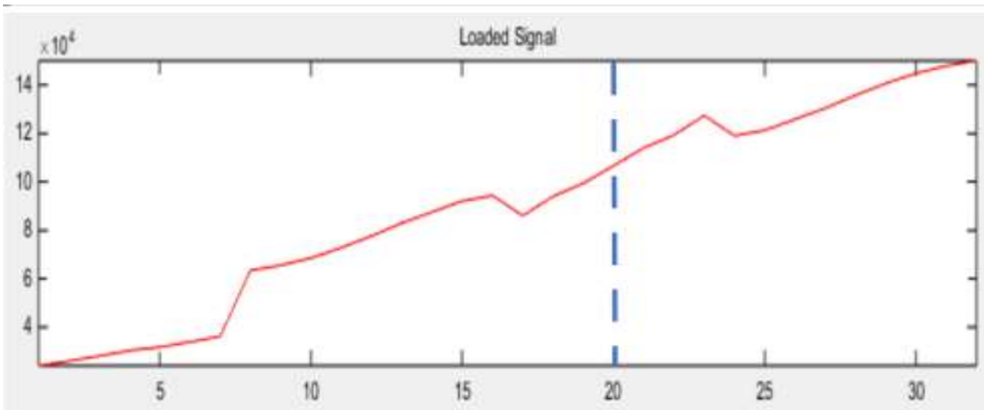

Figure 3: Predicted per capita income of India up to year 2035-36

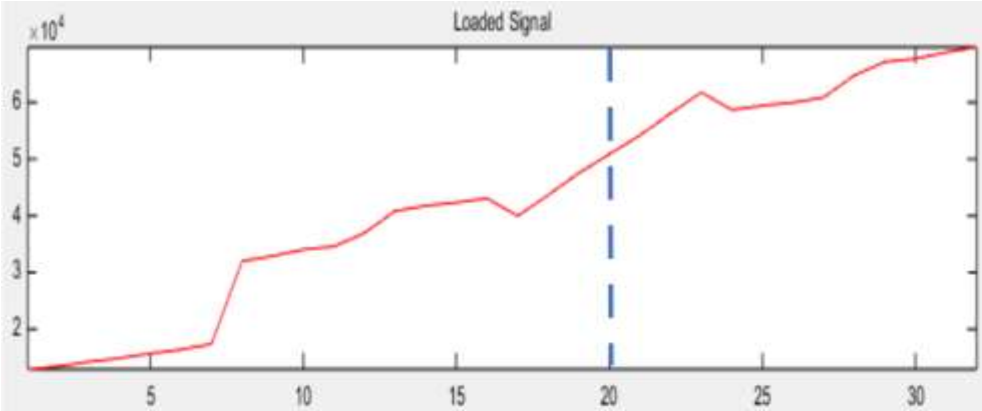

Figure 4: Predicted per capita income of U.P. up to year 2035-36

Some statistical parameters for the analysis of given and extended data are as follows: -

Table 1: Statistical parameters for the given and extended data

S. No.	Parameter	Given data		Extended data	
		India	U. P.	India	U. P.
1	Mean	6.509×10^4	3.128×10^4	8.997×10^4	4.303×10^4
2	Standard Deviation	2.86×10^4	1.311×10^4	4.024×10^4	1.875×10^4
3	L^1- norm	1.302×10^6	6.256×10^5	2.879×10^6	1.377×10^6
4	L^2- norm	3.167×10^5	1.511×10^5	5.561×10^5	2.648x105
	Skewness	-0.254	-0.293	-0.266	-0.276
	Kurtosis	-1.562	-1.537	-1.113	-1.149
	Correlation	0.998		0.998	

The approximation at highest scale value represents the trend or average behaviour of signal. The detail at each scale value represents the differential behaviour of the signal at each level. Figure 3 and 4 show an appreciable increasing trend of economic growth of India and Uttar Pradesh in near future. In the next years the trend in economic growth will also be maintained. The statistical parameters of present and future growth are given in table 1. The skewness represents asymmetry of data points about the mean value. The values of skewness are negative and low. The negative value of skewness represents that the data are skewed left. It indicates an increase in the economy of India and Uttar Pradesh. The kurtosis represents the peakedness of data points. The values of kurtosis are negative and low for both, which indicates the data is not much far distributed from mean value [16]. The values of the mean, L^1- norm, and L^2- norm indicate that the economy is likely to grow slightly faster than the current trend. The correlation represents the mode of dependence of two or more variables. The positive and high value of correlation coefficients indicates that the economic growth of India and Uttar Pradesh are linearly and strongly correlated.

5. Conclusion

The economic growth data of India and Uttar Pradesh from 2004–05 to 2023–24 exhibit periodic fluctuations over time with an increasing trend. The data is extended up to year 2035-36 with help of locally stationary wavelet analysis. Wavelet decomposition enables the separation of the signal into low and high-frequency components. The highest-level low-frequency component reflects the overall trend or average behaviour of the signal, while the high-frequency components highlight short-term variations and fluctuations. The wavelet and statistical analysis suggest a modest increasing rate in economic growth of India and Uttar Pradesh in the near future. Overall, wavelet transform offers a clear and effective approach for modelling the economic behaviour of both regions. The alignment between wavelet-based analysis and statistical measures reinforces the consistency and reliability of the findings.

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