International Journal of Research Publication and Reviews, Vol 6, Issue 6, pp 8544-8551 June 2025

International Journal of Research Publication and Reviews

Journal homepage: www.ijrpr.com ISSN 2582-7421

A Theoretical Viewpoint on How Quantitative Reasoning Helps Precalculus Students Understand the Fundamentals of Trigonometry

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ABSTRACT:

Previous studies have shown that the sine and cosine functions are hard for both teachers and students to grasp and use. Additionally, they have disjointed perceptions of the different contexts of trigonometry (e.g., unit circle and right triangle) and poor comprehension of concepts that are fundamental to learning trigonometry (e.g., angle measure and the unit circle). It is believed that the concept of angle measure—specifically, the radius as a unit of measurement for an angle—was fundamental to understanding and using the sine and cosine functions. In order to simulate the periodic behavior between the sine and cosine functions, students had to consider how an angle measure and a changing distance change together. Understanding and applying the sine and cosine functions also required a process view of function.

Keywords: Precalculus, Reasoning, Students, Trigonometry.

INTRODUCTION

For the past century, trigonometry and trigonometric functions have played a significant role in the mathematics curricula of both high schools and universities. Trigonometric functions are used in many scientific fields (such as projectile velocity and wave behavior modeling) as well as many mathematical fields (such as Fourier series and integration techniques).

One of the earliest mathematical experiences that combines geometric, symbolic, and graphical thinking about functions that cannot be determined using algebraic computations is provided by trigonometry and trigonometric functions. Despite the fact that trigonometry has been a staple of science and math curricula for more than a century, it frequently happens that teachers and students struggle to reason about subjects that rely on trigonometric functions (Brown, 2005; Fi, 2003; Thompson, Carlson, & Silverman, 2007; Weber, 2005). Few researches have examined the thinking skills required to comprehend and use trigonometric functions, which further complicates the matter.

The method used by instructional materials to introduce the sine and cosine functions may be connected to students' challenges with trigonometric functions. Trigonometric functions are covered in a variety of scenarios in many popular American textbooks (such as unit circle trigonometry and triangle trigonometry). Textbooks outline several uses for trigonometric functions in each of these situations (e.g., determining the side of a triangle or finding a position). Additionally, these situations are usually treated as unrelated or just marginally related in curriculum. For example, right triangle trigonometry is frequently utilized as a springboard for unit circle trigonometry; yet, the curriculum does not seem to take advantage of the shared underpinnings of both contexts (such as angle measure) to foster coherence between them. This approach to trigonometry may prevent pupils from gaining knowledge that has deep linkages between the different contexts of trigonometry. In order to support students in developing coherent and adaptable understandings of trigonometric functions, Thompson (2008) recently advocated that curricula should be created so that they build upon meanings and foundations that are shared across the many settings. The idea of expanding on shared meanings and underpinnings among the many trigonometries is a primary focus of this paper.

BACKGROUND FOR THE STUDY

There is a dearth of research on students' thinking in trigonometry; Markel (1982) characterizes trigonometry as "forgotten and abused" since it receives little consideration in mathematics education research and instruction. Although the number of students enrolled in trigonometry courses has increased consistently over the past century, the relatively small percentage of the mathematics curriculum that trigonometry occupies may be the reason for the lack of attention paid to students' views of the subject (Brown, 2005). The available study focuses on how students and teachers comprehend trigonometry.

STATEMENT OF THE PROBLEM

Despite the ongoing challenges that students face, trigonometry has received little attention from math teachers and curriculum designers (Brown, 2005; Weber, 2005). Researchers who have studied trigonometry frequently agree that more attention should be paid to fostering coherence among the various contexts of trigonometry and creating foundational understandings (Brown, 2005; Thompson, 2008; Thompson, et al., 2007; Weber, 2005).

Students probably experience a variety of challenges when trying to build cohesive trigonometric understandings. First, advanced reasoning in relation to the function notion is necessary for trigonometric functions. One of a student's first encounters with functions that cannot be computed is frequently with trigonometric functions. Trigonometric function reasoning is based on the ability to predict input values for a function being evaluated and output values being generated without the need for numerical calculations. According to several studies (M. Carlson, 1998; M. Carlson & Oehrtman, 2004; Harel & Dubinsky, 1992; Oehrtman, Carlson, & Thompson, 2008; Sierpinska, 1992; Thompson, 1994b), students find it challenging to conceptualize function as a process. Although trigonometric functions provide the chance to foster and advance students' reasoning in this manner, it doesn't seem that this fundamental mode of reasoning is addressed or developed in traditional mathematics curriculum.

Students may also find the sine and cosine functions challenging because of their insufficient comprehension of the concepts that underlie these functions. Students lack the skills required to develop meaningful and cohesive understandings of the sine and cosine functions as a result of these incomplete understandings. According to Moore (2009), for example, students must develop concepts of angle measure and the radius as a unit of measurement in a way that supports related trigonometric understandings. Students must also be able to use the geometric objects of trigonometry, such as the unit circle and right triangles, to bolster their arguments regarding the links that the sine and cosine functions codify, as suggested by Weber (2005).

THEORETICAL PERSPECTIVE

The radical constructivism tenet that all learning starts and ends with the learner forms the basis of the study (Glasersfeld, 1995). This method of teaching sees the classroom as an exploratory space where everyone asks questions, makes hypotheses, finds answers, and develops understandings. Each student's knowledge is regarded as essentially unknown to everybody else. Individual experiences are the source of knowledge, or knowing; in a class of thirty pupils, thirty distinct sets of experiences will take place. Knowing is attained by a person's reflection on their experiences, yet this knowledge is about nothing, and there is no direct correlation between knowledge and its subject. Knowledge is the culmination of a person's processes of modifying their mental schema, or way of knowing, in reaction to a cognitive disturbance or imbalance. Accommodation is the process by which a person responds to a disturbance by rearranging and creating cognitive structures and the connections between them (Glasersfeld, 1995).

It could seem that giving the learner sole responsibility for their education puts a barrier in the way of instruction since it suggests that the teacher's job is not crucial. In contrast to this view, this position may be seen as suggesting that teaching does not entail the direct transfer of ideas. A teacher can impact the classroom by serving as a catalyst for learning. It is their responsibility to establish circumstances that allow students to learn by applying and reflecting on repeated reasoning.

In order to give different formalisms (such as the notation of $\sin(\theta)$), a teacher must frequently lecture in the classroom. However, the student is best prepared for lecturing when they have the mental structures necessary to absorb the material that the instructor is describing. The foundations and cognitive structures needed for the pupils to be able to analyze, interpret, and create meaning from the teacher's words and deeds must be in place.

One of the most significant components of learning and perhaps the most crucial component of knowledge construction is reflection (Piaget, 2001). Reflection relates learning to the mind's capacity to "stand still" and try to make sense of an experience, in contrast to empiricists who reject the mind and its workings and reduce all knowing to the reception of "sense data." According to Ernst von Glasersfeld (1995), reflection is the enigmatic skill that enables us to exit the stream of direct experience, re-present a portion of it, and view it as though it were direct experience while still being conscious that it is not.

Apart from the concept of reflection, Piaget, Glasersfeld, and other scholars have recognized the idea of abstraction, which is enabled by the separation, comparison, and linking of experiences. According to Ernst von Glasersfeld (1995), John Locke described abstraction as follows:

Abstraction is the process by which concepts derived from specific entities are transformed into universal representations of the same kind, and their names are general terms that apply to anything that complies with these abstract concepts.

Cognitive structures are reorganized and constructed through the mental processes of abstraction and reflection. It is crucial to stress once more that these processes are entirely dependent on the individual and invisible to outside observers. Additionally, keep in mind that a student's experiences are based on their current schema, or model of knowing, which determines the experience.

The need for abstraction and reflection suggests that the goal of education is to give people the chance to engage in (mental) activities and think back on them so that mathematical meanings and structures can be developed.

Teachers must set up scenarios where students encounter disruptions in their reasoning—what onlookers could refer to as flawed or underdeveloped reasoning—and learn to recognize and overcome these conflicts. It is challenging and unnatural for a person to reflect on their own thoughts in order to become aware of and confront these conflicts.

Function, Covariational Reasoning, and Quantitative Reasoning

One of a student's first mathematics encounters is usually with trigonometric functions, which demand the student to reason about a relationship between the (changing) values of two quantities that is difficult to compute by hand.

In order to reason about these linkages, it is necessary to establish an understanding of trigonometric functions as processes. A process notion of function entails a dynamic transformation of quantities via some repeatable means that, given the same original quantity, will always generate the same transformed amount, as explained by Dubinsky and Harel (1992). The transformation might be viewed by the subject as a whole process that starts with some sort of object, involves altering these objects, and ends with new objects.

A learner who has a process conception of function views an expression or function as representing a relationship or mapping that is "self-evaluating," rather than as a call to evaluate. Put differently, a learner who has a process conception may predict how input values will be evaluated and what output values will be produced without actually doing the calculations. This is the exact logic required to intellectually comprehend trigonometric functions since they cannot be efficiently evaluated computationally without the use of a calculator or data that have been committed to memory.

The fact that a student can only reason by carrying out particular tasks, like math, runs counter to a process notion of function. It becomes extremely difficult for students to reason about trigonometric functions if this is their dominant function notion. Trigonometric functions can be thought of as accepting an input and creating an output, even though a learner can memorize a portion of their input-output values. An action-concept of function would include, for instance, the capacity to enter numbers into an algebraic expression and do calculations, claim Harel and Dubinsky (1992). Because the subject will typically consider it step-by-step (e.g., one evaluation of an expression), it is a static conception.

When thinking about functions, a student who has an action conception of function concentrates on steps and computations. their kids are quick to make calculations without taking into account the contextual significance of their calculations, instead of thinking critically about the quantities of a scenario. Regardless of the algorithm or representation, a learner who has an action conception of function cannot see beyond particular calculations to see a function as taking inputs and returning outputs. Students that have an action conception are unable to reason dynamically about the link between two quantities, such as by visualizing a changing length and a changing angle measure (M. Carlson, 1998; M. Carlson et al., 2002; Oehrtman et al., 2008).

A student's capacity to plan input-output intervals and reason dynamically about an input-output connection is supported by a self-evaluating, or process, perspective of function. Covariational reasoning is defined as "the cognitive activities [of an individual] involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other" (M. Carlson, et al., 2002, p. 354). It has been discovered that comprehension of the function concept depends on the mental processes involved in covarying quantities (M. Carlson et al., 2002; M. Carlson & Oehrtman, 2004; Oehrtman et al., 2008).

Saldanha and Thompson (1998) contended that representations of covariation are developmental in addition to Thompson's (1994b) explanation of covariation. To put it another way, a student first coordinates the values of two quantities (for example, consider an angle measurement, then a length, and so on). Then, when a student's perception of the covariational relationship grows, their comprehension of covariation starts to entail visualizing continuously changing quantities at the same time (for example, when the measure of an angle varies, one realizes that a length changes at the same time). When two quantities change simultaneously, continuous covariation suggests an image that contains a knowledge that all intermediate values of the quantities are achieved. Moreover, continuous covariation suggests that any input interval can be broken down into smaller intervals where the quantities fluctuate continuously. Although the student does not actually consider all intermediate values, they understand that all of the intermediate values occur independently of calculating these values. This ability to envision continuously changing quantities also resembles a process conception of function in that it allows the student to envision or anticipate simultaneous changes without having to determine the changes in one quantity and then the changes in the other quantity.

Carlson et al. (2002) conducted a study that provided more information about the intricacy of students' mental processes during covariational reasoning. The authors first noted a variety of responses among undergraduate students as they tried to comprehend and depict dynamic function scenarios. A framework comprising five mental acts and five stages of covariational reasoning was created in order to categorize the various behaviors displayed. The five mental processes correspond to particular behaviors. However, it was discovered that the mental processes by themselves were not enough to categorize the group of behaviors a student displayed. Students, for example, performed higher mental acts but were unable to deconstruct them into lower mental actions. Therefore, the framework was expanded to incorporate five degrees of covariational reasoning that correspond to the five mental activities in order to characterize a student's capacity for covariational thinking in relation to a scenario or issue. When a student can reason utilizing all mental actions associated with lower levels in addition to the mental action associated with that level, it is said that they are covariating at that level.

Carlson et al. (2002) placed the problems that were given to the students in their study within a specific context. As a result, the students were required to use covariational reasoning when constructing quantities from a hypothetical scenario. The students in the study created mental images of the problem circumstances, even though this was not the main focus of Carlson et al.'s analysis. It's possible that the quantities that made up these scenarios affected the students' capacity for covariational reasoning.

Situation-sensitive reasoning that emphasizes students creating conceptual objects (quantities) that may be reasoned about is known as quantitative reasoning (Smith III & Thompson, 2008; Thompson, 1989). The focus of quantitative reasoning is on how students reason about relationships between quantities, create an image of the quantities in a situation, and make sense of it. A foundation of quantities and their relationships forms a mental structure upon which the student can reflect, leading to the development of mathematical reasoning and conceptual understanding.

Although Thompson (1989) contended that students can develop mathematical understandings through quantitative reasoning, he cautioned that this kind of reasoning is not always common or natural in mathematics curricula and instruction. For example, algebraic tools are frequently taught as a method for handling challenging mathematical issues, where they are employed to both simplify and resolve the issue. However, this method may cause the problem-solving process to become divorced from the problem's true context, which includes the linkages and intuitive characteristics of the issue.

While some people may find this to be a natural procedure, others who consider certain mathematical and numerical manipulation requirements to be "magical" and situationally irrelevant may find this to be neither suitable nor natural. Furthermore, reasoning about the context of a problem becomes essential to developing profound and related understandings in situations where algebraic manipulations are difficult or impossible, such as in the context of trigonometry and the usage of trigonometric functions.

A few working definitions of quantitative reasoning are provided and illustrated using a variety of trigonometric subjects in order to further clarify the use of quantitative reasoning in reasoning about trigonometric functions and angle measure. According to Thompson (Thompson, 1989), the definitions offered are meant to be components of a system made up of algebraic and quantitative reasoning concepts. It is crucial to remember that this working definition system makes no claims to accurately reflect how individuals think numerically. Instead, it is a model that describes the conceptual frameworks and mental operations that make quantitative reasoning possible.

Analyzing a problem into a quantitative framework is known as quantitative reasoning. The network of numbers and quantitative connections created that serves as a basis for thought and reasoning is known as a quantitative structure. Quantitative thinking, determining the proper operations (inferred from relationships among quantities) to compute quantities' values, and propagating calculations are the components of quantity-based arithmetic. Furthermore, it is now possible to characterize quantity-based algebra as being identical to quantity-based arithmetic, with the exception that formulas are propagated rather than values being propagated, some values are represented symbolically, and representations of situations are under-constrained in terms of quantities' values (i.e., there is insufficient numerical information to propagate calculations).

All things considered, quantitative reasoning emphasizes how crucial it is for students to conceptualize scenarios and quantifiable aspects of those scenarios as the basis for mathematical reasoning. Even while every student in a classroom forms their own understandings and images, this does not mean that students cannot form understandings and images that align with the learning objectives. Since research has shown that students frequently struggle to generate circumstances consistent with the aim of a problem, instruction must take into account this first generation of situations and quantities that the students are to reason about (Moore et al., 2009).

Formulas and functions can also emerge in a meaningful way when students are given the chance to create a scenario in which they can grasp quantities and their relationships. Formulas and representations of functions can arise as a reflection and generalization of these relationships by first encouraging the cognitive growth of quantities and their interactions so that these images encompass how the quantities vary (Moore et al., 2009). This contrasts with the method of creating a function or formula and then trying to give it interpretations and meanings.

Quantitative Reasoning in Problem Solving

Quantitative reasoning refers to a student identifying and conceptualizing quantities that compose a situation, which is an action central to contextual problem solving. When engaged in a novel contextual problem, the mental processes of creating a mental image occur. Objects of this mental image may be imagined and attributes of these objects can be identified and quantified. These mental actions (which are unique to each individual) of constructing an image of the problem’s context may be part of the orientation phase of problem solving (M. P. Carlson & Bloom, 2005).

A study that looked at the information and thought processes influencing mathematicians' problem-solving behaviors gave rise to Carlson and Bloom's Multidimensional Problem Solving Framework (2005). Four separate stages of issue solution were identified by data analysis: planning, executing, checking, and orienting. A problem solver places himself in relation to a problem and creates an initial mental picture of the problem's environment during the orientation phase. The solver organizes, constructs, and makes meaning during this stage. Hypotheses on the solution's methodology are developed and tested during the planning stage. The sub-cycle of conjecture—imagine—verify occurs during the planning stage. The following is a definition for this sequence: a) formulate a hypothesis, b) envision the outcome of the solution, and c) assess the feasibility of the hypothesized strategy. Because it avoids officially running each conjecture, this sub-cycle enables the problem solver to approach the problem more efficiently. The problem solver performs calculations and formal constructions during the execution phase. Verification is one way to characterize the checking phase. In this stage, the issue solver examines if his calculations and solution make sense. In the event that the problem is not solved, this leads to either an acceptance (and a cycle back to orienting or planning) or a rejection (and a cycle back to addressing the problem).

The several orientation engagement processes can be divided into three categories, according to Carlson and Bloom (2005): sense making, organization, and construction. These behaviors are influenced by affect, heuristics, and resources. Resources are facts, techniques, and both formal and informal knowledge that are employed for solving problems. Actions like creating a diagram or trying to solve a parallel problem are examples of heuristics. Affect describes a problem solver's attitudes and views about the nature of mathematics, problem solving, testing, etc. Mathematical integrity, happiness, confidence, and dissatisfaction are a few examples of affect.

Making meaning is another facet of orientation. The process by which a problem solver studies and analyzes a problem is known as sense making. A problem solver determines the qualities of an item and a scenario to be modeled during the sense-making process. In the course of future mental operations, these qualities may be attributes of an object that must be measured. Additionally, the person solving the problem could just see the physical scenario being interpreted—for example, a dog chasing a fox. In the process of making sense, the issue solver also determines which questions need to be addressed and creates an image of one or more objectives to achieve.

Using preexisting ideas and experiences to try to connect them to the scenario being interpreted is another mental process that could take place during sense-making. The word experiences is not used carelessly here; rather, it is utilized to draw attention to physical experiences and observances rather than just using the word concepts, which would simply suggest mathematical concepts to the reader. For example, while answering a question about constructing a box and adding volume, the solver might conjure up a picture of a previous box-building and box-forming event. Alternatively, a student might remember a visible action or calculation that they or another person did in the course of a related issue or circumstance.

Although Carlson and Bloom (2005) highlighted that the orientation phase was a crucial part of a person's problem-solving activities, they did not go into great detail on what the solver's mind was seeing and creating during this phase. The authors also pointed out that many of the mental processes involved in problem solving were invisible due to the superior mathematical skills of the participants, especially during the orientation phase.

Students' Trigonometric Understandings

Although there is a dearth of research on students' comprehension of trigonometric functions, some studies have produced data relevant to this inquiry (Brown, 2005, 2006; Weber, 2005). These investigations into students' thought processes have shown that they have constrained and limited understandings. Additionally, students have been accused of having a brittle understanding of angle measure (Brown, 2005).

Weber (2005) compared experimental instruction with lecture-based instruction in two undergraduate trigonometry courses in an effort to understand how students think about the subject. Through the physical (or mental) creation of scenarios and the making (or guessing) of measurements from these constructions, the experimental teaching centered on exploring trigonometric functions. This experimental instruction was based on the idea that, regardless of the situation, students must have a clear understanding of the geometric processes that are employed to arrive at the values of trigonometric functions or expressions when they are reasoning about them. Weber's position is in line with the recommendations of quantitative reasoning; pupils need to form ideas about the things and numbers they are expected to reason about. In Weber's study, it was discovered that students who had the experimental training gained a deeper and more comprehensive comprehension of trigonometric functions.

It was frequently impossible for the students who made up the conventional, lecture-based group of Weber's study (2005) to estimate the output values of trigonometric functions for different input values or to analyze different aspects of these functions. These students' inability to form the geometric objects required to reason about trigonometric functions was noted by the author. Students in the traditional group, for example, were unable to approximate $\sin(\theta)$ for different values of θ. Rather, the students asserted that they required a triangle with the proper name and that they were not provided enough information to complete this job. The author also disclosed that rather than describing the sine function as a function or process between the values of two numbers, the students in the lecture-based course frequently referred to it as a cue for determining an answer.

Additionally, Weber (2005) noted that none of the students in the lecture-based class could meaningfully respond to the question of why $\sin(x)$ is a function. According to research, students struggle to reason about function as a process (M. Carlson, 1998; M. Carlson & Oehrtman, 2004; Harel & Dubinsky, 1992; Oehrtman, et al., 2008; Sierpinska, 1992; Thompson, 1994b). This finding is in line with those findings. Three of the four students explained the sine function in terms of a process between an input and an output quantity in relation to the experimental course. Weber credited the pupils' application of the unit circle with improving their performance and thinking. In other words, students who performed better frequently displayed arguments that were grounded in the unit circle. He did point out, nevertheless, that not all methods of teaching trigonometric functions that make use of the unit circle will lead to better pupil comprehension. He emphasized how crucial it is that students comprehend how the unit circle is made in connection to the matching trigonometric functions. Kendal and Stacey (1997) found that students who were taught using a unit circle model learned less than those who were taught using a right triangle model, and Brown (2005) found that students had trouble connecting a point on the unit circle to the graph of the sine or cosine function. These findings may be explained by Weber's crucial suggestion.

Reports on pupils' trigonometric comprehension have often shown that they struggle greatly with trigonometric function reasoning. It is frequently noted that students exhibit weak cognitive links between the different trigonometric situations. Additionally, it seemed that students lacked the fundamental knowledge required to create these connections. Concepts of angle measure, the radian as a unit of measurement, and the function of the unit circle in trigonometry are some examples of these fundamental ideas. These results imply that more time should be spent on fostering the fundamental knowledge required for trigonometry and that research is required to ascertain the best ways to encourage coherence among the different trigonometric contexts. As recommended by Weber (2005), great care must be taken to encourage students to construct the unit circle and its connection to trigonometric functions so that this context serves as a tool for thinking. This idea can be extended to any comprehension that is thought to be fundamental to trigonometry and trigonometric functions (such as the radian and angle measure).

Teachers' Trigonometry Understandings

According to a number of studies, instructors' knowledge of trigonometry is constrained, small, and deeply ingrained (Akkoc, 2008; Fi, 2003, 2006; Thompson et al., 2007; Topçu, Kertil, Akkoç, Kamil, & Osman, (2006)). These studies discovered that teachers were far more at ease with degree angle measures and that they lacked a meaningful comprehension of the radian as an angle measure unit. For example, while converting between radian and degree angle measures, secondary teachers employed meaningless approaches, according to Fi (2003, 2006). Beyond these conversion processes, the professors were also unable to explain what radian measure meant. According to several research (Akkoc, 2008; Fi, 2003, 2006; Tall & Vinner, 1981; Topçu, et al., 2006), when! is brought up in a trigonometry setting, teachers do not consider it to be a real number. Instead, other professors defined! as the unit for radian measure (e.g., a radian equals so many multiples of π), whereas these teachers were seen graphing! radians as equal to 180 (as a number, not degrees).

Additionally, Akkoc (2008) found that pre-service teachers who had the most developed understanding of the radius as a unit of measurement related different trigonometric concepts using the unit circle, whereas teachers with less developed understandings used a right triangle—which does not have a circle—to explain trigonometric concepts. The author proposed that teachers' degree-dominated perceptions of angle measure may have their roots in the geometric foundations of trigonometry and the introduction of the sine and cosine functions in the setting of right triangles. In right triangle trigonometry, the degree is the common unit of angle measurement that forms the mental image that predominates in a person's thinking. Akkoc responded to this observation by proposing that educational activities foster ideas that make it possible to comprehend the radian as a unit of measurement.

Thompson, Carlson, and Silverman (2007) challenged instructors to rethink the mathematics they teach in order to address the constrained and disjointed understandings of trigonometry that they frequently develop. The writers concentrated on employing magnitude, often known as length, as a fundamental idea. To encourage teachers to build a method for measuring an angle, for example, the angle measure was created in terms of the proportion of a circle's diameter that represents the length of an arc. The teachers' engagement with the assignments suggested that they were deeply committed to the high school curriculum they were using at the time and the meanings they ascribed to it. In particular, rather than utilizing the unit circle and angle measure to introduce trigonometry, the professors were committed to using right triangles. Additionally, the teachers continued to hold the view that trigonometry is primarily about calculating a triangle's measures. Even after the authors made the teachers aware of the incoherence of these meanings, these understandings—regardless of their incoherence—dominated what the teachers envisioned themselves teaching. This highlights how crucial it is to constantly pay attention to the ideas that students—who may eventually become teachers—form, particularly when a mathematical topic is only getting started.

All of the aforementioned research papers have one thing in common: teachers were shown to be steadfastly devoted to interpretations of trigonometry that did not align with logical understandings of the subject. According to these findings, most of the teachers lacked the fundamental knowledge required for trigonometry, which may have contributed to the meanings' incoherence. Teachers' perceptions of angle measure were often dominated by degree measures, and most of them had a very limited understanding of the radian as a unit of measurement. The reasoning required for significant and related comprehensions of trigonometric functions did not seem to be supported by these pictures.

SUGGESTIONS FOR CURRICULUM AND INSTRUCTION

Giving students contextual problems alone does not guarantee that they will use covariational or quantitative reasoning. Curriculum and instruction must therefore encourage problem-solving techniques that align with the logic that students are supposed to use. Students must first create a problem scenario and quantities to reason about in order to use covariational reasoning and quantitative reasoning to build mathematical understandings. Students should then think about or anticipate calculations related to the situation's quantities as they work through a problem (e.g., quantitative operations; Thompson, 1989) rather than executing steps right away or utilizing formulas that don't make sense to them quantitatively. Students are able to develop and use a quantitative framework to verify, edit, and consider their answers as they reply to a new topic by keeping a quantitative perspective.

Any mathematical topic can benefit from the development of problem-solving techniques that align with the understandings that students are expected to build. A student's prior experiences in mathematics classes have an impact on how they approach mathematics and problem solving. Therefore, curriculum and instruction at all educational levels should emphasize developing thinking and problem-solving skills that are advantageous to the learner's current and long-term growth.

In particular, a student's understanding of the radius as a unit of measure and the angle measure are crucial for reasoning about trigonometric functions. A learner must develop numbers and methods of measuring them in order to support their understanding of the trigonometric functions, as the sine and cosine functions define a quantitative and covariational relationship. The foundation for a learner to use the unit circle to reason about a changing arc length and other quantities measured in terms of the radius is established by developing quantitatively based understandings of angle measure and the radius as a unit of measure. Additionally, a learner can build a basis for indefinite reasoning about the input-output processes codified by the sine and cosine functions by comprehending quantitative relationships within the framework of circular motion. Students can now predict an input-output process without having to assess the sine or cosine function thanks to this.

Giving students the chance to solve problems and consider their own thinking is crucial, as opposed to giving them answers and steps that someone else has already developed.

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