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A Study of Graph Theoretic Properties on Clean Graph of Rings and their Complements for Some Rings of Prime Orders

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ABSTRACT

Let R be a ring of order ' p' such that ' p' is a prime number, and p ⩾ 3. A clean element x of R is an element that can be written as a sum of an idempotent and a unit of R. A ring R is clean if all its elements are clean. A clean graph of a ring Cl(R) is graph whose vertices are of the form (e, u) where e is an idempotent and u is a unit in R and two vertices (e, u) and (f, v) are connected if and only if either ef = 0 or uv = vu = 1 .In this paper, we constructed clean graph of rings for rings of prime order. We also study some graph theoretic properties of Cl(R) for commutative unitary rings of prime orders. The results gave more insight on the relation connection between ring theoretic and graph theoretic properties of clean graph of rings

Keywords; Idempotent, Unit, Clean element, Clean graph

1. INTRODUCTION

A graph is conceptually speaking a collection of points or nodes or vertices and lines connecting those vertices under certain condition(s). An algebraic graph is a graph constructed to visualise an algebraic structure where the elements of the structure serve as vertices of the graph and a particular property of the structure will serve as an edge connecting the concerned vertices (elemnts) see [7] , [14]

The study of algebraic structures using properties of graph has become an interesting research topic over the past few decades. There are many papers which study the graphs associated with rings(Pongthana 2022) Many graphs have been constructucted and studied about about different algebraic structures such as semi- groups, groups rings and fields, the essence of these graphs is to visualise an algebraic structure and study its properties. Some of the most popular graphs among them are Cayley graph of a group, identity graph of groups, commuting graph of groups, power graph of group, subgroup graph of groups, zero divisor graph of a rings, prime graph of a rings, graph of units of a ring, ideal graph of rings and clean graph of rings. see [1, 2, 4, 8, 10, 9, 5, 6, 13, 3, 11, 12] and [15]

The concept of zero divisor graph and that of graph theoretic properties of commutative rings was first introduced by Beck (), where all elements of the ring serve as vertices of the graph. Also, Anderson and Vingston introduced zerodivisor graph of commutative rings where nonzero zero divisors of a ring serve as vertices of the graph and two vertices a and b are adjacent iff ab = 0.After that, the attention of researchers was then drown to the area of study, and different studies have been conducted with respect to graphs associated to rings. This research will focus on one of these graphs namely clean graph of a ring and study some of its graph theoretic properties. Many researches have recenly been conducted with respect to the clean graph of a ring. Habib et al [10] introduced the graph, he constructed and study subgraphs of clean graphs for only some rings of even order (Cl2(R)) and determined its clique number , Matching number, independent number and domination number, he also study connectedness of this subgraphs but does not consider the general structure of the graph which gives a room to further determine these properties and even more uninvestigated properties of the structure of other rings with different order such as rings of odd or prime order Cl(R).

A ring R is also an algebraic structure but with two binary operations i.e (R, +, ·) where R is a nonempty set and the two binary operations are namely addition and multiplication respectively such that

(R,+) is an abelian group

(R, .) is a semigroup

Left and right distributivity holds over R.

Throughout this paper, R will be a associative unital ring. The ring of integers and the ring of integers modulo n will be respectively denoted by Z and Zn. An element a ∈ R is said to be a unit u in R if ∃x ∈ R such that a · x = 1, where 1 is called the unity of R, an idempotent (e) in R if a2 = a. and a clean if a = e + u. here, we adopt the convention in some context and represent a clean element a of R in the form a = (e, u). A ring is commutative/ abelian if ∀x, y ∈ R, x.y = y.x

As idempotent elements and units play a major role in the clean graph, we are interested in the number of idempotent elements in Zpm (Pongthana, 2022)

A graph of a clean ring also called clean graph of a ring is a graph whose vertices are clean elements of the ring

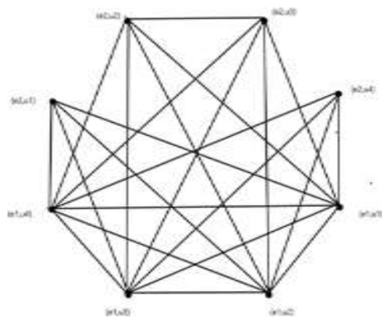

Figure 1: Clean graph of the ring Z5

R usually in the form (e, u) where e and u are respectively an idempotent and a unit of R, two vertices (e, u) and (f, v) are connected if and only if either ef = 0 or uv = vu = 1

2. METHODOLOGY

This research is not variable based research; propositions, lemmas and theorems from the reviewed literature are going to be used to obtain the results with help of graphical structure analysis. In the process of obtaining results, this research will make a systematic analysis of the structure of clean graphs of the commutative rings Zpn based on degrees of vertices, distances between a pair of vertices in the graph and eccentricity of the vertices. Partition of the vertex set and edge set will be done based on the use of theorems, propositions and lemmas from the reviewed literature in order to achieve our results or the stated objectives the results will provide quantitative measures that will reflect spatial distribution of vertices and their connectivity withing these graphs. Below is a list of graph structural properties and the topological indices to be investigated and obtained from the graph.

Proporsition 2.1. Let R be a finite commutative ring with unity of order n with n a prime number, let cl(R) be clean graph of R, then the degree of the vertices of cl(R) partitioned V (cl(R)) in to three subsets namely S_1, S_2 and S_3 where

$$S_1 = \{(0,x)|x \in U(R)\} = \{(0,x_1),(0.x_2), \quad .., \quad (0,x_{p-1})\} \Rightarrow |S_1| = p-1$$

$$S_2 = \{(1,x)|x \in U''(R)\} = \{\,(1.x_2),(1,x_2)\,..,(0,x_{p-3})\} \Rightarrow |S_2| = p-3$$

$$S_3 = \{(1,x)|x \in U(R)\} = \{(1,x_1),(1.x_2), \quad .., \quad (1,x_{|id(R)|})\} \Rightarrow |S_1| = |id(R)|$$

Proporsition 2.2. Let R be a finite commutative ring with unity of order n with n a prime number, let Cl(R) be clean graph of R, then the degree of each vertex vi ∈ Si is given by;

$$\deg(v) = \begin{cases} (2\text{p} - 3) \text{ if} \text{v} \in S_1 \\ (\text{p}-1) \text{ if v} \in S_1 \\ \text{p} \quad\quad \text{if v} \in S_1 \end{cases}$$

Proof. Let R be a finite commutative ring with unity of order n with n a prime number, let cl(R) be clean graph of R,

Since order of R is prime, by lemma 3 |id(R)|=2 and there are (p-1) units in R (because all nonzero elements of R are units in R).Hence by lemma 2 and defination of cl(R), the order of V (cl(R)) follows.

Based on the information above, observe that by defination of cl(R), the element 0 ∈ id(R) will dorm p − 1 vertices with all elements of U (R) so also the element 1 ∈ id(R) will form same number of vertices with elements of U (R), this will make a sum of 2(p − 1) = 2p − 2 vertices which is exactly |V (cl(R))|

Now, each of the first p − 1 vertices of the form (0, x) (i.e elements of S1)will be adjacent to all of the vertices in V (Cl(R)) except itself because 0.e = 0∀e ∈ id(R) making its degree to be 2p − 2 − 1 = 2p − 3 and that is the degree for each and every one of the vertices of S1.

Secondly, observe that |S2| is given by proposition 2.1 and each of the vertices in S2 is adjacent to all of the vertices in S1 and is also adjacent to one more vertex (1, xi) of S2 such that xi is inverse of x in U ′ ” (R) thus

making deg((1, x)) in S2 to be |S1| + 1 = (p − 1) + 1 = p and that is the degree of each vertex in S2.

Lastly, the vertices in S3 are of the form (1, x)|x ∈ U ′ (R). This implies that the unit part of each vertex inS3 is self-invertible in R and therefore each of them will only be adjacent to the vertices of S1. Thus, deg(v)|v ∈ S3 = |S1| = p − 1.

The proof is complete.

Remark 2.1.i From proposition 4.4 and proposition 2.5, it is observed that the order of the underlined ring is equal to the degree of a vertex in S2 that is

| $$|R| = \deg(v|v \in S_2)$$

ii We use the following for degree of vertex in S_1,S_2 and S_3

$$\deg(v|v \in S_1) = d_1$$

$$\deg(v|v \in S_1) = d_2$$

$$\deg(v|v \in S_1) = d_3$$

iii $$|S_3| = (p-1) = d_3$$

For order of S_1,S_2,S_3 and values of d_1,d_2 and d_3 see proposition 2.1 and 2.2

Lemma 2.1. Let $Cl(R)$ be clean graph of a commutative ring with unity that has prime order, then

i sum of degrees of two adjacent vertices in S_1 is $SD_1 = 2d_1$

ii sum of degrees of two adjacent vertices between S_1 and S_2 is $SD_{12} = 3SD_{1,2} = 3d_3$

iii sum of degrees of two adjacent vertices between S_1 and S_3 is $SD_{13} = 3d_2 - 4$

iv sum of degrees of two adjacent vertices in S_2 is $SD_2 = 2d_2$

Proof.i Since from proposition 2.5 all vertices in S_1 have equal degrees, we have from thesame proposition the sum of degrees of a pair of adjacent vertices in S_1 as

$$SD_1 = (2p-3) + (2p-3) = 2(2p-3)$$

and by (ii) of remark 1 we have

$$SD_1 = 2d_1$$

ii From proposition 2.5, the degree of a vertex in $_1$ is $2p-3$ and degree of a vertex in S_2 is p. Therefore the sum of degrees of a pair of adjacent vertices between S_1 and S_2 is

$$SD_{1,2} = (2p-3) + p = 3p-3 = 3(p-1)$$

and by (ii) of remark 1 we have

$$SD_{1,2} = 3d_3$$

iii From proposition 2.5, the degree of a vertex in S_1 and S_3 are $(2p-3)$ and $(p-1)$ respectively. therefore

$$SD_{1,3} = (2p-3) + (p-1) = 3p-4$$

and by (ii) of remark 1 we have

$$SD_{1,3} = 3d_2 - 4$$

iv From proposition 2.5, the degree of a vertex S_2 is p. therefore

$$SD_2 = p + p = 2p$$

and by (ii) of remark 1 we have

$$SD_2 = 2d_2$$

The proof is complete

Lemma 2.2. Let Cl(R) be clean graph of a commutative ring with unity that has prime order, then Product of degrees of two adjacent vertices in S_1 is $PD_1 = d^2$

i Product of degrees of two adjacent vertices between S_1 and S_2 is $PD_{1,2} = d_1d_2$

ii Product of degrees of two adjacent vertices between S_1 and S_3 is $PD_{1,3} = d_1d_3$

iii Product of degrees of two adjacent vertices in S_2 is $PD_2 = d^2$

Proof. i From proposition 2.5, the product of degrees of a pair of adjacent vertices in S_1 is

$$PD_1 = (2p-3)(2p-3) = (2p-3)^2$$

and by (ii) of remark 1 we have

$$PD_1 = {}_1 2d_2$$

ii From proposition 2.5, the product of degrees of a pair of adjacent vertices between S_1 and S_2 is

$$PD_{1,2} = (2p-3)p$$

and by (ii) of remark 1 we have

$$PD_{1,2} = d_1d_2$$

iii From proposition 2.5, the product of degrees of a pair of adjacent vertices between S_1 and S_3 is

$$PD_{1,3} = (2p-3)(p-1)$$

and by (ii) of remark 1 we have

$$PD_{1,3} = d_1d_3$$

iv From proposition 2.5, the product of degrees of a pair of adjacent vertices in S_2 is is

$$PD_2 = p \times p = p^2$$

And by (II) of remark 1 , we have

$PD_2 = d^2$

The proof is complete

Lemma 2.3. Let Cl(R) be clean graph of a commutative ring with unity that has prime order, then

i Number of edges between vertices of S_1 is $N_1 = \frac{d_3(d_2-2)}{2}$

ii Number of edges between vertices of S_1 and vertices of S_2 is $N_{1,2} = d_3(d_2-3)$

iii Number of edges between vertices of S_1 and vertices of S_3 is $N_{1,3} = 2d_3$

iv Number of edges between vertices of S_2 only is $N_2 = \frac{d_2-3}{2}$

Proof. Let Cl(R) be clean graph of a commutative ring with unity that has prime order, then

i From proposition 2.5, we have the number of edges between vertices of S_1 is given by the size of S_1 that is

$$N_1 = \frac{(p-1)(p-2)}{2}$$

From proposition 2.4 and 2.5, the number of edges between vertices of S_1 and vertices of S_2 is given by

$N_{1,2} = (p-1)(p-3)$

because all the $(p-1)$ vertices of S_1 are adjacent to all the $(p-3)$ vertices of S_2. Therefore by (ii) of remark 1, we have

$N_{1,3} = d_3(d_2-2)$

ii From proposition 2.4 and 2.5, the number of edges between vertices of S_1 and vertices of S_3 is given by

$N_{1,3} = (p-1)2$

because all the $(p-1)$ vertices of S_1 are adjacent to all the 2 or |id(R)| vertices of S_3. Therefore by (ii) of remark 1, we have

$N_{1,3} = 2d_3$

iii From proposition 2.4 and 2.5, the number of edges between vertices of S_2 is given by $N_2 = \frac{p-3}{2}$

because only half of vertices of S_2 are adjacent. Therefore by (ii) of remark 1, we have

$$N_1 = \frac{p-3}{2}$$

Lemma 2.4. Let Cl(R) be clean graph of a commutative ring with unity that has prime order, then

i The number of pairs of vertices at distance 1 is $NP_1 = \frac{1}{2}[d_3(3d_2-4) + (d_2-3)]$

ii The number of pairs of vertices at distance 2 is $NP_2 = \frac{5d_2-13}{2}$

Remark 2.2. Number of vertices at distance 1 is equal to the size of the clean graph of R. That is

$NP_1 = |Cl(R)|$

Proporsition 2.3. Let R be a finite commutative ring with unity of order n with n a prime number, let Cl(R)

be clean graph of R, then the size of Cl(R) is

$\left|E\big(Cl(R)\big)\right| = \frac{1}{2}(3p^2 - 6p + 1)$ (1)

Proof. Let R be a finite commutative ring with unity of order n with n a prime number, let Cl(R) be clean graph of R, since proposition 2.5 tells us that the degrees of vertices of Cl(R) partitioned V (Cl(R)) in to S_1, S_2 and S_3 where all vertices of S_i, i ∈ {1, 2, 3} have equal degree, we have the total number of vertices of Cl(R) given by

$|S_1| + |S_2| + |S_3|$

and the total degrees of vertices of Cl(R) given by

$$\sum d\left(v\middle|v \in V\big(Cl(R)\big)\right) = |S_1| \times d(v|v \in S_1) + |S_2| \times d(v|v \in S_2) + |S_3| \times d(v|v \in S_3)$$

$$= \sum_{p-1}(2p-3) + \sum_{p-3} p + \sum_{|id(R)|}(p-1)$$

$$= (p-1)(2p-3) + (p-3)p + 2(p-1)$$

$$= (p-1)[(2p-3)+2] + p(p-3)$$

$$= (p-1)(2p-3) + p(p-3)$$

$$= 3p^2 - 6p + 1$$

Now, using **lemma 1** we have

$$2\left|E\big(Cl(R)\big)\right| = 3p^2 - 6p + 1$$

$$\therefore \left|E\big(Cl(R)\big)\right| = \frac{1}{2}(3p^2 - 6p + 1)$$

The proof is complete $\bar{2}$

Proporsition 2.4. Let R be a finite commutative ring with unity of order n with n a prime

number, let Cl(R)

and Cl′(R) be clean graph of R and its complement respectively, then the size of cl′(R) is

$\left|E\big(Cl(R)\big)\right| = \frac{1}{2}(P^2 - 4p + 5)$ (2)

Proof. Let R be a finite commutative ring with unity of order n with n a prime number, let Cl(R) and Cl′(R) be clean graph of R and its complement respectively, then, since |V (cl(R))| = 2p − 2, by definition ...(of complete graph on n vertices K_n), each vertex has degree n − 1. in this case n = 2p − 2 which implies that in $K_{|V(Cl(R))|} = K_{2p-2}$, each vertex has degree 2p – 3

Now for Cl(R), consider vertices in the set S_1 where degree of each is 2p − 3 (they contain thier maximum degrees in the graph), by proposition 2.3 S_1 is a clique of cl(R) and will therefore be an independent set in Cl′(R).

Thus, in Cl ′(R) deg(v)|v ∈ S_1 is 0

Secondly, in $Cl(R),\ deg(u)|u \in S_2\ is\ p$ which implies that deg(u)|u ∈ S_2 will be

$(2p-3) - p = p - 3\ in\ Cl'(R)$

Lastly, the vertices in S_3 are 2 and deg(v)|v ∈ S_3 ⊂V(Cl(R)) is p − 1 in Cl(R) which implies that deg(v)|v ∈ S_3 ⊂V(Cl'(R))= (2p − 3) − (p − 1) = p − 2 in Cl′(R)

We therefore have the sum of the degrees of vertices of cl′(R) as follows;

E(Cl'(R))=|S_1|(deg(v)|v ∈ S_1 ⊂V(Cl(R)))+ |S_2|(deg(v)|v ∈ S_2 ⊂V(Cl(R))) + |S_3|(deg(v)|v ∈ S_3 ⊂V(Cl(R)))

$$\Rightarrow 2\left|E\big(Cl(R)\big)\right| = (p-1)0 + (p-3)(p-3) + 2(p-2)$$

$$= p^2 - 4p + 5$$

And using **Lemma 1,** we have

$$\left|E\big(Cl(R)\big)\right| = \frac{p^2 - 4p + 5}{2}$$

Or equivalently

$$\left|E\big(Cl'(R)\big)\right| = \left|E\left(K_{|V(Cl(R))|}\right)\right| - |E(Cl(R))|$$

Where $K_{|V(Cl(R))|}$ is the complete graph on $|V(Cl(R))|$ vertices

$$= \frac{(2p-2)(2p-3)}{2} - \frac{3p^2 - 6p + 1}{2}$$

After simplification, we have

$$|E(Cl'(R))| = \frac{p^2 - 4p + 5}{2}$$

The proof is complete

The subgraph $Cl_2(R)$)

$Cl_2(R)$ is a subgraph of Cl(R) induced by the set id^* (R) of nonzero idempotent of the ring R. Baesd on the nature of the order of the ring we are considering in this work, this subgraph has $p - 1$ vertices because the only nonzero idempoteent of this ring is the element 1. We therefore explore properties of this type of the induced subgraph.

$Cl_2(R)$ is a simple and disconnected graph with $\frac{p+1}{}$ components

Proporsition 2.5. Let R be a finite commutative ring with unity of order n with n a prime number, let Cl(R)

be clean graph of R, then the subgraph $Cl_2(R)$ of Cl(R) is a disconnected graph with $(p-1)$ components.

Proporsition 2.6. Let R be a finite commutative ring with unity of order n with n a prime number, let Cl(R)

be clean graph of R, then two vertices (e, u) and (f, v) are adjacent in $Cl_2(R)$ if and only if uv = vu = 1

Proporsition 2.7. Let R be a finite commutative ring with unity of order n with n a prime number, let $Cl_2(R)$ be be induced subgraph of the clean graph Cl(R), then the degrees of the vertices of $Cl_2(R)$ partitioned V (Cl(R)) into two subsets V_1 and V_2 where all the vertices of a given subset contain equal degrees and $|V_1| = 2$ and

$|V_2| = (p - 3)$. Furthermore, for any vertex v, $\deg(v)|v \in V_1 = 0$ and $\deg(v)|v \in V_2 = 1$.

Proof. Let R be a finite commutative ring with unity of order n with n a prime number, let $Cl_2(R)$ be be induced subgraph of the clean graph Cl(R), then, clearly $Cl_2(R)$ has $p - 1$ vertices because $|id^*| = 1$ and $|U(R)| = p - 1$. Secondly, since p is prime, $p - 3$ out of p vertices of Cl(R) will satisfy the condition in proposition 2.8 (i.e excluding vertices of the form $(1, u)|u \in U'(R)$)and using the fact that $0 \notin U(R)$, thus $V_2 = S_1$ and since

$|V_2|$ is even, by proposition 2.8 and lemma 1 $\frac{p-3}{2}$ pairs of adjacent vertices will be formed in $cl_2(R)$ hence for each vertex $v \in Cl_2(R)$, deg(v) = 1 and therefore $|V_2| = p-, 3$. Lastly these excluded vertices are (1, 1) and (1, p − 1) and are isolated in $Cl_2(R)$, hence d((1, 1)) = deg ((1, p − 1)) = 0. This forms the first partition of V i,e V_1 and it is obvious that

$|V_1| = |\{(1, 1), (1, p - 1)\}| = 2$

The proof is complete.

Proporsition 2.8. Let R be a commutative ring with inity If Cl(R) is a clean graph of R, then the size of

$Cl_2(R)$ is given by

$$E(Cl_2(R))=(p-3)/2 \qquad (3)$$

Proof. From proposition 2.4, elements of S_1 are exacly the vertices of $Cl_2(R)$ i.e $p - 1$ vertices, out of these $p - 1$ vertices, (1, 1) and $p - 1$ are self-inverses and by proposition 2.9, they are not adjacent to any other vertex in rhe graph, thus the remaining $p - 3$ will by proposition 2.2 form $\frac{p-3}{}$ pairs of adjacent vertices.

Hence the proposition

Proporsition 2.9. Let R be a commutative ring with inity If Cl(R) is a clean graph of R, then the size of the complement of $Cl_2(R)$ is given by

$$|E(Cl_2(R))| = \frac{p^2 - 4p + 5}{2}$$

Proof. Let R be a commutative ring with inity If Cl(R) is a clean graph of R,then Since $Cl_2(R)$ has $p - 1$ vertices, so by **Proposition 2.8** The size of complete graph on $p - 1$ vertices is given by

$$ECl2(R) = \frac{(p-1)(p-2)}{2}$$

Therefore, the size of the complement of $Cl_2(R)$ is given by

$$ECl_2(R) = |E(K_{|Cl_2(R)|})| - |E(Cl(R))| = \frac{(p-1)(p-2)}{2} - \frac{p-3}{2} = \frac{p^2-4p+5}{2}$$

3. RESULTS

The table below shows the results obtained from the above lemmas and proposition in this article.

Table 4.1 Order of the ring R, the graph Cl(R) and that of the partition of vertex set V

$\|R\|$	$\|Cl(R)\|$	$\|S_1\|$	$\|S_2\|$	$\|S_3\|$
p	$2(p-1)$	$p-1$	$p-3$	2

Table 4.2 Degree of vertices in each partition of $V(S_i)$

$\|S_i\|$	S_1	S_2	S_3
$d(v\|v \in S_i)$	$2p-3$	p	$p-1$

$\|S_i\|$	S_1	S_2	S_3	Between S_1 and S_2	Between S_1 and S_3
Number of edges	$\frac{(p-1)(p-2)}{2}$	$\frac{p-3}{2}$	0	$(p-1)(p-3)$	$2(p-1)$

Table 4.3 Number of edges within and between each partitions of V

4. CONCLUSION

This paper reports construction and graphical structure analysis of clean graphs of those rings having prime order It was found that the higher the order of a ring the more the number of edges and that is due to the increment in number of vertices in the graph. Thus, the research would help in analysing physical or chemical properties of drugs or chemical compounds that are having ring like structure

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