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On Finding Integer Solutions to Non-homogeneous Ternary Cubic Equation $(x-1)^2+(y-1)^2=8z^3$

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ABSTRACT :

The thrust of this paper is to determine plenty of non-zero distinct integer solutions to the non-homogeneous cubic equation with three unknowns given by $(x-1)^2+(y-1)^2=8z^3$. Substitution technique and factorization method are utilized to obtain the same.

Keywords: Non-homogeneous cubic, Ternary cubic, Integer solutions ,Substitution technique, Factorization method

Introduction

The fascinating branch of Mathematics is the Theory of Numbers in which the subject of Diophantine equations requiring only the integer solutions is an interesting area to various mathematicians and to the lovers of mathematics because it is a treasure house in which the search for many hidden connections is a treasure hunt. In other words, the theory of Diophantine equations is an ancient subject that typically involves solving, polynomial equation in two or more variables or a system of polynomial equations with the number of unknowns greater than the number of equations, in integers and occupies a pivotal role in the region of mathematics. It is worth to mention that the Diophantine problems are plenty playing a significant role in the development of mathematics because the beauty of Diophantine equations is that the number of equations is less than the number of unknowns . For the sake of clear understanding by the readers, one may refer the varieties of cubic Diophantine equations with multi variables [1-18]. This paper aims at determining many integer solutions to non-homogeneous polynomial equation of degree three with three unknowns given by $(x-1)^2+(y-1)^2=8z^3$.

Method of analysis

The non-homogeneous third degree equation with three unknowns to be solved for its integer solutions is

$$(x-1)^2+(y-1)^2=8z^3 \qquad (1)$$

The procedure to obtain many non-zero distinct integer solutions to (1) is illustrated below:

Procedure 1

The substitution

$$x=X+1,\, y=Y+1,\, 2z=P \qquad (2)$$

in (1) leads to the non-homogeneous ternary cubic equation given by

$$X^2+Y^2=P^3 \qquad (3)$$

By scrutiny , it is noted that (3) is satisfied by

$$X=m(m^2+n^2)\,,\, Y=n(m^2+n^2)\,,\, P=(m^2+n^2) \qquad (4)$$

From (2) , it is seen that

$$x=X+1=m(m^2+n^2)+1,$$
$$y=Y+1=n(m^2+n^2)+1,$$
$$z=\frac{P}{2}=\frac{(m^2+n^2)}{2}.$$

As integer solutions are required , it is observed that the above values of x, y, z are integers when the values of m, n belong to the same parity. That is , the values of m, n are both even or both odd. For the benefit of the readers , the corresponding integer solutions to (1) are exhibited as below:

Set 1

$$m = 2M, n = 2N$$
$$x = 8M(M^2 + N^2) + 1$$
$$y = 8N(M^2 + N^2) + 1$$
$$z = 2(M^2 + N^2)$$

Set 2

$$m = 2M + 1, n = 2N + 1$$
$$x = (2M + 1)(4M^2 + 4N^2 + 4M + 4N + 2) + 1$$
$$y = (2N + 1)(4M^2 + 4N^2 + 4M + 4N + 2) + 1$$
$$z = (2M^2 + 2N^2 + 2M + 2N + 1)$$

Note 1

In addition to (4) , there is an another solution to (3) given by

$$X = m(m^2 - 3n^2), Y = n(3m^2 - n^2), P = (m^2 + n^2)$$

From (2) , it is seen that

$$x = X + 1 = m(m^2 - 3n^2) + 1,$$
$$y = Y + 1 = n(3m^2 - n^2) + 1,$$
$$z = \frac{P}{2} = \frac{(m^2 + n^2)}{2}.$$

Following the above analysis ,the corresponding two sets of integer solutions to(1) are given below:

Set 3

$$m = 2M, n = 2N$$
$$x = 2M(4M^2 - 12N^2) + 1$$
$$y = 2N(12M^2 - 4N^2) + 1$$
$$z = 2(M^2 + N^2)$$

Set 4

$$m = 2M + 1, n = 2N + 1$$
$$x = (2M + 1)(4M^2 - 12N^2 + 4M - 12N - 2) + 1$$
$$y = (2N + 1)(12M^2 - 4N^2 + 12M - 4N + 2) + 1$$
$$z = (2M^2 + 2N^2 + 2M + 2N + 1)$$

Procedure 2

The option

$$x = 2X + 1, y = 2Y + 1 \quad (5)$$ in (1) gives

$$X^2 + Y^2 = 2z^3 \quad (6)$$

Assume

$$z = a^2 + b^2 \quad (7)$$

Write the integer 2 in (6) as

$$2 = (1 + i)(1 - i) \quad (8)$$

Substituting (7) & (8) in (6) and applying factorization ,consider

$$\begin{aligned} X + iY &= (1 + i)(a + ib)^3 \\ &= (1 + i)[f(a, b) + ig(a, b)] \end{aligned} \quad (9)$$

Where

$$f(a, b) = a^3 - 3ab^2$$
$$g(a, b) = 3a^2b - b^3$$

Equating the real and imaginary parts in (9),we have

$$X = f(a,b) - g(a,b)$$
$$Y = f(a,b) + g(a,b)$$

From (5),one obtains

$$x = 2[f(a,b) - g(a,b)] + 1,$$
$$y = 2[f(a,b) + g(a,b)] + 1. \quad (10)$$

Thus,(1) is satisfied by (7) & (10).

Note 2

It is to be noted that the integer 2 in (6) may also be represented as

$$2 = \frac{(7+i)(7-i)}{25}$$
$$2 = \frac{(1+7i)(1-7i)}{25}$$

Utilizing the above process , two more sets of integer solutions to (1) are obtained.

Procedure 3

Taking

$$Y = kz, k > 0 \quad (11)$$

in (6), we get

$$X^2 = z^2(2z - k^2) \quad (12)$$

The expression $(2z - k^2)$ in (12) is a perfect square when

$$z = (2s^2 - 2s + 1)k^2 \quad (13)$$

From (12) , observe that

$$X = (2s-1)(2s^2 - 2s + 1)k^3 \quad (14)$$

Using (13) in (11) ,we have

$$Y = (2s^2 - 2s + 1)k^3 \quad (15)$$

Employing (14) & (15) in (5) ,one has

$$x = 2(2s-1)(2s^2 - 2s + 1)k^3 + 1,$$
$$y = 2(2s^2 - 2s + 1)k^3 + 1. \quad (16)$$

Thus , (1) is satisfied by (13) & (16) .

Procedure 4

We present below an alternate way of solving (12).

Let

$$\alpha^2 = 2z - k^2 \quad (17)$$

The smallest positive integer solution to (17) is

$$z = z_0 = k^2, \alpha_0 = k$$

Assume the second solution to (17) as

$$z_1 = h + z_0, \alpha_1 = h - \alpha_0 \quad (18)$$

where h is an unknown to be determined. Using (18) in (17) and simplifying, we get

$$h = 2\alpha_0 + 2$$

and from (18) ,one has

$$z_1 = z_0 + 2\alpha_0 + 2,$$
$$\alpha_1 = \alpha_0 + 2.$$

The repetition of the above process leads to the general solution (z_n, α_n) to (17) as

$$\alpha_n = \alpha_0 + 2n = k + 2n$$
$$z_n = z_0 + 2\alpha_0 n + 2n^2 = k^2 + 2kn + 2n^2 \quad (19)$$

From (12) ,observe that

$$X_n = z_n * \alpha_n = (k^2 + 2kn + 2n^2)(k + 2n)$$

From (11) ,we have

$$Y_n = k z_n = k(k^2 + 2kn + 2n^2)$$

In view of (5) , one has

(20)

$$x_n = 2X_n + 1 = 2(k^2 + 2kn + 2n^2)(k + 2n) + 1$$
$$y_n = 2Y_n + 1 = 2k(k^2 + 2kn + 2n^2) + 1$$

Thus , (1) is satisfied by (20) jointly with z_n given by (19) .

Conclusion

In this paper, we have made an attempt to obtain many integer solutions to the non-homogeneous cubic equation with three unknowns given by $(x-1)^2 + (y-1)^2 = 8z^3$. As cubic equations are plenty ,one may search for integer solutions to other choices of homogeneous or non-homogeneous third degree Diophantine equations with multivariables.

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