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# Curvature Properties and Generalizations of $W_{|h}$-Trirecurrent Spaces in Finsler Geometry

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**ABSTRACT:**

In this paper, we introduce and investigate a new class of Finsler spaces, termed generalized $W_{|h}$-trirecurrent Finsler spaces, which extend the concept of birecurrence to higher-order covariant derivatives. These spaces are characterized by the third-order h-covariant derivative of Weyl's projective curvature tensor $W^i_{jkh}$, satisfying a specific recurrence condition. We derive and analyze several key characterizations of this class by formulating higher-order derivative relations for the curvature, torsion, and deviation tensors. Furthermore, we prove that every generalized $W_{|h}$-birecurrent Finsler space is inherently a generalized $W_{|h}$-trirecurrent Finsler space. Additionally, we explore the connection between Weyl's curvature tensor and Cartan's third curvature tensor within this framework, establishing necessary and sufficient conditions for their recurrence behavior.

**Key words:** Generalized trirecurrent Finsler space, Covariant derivative, Wely's tensor $W^i_{jkh}$, Cartan's curvature tensors $R^i_{jkh}$, $K^i_{jkh}$, Torsion tensor, Ricci tensor.

## I. Introduction

Finsler geometry, as a generalization of Riemannian geometry, has found increasing relevance in both pure mathematics and theoretical physics due to its capacity to model anisotropic spaces. Among the many curvature tensors used to describe the geometric properties of Finsler spaces, Weyl's projective curvature tensor and Cartan's third curvature tensor play a pivotal role in understanding the intrinsic and extrinsic structure of such spaces. While previous works have addressed the recurrence of curvature tensors through first and second-order covariant derivatives, the study of higher-order recurrence, particularly in the context of the third-order h-covariant derivative, remains an area of ongoing research.

The study of curvature tensors in differential geometry, particularly within the framework of Finsler and pseudo-Riemannian geometries, has attracted substantial interest due to its deep mathematical structure and profound implications in theoretical physics. In recent years, numerous researchers have extended classical notions of curvature to more generalized and higher-order settings, providing a richer geometric understanding and opening new avenues for mathematical and physical interpretations. Significant contributions have been made in exploring relationships between various curvature tensors in Finsler spaces, such as the work by Abdallah [1], which investigates connections between two distinct curvature tensors, and the extension of third-order P-generalized Finsler spaces by Abdallah and Hardan [2]. These efforts highlight the importance of higher-order tensor structures in enhancing our understanding of Finsler geometry.

Moreover, the inheritance properties of curvature tensors in general relativity have been addressed by Ali et al. [3] and others [4, 5], where the role of conharmonic, and related curvature tensors have been analyzed in spacetime models. These studies bridge the gap between abstract geometry and physical models, particularly in relativistic contexts. A large body of work by Al-Qashbari and collaborators [6–14] has focused on the generalization and decomposition of curvature tensors in Finsler spaces, incorporating Berwald and Cartan derivatives, Lie derivatives, and various forms of recurrence. These contributions have systematically constructed generalized recurrent spaces of different orders and studied the behavior of specialized curvature tensors like the M-projective and R-projective tensors.

Further developments include studies of Weyl's curvature tensor in the context of higher-order derivatives [10], trirecurrent and $\mathcal{BR}$ −trirecurrent spaces [11, 14, 20], and decompositions using covariant derivatives [7, 8]. Foundational works by Goswami [15], Hamoud et al. [16], Misra et al. [17] and Rund [19] provide the theoretical backdrop for these modern generalizations. Additionally, recent theses and dissertations [18] have expanded on the projective and birecurrent structures in Finsler geometry, reflecting growing academic interest in this domain.

Collectively, these studies form a robust foundation for the current investigation, which aims to further analyze the structure and recurrence behavior of generalized curvature tensors in Finsler spaces, using advanced differential operators and tensorial decompositions. In this work, we define and explore a novel class of Finsler spaces known as generalized $W_{|h}$-trirecurrent spaces, denoted $G^{2nd}\, W_{|h} - TRF_n$, characterized by specific third-order recurrence relations of the Weyl curvature tensor. We derive explicit conditions under which these recurrence relations hold and construct corresponding identities for curvature, torsion, and Ricci-type tensors. The study is further extended by establishing connections between the Weyl tensor and Cartan's curvature tensor, leading to new insights into their interrelations and generalization conditions.

The remainder of this paper is structured as follows: In Section 3, we introduce the generalization of trirecurrence and formulate the defining relations for $G^{2nd}\, W_{|h} - TRF_n$. Several theorems are proved to characterize these spaces geometrically and algebraically. Section 4 explores the relationship between the Weyl and Cartan curvature tensors within this setting, leading to a series of necessary and sufficient conditions for trirecurrence based on both tensor forms. The results obtained deepen the understanding of Finsler curvature behavior under extended derivative operations and open avenues for applications in mathematical physics, particularly in theories involving higher-order geometric structures. Two vectors $y_i$ and $y^i$ meet the following conditions

a) $y_i = g_{ij}\, y^j$ , b) $y_i\, y^i = F^2$ , c) $\delta^k_j\, y^j = y^k$ , d) $g_{ir}\, \delta^i_j = g_{rj}$ ,

e) $g^{jk}\, \delta^i_k = g^{ji}$ , f) $\dot\partial_i\, y^i = 1$ and g) $\dot\partial_j\, y_h = g_{jh}$ . (1.1)

The quantities $g_{ij}$ and $g^{ij}$ are related by

a) $g_{ij}\, g^{jk} = \delta^k_i = \begin{cases} 1 \ , & if \quad i = k \ , \\ 0 \ , & if \quad i \neq k \ . \end{cases}$

b) $g^{jk}{}_{|h} = 0$ and c) $g_{ij|h} = 0$ . (1.2)

Tensor $C_{ijk}$ is known as (h)hv-torsion tensor defined by

a) $C_{ijk} = \frac{1}{4}\dot\partial_i\dot\partial_j\dot\partial_k\, F^2$, b) $C_{ijk}\, y^i = C_{ijk}\, y^j = C_{ijk}\, y^k = 0$ and c) $C_{rjp} = g_{ir}\, C^i_{jp}$ . (1.3)

The vector $y^i$ and metric function $F$ are vanished identically for Cartan's covariant derivative.

a) $F_{|h} = 0$ and b) $y^i{}_{|h} = 0$ . (1.4)

The h-covariant derivative of second order for an arbitrary vector field with respect to $x^k$ and $x^j$, successively, we get

$X^i_{|k|j} = \partial_j\big(X^i_{|k}\big) - \big(X^i_{|r}\big)\, \Gamma^{*r}_{kj} + \big(X^r_{|k}\big)\, \Gamma^{*i}_{rj} - \big(\partial_j X^i_{|k}\big)\, \Gamma^{*i}_{js}\, y^s$ . (1.5)

In view (1.5) and by taking skew-symmetric part with respect to the indices j and k, we get the commutation formula for Cartan is covariant differentiation as follows:

$X^i_{|k|j} - X^i_{|j|k} = X^r K^i_{rkj} - (\partial_r X^i)\, K^i_{skj}\, y^s$ . (1.6)

$K^i_{jkh} = \partial_j\, \Gamma^{*i}_{kr} + \big(\dot\partial_l\, \Gamma^{*i}_{rj}\big) G^l_k + \Gamma^{*i}_{mj}\, \Gamma^{*m}_{kr} - \partial_k \Gamma^{*i}_{jr} - \big(\dot\partial_l\, \Gamma^{*i}_{rk}\big) G^l_j - \Gamma^{*i}_{mk}\, \Gamma^{*m}_{jr}$ . (1.7)

The tensor $K^i_{jkh}$ as defined above is called Cartan's fourth curvature tensor, this tensor is positively homogeneous of degree zero in the directional arguments $y$ .

The tensor $W^i_{jkh}$ , torsion tensor $W^i_{jk}$ and deviation tensor $W^i_j$ are defined by:

$W^i_{jkh} = H^i_{jkh} + \frac{2\, \delta^i_j}{(n+1)} H_{[hk]} + \frac{2\, y^i}{(n+1)} \dot\partial_j H_{[kh]} + \frac{\delta^i_k}{(n^2-1)}\big(n\, H_{jh} + H_{hj} + y^r \dot\partial_j H_{hr}\big)$

$- \frac{\delta^i_h}{(n^2-1)}\big(n\, H_{jk} + H_{kj} + y^r \dot\partial_j H_{kr}\big)$ , (1.8)

$W^i_{jk} = H^i_{jk} + \frac{y^i}{(n+1)} H_{[jk]} + 2\left\{\frac{\delta^i_{[j}}{(n^2-1)}\big(n\, H_{k]} - y^r H_{k]\, r}\big)\right\}$ . (1.9)

$W^i_j = H^i_j - H\, \delta^i_j - \frac{1}{(n+1)}\big(\dot\partial_r H^r_j - \dot\partial_j H\big)\, y^i$, respectively. (1.10)

The tensors $W^i_{jkh}$ and $W^i_{jk}$ satisfy the following identities

a) $W^i_{jkh}\, y^j = W^i_{kh}$ , b) $W^i_{kh}\, y^k = W^i_h$ ,

c) $W^i_{jki} = W_{jk}$ and d) $g_{ir}\, W^i_{jkh} = W_{rjkh}$ . (1.11)

Also, if we suppose that the tensor $W^i_j$ and $W_{jk}$ satisfy the following identities

a) $W^i_k\, y^k = 0$ , b) $W^i_i = 0$ , c) $g_{ir}\, W^i_j = W_{rj}$ ,

d) $g^{jk} W_{jk} = W$ and e) $W_{jk}\, y^k = 0$ . (1.12)

Cartan's third curvature tensor $R^i_{jkh}$ , Ricci tensor $R_{jk}$ , $H^i_{kh}$ , the vector $H_k$ and scalar curvature $H$ are defined as

a) $R^i_{jkh} = \Gamma^{*i}_{hjk} + \big(\Gamma^{*i}_{ljk}\big)\, G^l_h + C^i_{jm}\big(\, G^m_{kh} - G^m_{kl}\, G^l_h\,\big) + \Gamma^{*i}_{mk}\, \Gamma^{*m}_{jh} - \frac{k}{h}$ , b) $R^i_{jkh}\, y^j = H^i_{kh}$ ,

c) $R_{jk}\, y^j = H_k$ , d) $R_{jk}\, y^k = R_j$ , e) $R^i_i = R$ , f) $g_{ir}\, R^i_{jkh} = R_{rjkh}$ , g) $g^{jk} R_{jk} = R$ ,

h) $g^{jk}R^i_{jkh} = R^i_h$ , i) $R^i_{jki} = R_{jk}$ , t) $H_i\, y^i = H^i_i = (n-1)\, H$ , j) $H^i_{kh}\, y^k = H^i_h$ ,

k) $H^i_{ki} = H_k$ , $l$) $H^r_{rkh} = H_{hk} - H_{kh}$ and $m$) $H^i_{jki} = H_{jk}$ . (1.13)

The Cartan's $4^{th}$ curvature tensor $K^i_{jkh}$, the Ricci tensor $K_{jk}$ , the vector $K_k$, and the scalar curvature K are defined as follows

a) $K^i_{jkh}\, g_{ir} = K_{rjkh}$ , b) $K_{jk}\, y^j = H_k$ , c) $K_{jk}\, y^k = K_j$ , d) $g^{jk}K_{jk} = K$ ,

e) $K^i_{jki} = K_{jk}$ , f) $g^{jk}\, K^i_{jkh} = K^i_h$ , $g$) $H^i_{jkh} = \mathrm{K}^i_{jkh} + y^s\big(\dot{\partial}_j \mathrm{K}^i_{skh}\big)$

and $h$) $H^i_{jkh} - K^i_{jkh} = P^i_{jk|h} + P^r_{jk}P^i_{rh} - h/k$ . (1.14)

## 2. Preliminaries

We introduced the generalized by Cartan's covariant derivative for Wely's projective curvature tensor $W^i_{jkh}$ was given by

$$W^i_{jkh|m} = \lambda_m W^i_{jkh} + \mu_m\big(\delta^i_h g_{jk} - \delta^i_k g_{jh}\big). \tag{2.1}$$

A Finsler space $F_n$ which the curvature tensor $W^i_{jkh}$ satisfies the condition (2.1) is referred to as a generalized $W_{|h} -$ recurrent space and denoted by $G\, W_{|h} - RF_n$. $|m$ is called $h -$ covariant derivative of with respect to $x^m$.

By taking the $h -$ covariant derivative of (2.1), with respect to $x^l$, and using (1.2c), we obtain

$$W^i_{jkh|m|l} = \lambda_{m|l} W^i_{jkh} + \lambda_m W^i_{jkh|l} + \mu_{m|l}\big(\delta^i_h g_{jk} - \delta^i_k g_{jh}\big) .$$

By applying equation (2.1) in the above expression, we obtain

$$W^i_{jkh|m|l} = \big(\lambda_{m|l} + \lambda_m\lambda_l\big) W^i_{jkh} + \big(\mu_l + \mu_{m|l}\big)\big(\delta^i_h g_{jk} - \delta^i_k g_{jh}\big) . \tag{2.2}$$

The equation (2.2), can be expressed as

$$W^i_{jkh|m|l} = a_{ml}\, W^i_{jkh} + b_{ml}\big(\delta^i_h g_{jk} - \delta^i_k g_{jh}\big) . \tag{2.3}$$

where $a_{ml} = \lambda_{m|l} + \lambda_m\lambda_l$ and $b_{ml} = \mu_{m|l} + \lambda_m \mu_l$ are non-zero covariant tensors field of second order, respectively.

A Finsler space $F_n$ which the curvature tensor $W^i_{jkh}$ satisfies the condition (2.3) is referred to as a generalized $W_{|h} -$ birecurrent space and denoted by $G\, W_{|h} - BRF_n$ .

By taking the $h -$ covariant derivative of (2.3), with respect to $x^n$, and using (1.2c), we obtain

$$W^i_{jkh|m|l|n} = a_{ml|n}\, W^i_{jkh} + a_{ml}\, W^i_{jkh|n} + b_{ml|n}\big(\delta^i_h g_{jk} - \delta^i_k g_{jh}\big) .$$

By applying equation (2.1) in the above expression, we obtain

$$W^i_{jkh|m|l|n} = a_{ml|n} W^i_{jkh} + a_{ml}\left(\lambda_n W^i_{jkh} + \mu_n\big(\delta^i_h g_{jk} - \delta^i_k g_{jh}\big)\right)$$
$$+ b_{ml|n}\big(\delta^i_h g_{jk} - \delta^i_k g_{jh}\big). \tag{2.4}$$

The equation (2.4), can be expressed as

$$W^i_{jkh|m|l|n} = \big(a_{ml|n} + a_{ml}\lambda_n\big) W^i_{jkh} + \big(a_{ml}\mu_n + b_{ml|n}\big)\big(\delta^i_h g_{jk} - \delta^i_k g_{jh}\big). \tag{2.5}$$

The equation (2.5), can be expressed as

$$W^i_{jkh|m|l|n} = c_{mln} W^i_{jkh} + d_{mln}\big(\delta^i_h g_{jk} - \delta^i_k g_{jh}\big). \tag{2.6}$$

where $c_{mln} = a_{ml|n} + a_{ml}\lambda_n$ and $d_{mln} = a_{ml}\mu_n + b_{ml|n}$ are non-zero covariant tensors field of third order, respectively.

A Finsler space $F_n$ which the curvature tensor $W^i_{jkh}$ satisfies the condition (2.6) is referred to as a generalized $W_{|h} -$ trirecurrent space and denoted by $G\, W_{|h} - TRF_n$ .

From equation (1.3b), equation (2.1) can be rewritten in the following form:

$$W^i_{jkh|m} = \lambda_m W^i_{jkh} + \mu_m\big(\delta^i_h g_{jk} - \delta^i_k g_{jh}\big) + \gamma_m\big(W^i_h C_{ijk} y^i - W^i_k C_{ijh} y^i\big) . \tag{2.7}$$

Let us consider a Finsler space $F_n$ which the Wely's projective curvature tensor $W^i_{jkh}$ satisfies a generalization generalized $W_{|h} -$ recurrent space and denoted by $G^{2nd}\, W_{|h} - RF_n$. i.e. satisfies the following condition and by applying the conditions (1.3a), (1.1b), (1.1f), and (1.1g) to equation (2.7),we obtain:

$$W^i_{jkh|m} = \lambda_m W^i_{jkh} + \mu_m\big(\delta^i_h g_{jk} - \delta^i_k g_{jh}\big) + \tfrac{1}{4}\gamma_m\big(W^i_h g_{jk} - W^i_k g_{jh}\big) . \tag{2.8}$$

where $\lambda_m, \mu_m$ and $\gamma_m$ are non-zero covariant vectors of first order.

## 3. The Extension of Generalized $W_{|h}$– Trirecurrent Finsler Space

In this section, we introduce a new class of Finsler spaces, namely, generalized $W_{|h}$-trirecurrent spaces. These spaces generalize the concept of trirecurrence to a broader setting and exhibit interesting geometric properties. We investigate the curvature tensor of these spaces and establish several characterization theorems. Our work in this section we defined $|l|m|n$ is covariant derivative of third order.

By taking the $h-$ covariant derivative of (2.8), with respect to $x^l$, we obtain

$$W^i_{jkh|m|l} = a_{ml}W^i_{jkh} + b_{ml}\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right) + \frac{1}{4}c_{ml}\left(W^i_h g_{jk} - W^i_k g_{jh}\right) + \frac{1}{4}\gamma_m\left(W^i_h g_{jk} - W^i_k g_{jh}\right)_{|l}. \quad (3.1)$$

A Finsler space $F_n$ which the curvature tensor $W^i_{jkh}$ satisfies the condition (3.1) is referred to as the generalization generalized $W_{|h}-$ birecurrent space and denoted by $G^{2nd}\, W_{|h} - BRF_n$.

By taking the $h-$ covariant derivative of (3.1), with respect to $x^n$, we obtain

$$W^i_{jkh|m|l|n} = a_{ml|n}W^i_{jkh} + a_{ml}W^i_{jkh|n} + b_{ml|n}\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right) + \frac{1}{4}c_{ml|n}\left(W^i_h g_{jk} - W^i_k g_{jh}\right)$$
$$+\frac{1}{4}c_{ml}\left(W^i_h g_{jk} - W^i_k g_{jh}\right)_{|n} + \frac{1}{4}\gamma_{m|n}\left(W^i_h g_{jk} - W^i_k g_{jh}\right)_{|l} + \frac{1}{4}\gamma_m\left(W^i_h g_{jk} - W^i_k g_{jh}\right)_{|l|n}. \quad (3.2)$$

By applying equations (1.2c) and (2.8) to equation (3.2), we obtain:

$$W^i_{jkh|m|l|n} = \left(a_{ml|n} + a_{ml}\lambda_n\right)W^i_{jkh} + \left(a_{ml}\mu_n + b_{ml|n}\right)\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right)$$
$$+\frac{1}{4}\left(a_{ml}\gamma_n + c_{ml|n}\right)\left(W^i_h g_{jk} - W^i_k g_{jh}\right) + \frac{1}{4}c_{ml}\left(W^i_h g_{jk} - W^i_k g_{jh}\right)_{|n}$$
$$+\frac{1}{4}\gamma_{m|n}\left(W^i_h g_{jk} - W^i_k g_{jh}\right)_{|l} + \frac{1}{4}\gamma_m\left(W^i_h g_{jk} - W^i_k g_{jh}\right)_{|l|n}. \quad (3.3)$$

The equation (3.3), can be expressed as:

$$W^i_{jkh|m|l|n} = d_{mln}W^i_{jkh} + e_{mln}\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right) + \frac{1}{4}v_{mln}\left(W^i_h g_{jk} - W^i_k g_{jh}\right)$$
$$+\frac{1}{4}c_{ml}\left(W^i_h g_{jk} - W^i_k g_{jh}\right)_{|n} + \frac{1}{4}\eta_{mn}\left(W^i_h g_{jk} - W^i_k g_{jh}\right)_{|l} + \frac{1}{4}\gamma_m\left(W^i_h g_{jk} - W^i_k g_{jh}\right)_{|l|n}. \quad (3.4)$$

Where $d_{mln} = a_{ml}\gamma_n + c_{ml|n}$ , $e_{mln} = a_{ml}\mu_n + b_{ml|n}$ *and* $v_{mln} = a_{ml|n} + a_{ml}\lambda_n$ are non-zero covariant tensors field of third order , $c_{ml}$ and $\eta_{mn} = \gamma_{m|n}$ are non-zero covariant tensors field of second order and $\gamma_m$ are non-zero covariant vector of first order.

**Definition 3.1.** *In Finsler space $F_n$, which the Weyl's projective curvature tensor $W^i_{jkh}$ satisfies the condition (3.4) is called the generalization generalized $W_{|h}$- Trirecurrent space and the tensor will be called a generalization generalized h-Trirecurrent. These space and tensor denote them briefly by $G^{2nd}\, W_{|h} - TRF_n$ and $G^{2nd}\, h - TR$, respectively.*

**Result 3.1.** Every a $G^{2nd}\, W_{|h} -\ Bi$recurrent space is also a $G^{2nd}\, W_{|h}$- Trirecurrent spaces.

By transvecting condition (3.4) to a higher dimensional space using by $y^j$, and applying equations (1.1a), (1.3b), (1.5b) and (1.11a), we obtain:

$$W^i_{kh|m|l|n} = d_{mln}W^i_{kh} + e_{mln}\left(\delta^i_h y_k - \delta^i_k y_h\right) + \frac{1}{4}v_{mln}\left(W^i_h y_k - W^i_k y_h\right)$$
$$+\frac{1}{4}c_{ml}\left(W^i_h y_k - W^i_k y_h\right)_{|n} + \frac{1}{4}\eta_{mn}\left(W^i_h y_k - W^i_k y_h\right)_{|l} + \frac{1}{4}\gamma_m\left(W^i_h y_k - W^i_k y_h\right)_{|l|n}. \quad (3.5)$$

Again, transvecting condition (3.5) to a higher dimensional space using by $y^k$, and applying equations (1.1b), (1.2a), (1.2c), (1.4b), (1.12a) and (1.11b), we obtain:

$$W^i_{h|m|l|n} = d_{mln}W^i_h + e_{mln}\left(\delta^i_h F^2 - y^i y_h\right) + \frac{1}{4}v_{mln}\left(W^i_h F^2\right) + \frac{1}{4}c_{ml}\left(W^i_h F^2\right)_{|n}$$
$$+\frac{1}{4}\eta_{mn}\left(W^i_h F^2\right)_{|l} + \frac{1}{4}\gamma_m\left(W^i_h F^2\right)_{|l|n}. \quad (3.6)$$

Therefore, we can say

**Theorem 3.1.** *In the context of $G^{2nd}\, W_{|h} - TRF_n$, the $h-$covariant derivative of third-order for the torsion tensor $W^i_{kh}$ and deviation tensor $W^i_h$ are expressed by equations (3.5) and (3.6).*

By contracting the index space through summation over i and h in the condition (3.4), and applying relations (1.1d), (1.2a), (1.11c), (1.12b) and (1.12c), we obtain the following result

$$W_{jk|m|l|n} = d_{mln}W_{jk} + (n-1)e_{mln}g_{jk} + \frac{1}{4}v_{mln}W_{jk} + \frac{1}{4}c_{ml}W_{jk|n} + \frac{1}{4}\eta_{mn}W_{jk|l} + \frac{1}{4}\gamma_m W_{jk|l|n}. \quad (3.7)$$

By transvecting condition to a higher-dimensional space (3.4) by $g_{ir}$, and applying relations (1.1d), (1.2c), (1.11d) and (1.12c), we obtain

$$W_{rjkh|m|l|n} = d_{mln}W_{rjkh} + e_{mln}\left(g_{rh}\, g_{jk} - g_{rk}\, g_{jh}\right) + \frac{1}{4}v_{mln}\left(W_{rh}g_{jk} - W_{rk}g_{jh}\right)$$

$+\frac{1}{4}c_{ml}\left(W_{rh}g_{jk}-W_{rk}g_{jh}\right)_{|n}+\frac{1}{4}\eta_{mn}\left(W_{rh}g_{jk}-W_{rk}g_{jh}\right)_{|l}+\frac{1}{4}\gamma_m\left(W_{rh}g_{jk}-W_{rk}g_{jh}\right)_{|l|n}$.(3.8)

Therefore, we can say

**Theorem 3.2.** In the context of $G^{2nd}\,W_{|h}-TRF_n$ , the $W$ −Ricci $W_{jk}$ and associate tensor $W_{rjkh}$ represent a generalized trirecurrent Finsler space, as defined by equations (3.7) and (3.8), respectively.

By transvecting condition (3.7) with $g^{jk}$, and applying relations (1.1e) and (1.12d), we obtain the following result

$$W_{|m|l|n}=d_{mln}W+(n-1)e_{mln}+\frac{1}{4}v_{mln}W+\frac{1}{4}c_{ml}W_{|n}+\frac{1}{4}\eta_{mn}W_{|l}+\frac{1}{4}\gamma_m W_{|l|n}\ . \qquad (3.9)$$

From conditions (3.9), we show that the curvature scalar $W$ cannot equal to zero because if the vanishing of $W$ would imply $d_{mln}=0$ and $e_{mln}=0$, that is a contradiction.

Therefore, we can say

**Corollary 3.3.** *In the context of $G^{2nd}\,W_{|h}-TRF_n$ , the scalar $W$ in equation (3.9) is non-vanishing.*

## 4. Exploring the Relationship Between Weyl's Curvature Tensor and Cartan's Third Curvature Tensor in Finsler Geometry

This section examine the relationship between Weyl's curvature tensor and Cartan's third curvature tensor within the context of Finsler spaces. By analyzing their algebraic and geometric properties, we aim to derive new identities and inequalities that connect these two fundamental tensors. The curvature properties of the space are described by various curvature tensors, with Weyl's and Cartan's third curvature tensors being of particular significance. While the individual geometric and physical implications of these tensors have been widely studied, the connection between them has not been fully explored and remains an area of active research.

The results of this study are expected to enhance our understanding of the curvature structure of Finsler spaces and offer new insights into their potential applications, particularly in gravitational theories and cosmology. Some properties of $W^i_{jkh}$ curvature tensor was proposed by Ahsan and Ali [3, 4] in (2014). For $(n=4)$ a Riemannian space, it is known that Cartan's third curvature tensor $R^i_{jkh}$ and Wely's projective curvature tensor $W^i_{jkh}$ are connected by the formula

$$W^i_{jkh}=R^i_{jkh}+\frac{1}{3}\left(\delta^i_k R_{jh}-g_{jk}R^i_h\right). \qquad (4.1)$$

By taking the covariant derivative of (4.1), with respcet to $x^m$, $x^l$ and $x^n$ in the sense of Cartan and using (1.2c), we get

$$W^i_{jkh|m|l|n}=R^i_{jkh|m|l|n}+\frac{1}{3}\left(\delta^i_k R_{jh}-g_{jk}R^i_h\right)_{|m|l|n}. \qquad (4.2)$$

By substituting equations (3.4) and (4.1) in to (4.2), we obtain:

$$R^i_{jkh|m|l|n}=d_{mln}R^i_{jkh}+e_{mln}\left(\delta^i_h g_{jk}-\delta^i_k g_{jh}\right)+\frac{1}{4}v_{mln}\left(W^i_h g_{jk}-W^i_k g_{jh}\right)$$
$$+\frac{1}{4}c_{ml}\left(W^i_h g_{jk}-W^i_k g_{jh}\right)_{|n}+\frac{1}{4}\eta_{mn}\left(W^i_h g_{jk}-W^i_k g_{jh}\right)_{|l}+\frac{1}{4}\gamma_m\left(W^i_h g_{jk}-W^i_k g_{jh}\right)_{|l|n}$$
$$+\frac{1}{3}d_{mln}\left(\delta^i_k R_{jh}-g_{jk}R^i_h\right)+\frac{1}{3}\left(\delta^i_k R_{jh}-g_{jk}R^i_h\right)_{|m|l|n}. \qquad (4.3)$$

This demonstrates that

$$R^i_{jkh|m|l|n}=d_{mln}R^i_{jkh}+e_{mln}\left(\delta^i_h g_{jk}-\delta^i_k g_{jh}\right)+\frac{1}{4}v_{mln}\left(W^i_h g_{jk}-W^i_k g_{jh}\right)$$
$$+\frac{1}{4}c_{ml}\left(W^i_h g_{jk}-W^i_k g_{jh}\right)_{|n}+\frac{1}{4}\eta_{mn}\left(W^i_h g_{jk}-W^i_k g_{jh}\right)_{|l}+\frac{1}{4}\gamma_m\left(W^i_h g_{jk}-W^i_k g_{jh}\right)_{|l|n}. \qquad (4.4)$$

If and only if

$$\left(\delta^i_k R_{jh}-g_{jk}R^i_h\right)_{|m|l|n}=d_{mln}\left(\delta^i_k R_{jh}-g_{jk}R^i_h\right). \qquad (4.5)$$

Therefore, we can say

**Theorem 4.1.** *In the context of $G^{2nd}R_{|h}-TRF_n$ , Cartan's 3th curvature tensor $R^i_{jkh}$ defines a generalized trirecurrent Finsler space if and only if the tensor $\left(\delta^i_k R_{jh}-g_{jk}R^i_h\right)$ is a generalized trirecurrent Finsler space.*

By transvecting condition (4.3) with $y^j$, and utilizing equations (1.1a), (1.4b), (1.13b) and (1.13c), we obtain the following result

$$H^i_{kh|m|l|n}=d_{mln}H^i_{kh}+e_{mln}\left(\delta^i_h y_k-\delta^i_k y_h\right)+\frac{1}{4}v_{mln}\left(W^i_h y_k-W^i_k y_h\right)$$
$$+\frac{1}{4}c_{ml}\left(W^i_h y_k-W^i_k y_h\right)_{|n}+\frac{1}{4}\eta_{mn}\left(W^i_h y_k-W^i_k y_h\right)_{|l}+\frac{1}{4}\gamma_m\left(W^i_h y_k-W^i_k y_h\right)_{|l|n}$$
$$+\frac{1}{3}d_{mln}\left(\delta^i_k H_h-y_k R^i_h\right)+\frac{1}{3}\left(\delta^i_k H_h-y_k R^i_h\right)_{|m|l|n}. \qquad (4.6)$$

This demonstrates that

$$H^i_{kh|m|l|n}=d_{mln}H^i_{kh}+e_{mln}\left(\delta^i_h y_k-\delta^i_k y_h\right)+\frac{1}{4}v_{mln}\left(W^i_h y_k-W^i_k y_h\right)$$

$$+\frac{1}{4}c_{ml}\left(W_h^i y_k - W_k^i y_h\right)_{|n} + \frac{1}{4}\eta_{mn}\left(W_h^i y_k - W_k^i y_h\right)_{|l} + \frac{1}{4}\gamma_m\left(W_h^i y_k - W_k^i y_h\right)_{|l|n}. \quad (4.7)$$

If and only if

$$\left(\delta_k^i H_h - y_k R_h^i\right)_{|m|l|n} = d_{mln}\left(\delta_k^i H_h - y_k R_h^i\right). \quad (4.8)$$

Therefore, we conclude

**Theorem 4.2.** *In the context of $G^{2nd}\, R_{|h} - TRF_n$ , the covariant derivative of the third-orders for the torsion tensor $H_{kh}^i$ defines a generalized trirecurrent Finsler space if and only if (4.8) is satisfied.*

By transvecting condition (4.6) with $y^k$, and applying relations (1.1b), (1.1c), (1.4a), (1.4b), (1.12a) and (1.13j), we obtain the following result

$$H^i_{h|m|l|n} = d_{mln}H_h^i + e_{mln}\left(y^i y_h - \delta_h^i F^2\right) + \frac{1}{4}v_{mln}\left(W_h^i F^2\right) + \frac{1}{4}c_{ml}\left(W_h^i F^2\right)_{|n}$$
$$+\frac{1}{4}\eta_{mn}\left(W_h^i F^2\right)_{|l} + \frac{1}{4}\gamma_m\left(W_h^i F^2\right)_{|l|n} + \frac{1}{3}d_{mln}\left(y^i H_h - F^2 R_h^i\right) + \frac{1}{3}\left(y^i H_h - F^2 R_h^i\right)_{|m|l|n}. \quad (4.9)$$

This demonstrates that

$$H^i_{h|m|l|n} = d_{mln}H_h^i + e_{mln}\left(y^i y_h - \delta_h^i F^2\right) + \frac{1}{4}v_{mln}\left(W_h^i F^2\right) + \frac{1}{4}c_{ml}\left(W_h^i F^2\right)_{|n}$$
$$+\frac{1}{4}\eta_{mn}\left(W_h^i F^2\right)_{|l} + \frac{1}{4}\gamma_m\left(W_h^i F^2\right)_{|l|n}. \quad (4.10)$$

If and only if

$$\left(y^i H_h - F^2 R_h^i\right)_{|m|l|n} = d_{mln}\left(y^i H_h - F^2 R_h^i\right). \quad (4.11)$$

Therefore, we can say

**Theorem 4.3.** *In the context of $G^{2nd}\, R_{|h} - TRF_n$ ,the covariant derivative of the third-orders for the deviation tensor $H_h^i$ represents a generalized trirecurrent Finsler space if and only if the condition in equation (4.11) is satisfied.*

By contracting the indices i and h in equations (4.6) and (4.9), respectively and utilizing equations (1.2a), (1.1a), (1.1b), (1.4a), (1.13k), (1.13t), and (1.12b), we obtain the following result:

$$H_{k|m|l|n} = d_{mln}H_k + (1-n)e_{mln}\, y_k + \frac{1}{4}v_{mln}\left(W_k^i y_i\right) + \frac{1}{4}c_{ml}\left(W_k^i y_i\right)_{|n} + \frac{1}{4}\eta_{mn}\left(W_k^i y_i\right)_{|l}$$
$$+\frac{1}{4}\gamma_m\left(W_k^i y_i\right)_{|l|n} + \frac{1}{3}d_{mln}\left(H_k - y_k R\right) + \frac{1}{3}\left(H_k - y_k R\right)_{|m|l|n}. \quad (4.12)$$

This demonstrates that

$$H_{k|m|l|n} = d_{mln}H_k + (1-n)\, e_{mln}y_k + \frac{1}{4}v_{mln}\left(W_k^i y_i\right) + \frac{1}{4}c_{ml}\left(W_k^i y_i\right)_{|n} + \frac{1}{4}\eta_{mn}\left(W_k^i y_i\right)_{|l}$$
$$+\frac{1}{4}\gamma_m\left(W_k^i y_i\right)_{|l|n}. \quad (4.13)$$

If and only if

$$\left(H_k - y_k R\right)_{|m|l|n} = d_{mln}\left(H_k - y_k R\right). \quad (4.14)$$

And

$$H_{|m|l|n} = d_{mln}H + (1-n)e_{mln}F^2 + \frac{1}{3}d_{mln}\left(H - F^2 R\right) + \frac{1}{3}\left(H - F^2 R\right)_{|m|l|n}. \quad (4.15)$$

This demonstrates that

$$H_{|m|l|n} = d_{mln}H + (1-n)e_{mln}F^2. \quad (4.16)$$

If and only if

$$\left(H - F^2 R\right)_{|m|l|n} = d_{mln}\left(H - F^2 R\right). \quad (4.17)$$

Therefore, the proof of theorem is completed, we can say

**Theorem 4.4.** *In the context of $G^{2nd}\, R_{|h} - TRF_n$ , the vector $H_k$ and scalar $H$ are defined in equations (4.13) and (3.16), respectively, provided that the conditions (4.14) and (3.17) are satisfied.*

By contracting the indices i and h in equation (4.3) and utilizing equations (1.1d), (1.1b), (1.13i), (1.13e), (1.12d) and (1.12b), we obtain the following result:

$$R_{jk|m|l|n} = d_{mln}R_{jk} + (1-n)e_{mln}\, g_{jk} + \frac{1}{4}v_{mln}W_{jk} + \frac{1}{4}c_{ml}W_{jk|n} + \frac{1}{4}\eta_{mn}W_{jk|l} + \frac{1}{4}\gamma_m W_{jk|l|n}$$
$$+\frac{1}{3}d_{mln}\left(R_{jk} - g_{jk}R\right) + \frac{1}{3}\left(R_{jk} - g_{jk}R\right)_{|m|l|n}. \quad (4.18)$$

This demonstrates that

$$R_{jk|m|l|n} = d_{mln}R_{jk} + (1-n)e_{mln}\, g_{jk} + \frac{1}{4}v_{mln}W_{jk} + \frac{1}{4}c_{ml}W_{jk|n} + \frac{1}{4}\eta_{mn}W_{jk|l}$$

$$+\frac{1}{4}\gamma_m W_{jk|l|n}. \quad (4.19)$$

If and only if

$$\left(R_{jk} - g_{jk} R\right)_{|m|l|n} = d_{mln}\left(R_{jk} - g_{jk} R\right). \quad (4.20)$$

In conclusion, we can say

**Theorem 4.5.** *In the context of $G^{2nd}R_{|h} - TRF_n$, R-Ricci tensor $R_{jk}$ is defined in equation (4.19), provided that the condition (4.20) is satisfied.*

By transvecting condition (4.18) with $y^k$, and applying relations (1.1a), (1.1c), (1.4b), (1.12e) and (1.13d), we obtain the following result

$$R_{j|m|l|n} = d_{mln}R_j + (1-n)e_{mln}y_j + \frac{1}{3}d_{mln}\left(R_j - y_j R\right) + \frac{1}{3}\left(R_j - y_j R\right)_{|m|l|n}. \quad (4.21)$$

This demonstrates that

$$R_{j|m|l|n} = d_{mln}R_j + (1-n)e_{mln}y_j. \quad (4.22)$$

If and only if

$$\left(R_j - y_j R\right)_{|m|l|n} = d_{mln}\left(R_j - y_j R\right). \quad (4.23)$$

By transvecting conditions (4.3) and (4.18) with $g^{jk}$, respectively, and applying relations (1.2a), (1.2b), (1.12d), (1.13g), and (1.13h), we obtain:

$$R^i_{h|m|l|n} = d_{mln}R^i_h + (n-1)e_{mln}\delta^i_h + \frac{1}{4}(n-1)v_{mln}W^i_h + \frac{1}{4}(n-1)c_{ml}W^i_{h|n}$$
$$+\frac{1}{4}(n-1)\eta_{mn}W^i_{h|l} + \frac{1}{4}(n-1)\gamma_m W^i_{h|l|n} + \frac{1}{3}(1-n)d_{mln}\left(R^i_h\right) + \frac{1}{3}(1-n)\left(R^i_h\right)_{|m|l|n}. \quad (4.24)$$

This demonstrates that

$$R^i_{h|m|l|n} = d_{mln}R^i_h + (n-1)e_{mln}\delta^i_h + \frac{1}{4}(n-1)v_{mln}W^i_h + \frac{1}{4}(n-1)c_{ml}W^i_{h|n}$$
$$+\frac{1}{4}(n-1)\eta_{mn}W^i_{h|l} + \frac{1}{4}(n-1)\gamma_m W^i_{h|l|n}. \quad (4.25)$$

If and only if

$$(R^i_h)_{|m|l|n} = d_{mln}R^i_h. \quad (4.26)$$

And

$$R_{|m|l|n} = d_{mln}R + (1-n)e_{mln}n + \frac{1}{4}v_{mln}W + \frac{1}{4}c_{ml}W_{|n} + \frac{1}{4}\eta_{mn}W_{|l} + \frac{1}{4}\gamma_m W_{|l|n}$$
$$+\frac{1}{3}(1-n)d_{mln}R + \frac{1}{3}(1-n)(R)_{|m|l|n}. \quad (4.27)$$

This demonstrates that

$$R_{|m|l|n} = d_{mln}R + (1-n)ne_{mln} + \frac{1}{4}v_{mln}W + \frac{1}{4}c_{ml}W_{|n} + \frac{1}{4}\eta_{mn}W_{|l} + \frac{1}{4}\gamma_m W_{|l|n}. \quad (4.28)$$

If and only if

$$(R)_{|m|l|n} = d_{mln}R. \quad (4.29)$$

In conclusion, we can say

**Theorem 4.6.** *In the context of $G^{2nd}R_{|h} - TRF_n$, the vector $R_j$, projective deviation tensor $R^i_h$ and scalar R are defined in equations (4.22), (4.25) and (4.28), respectively, provided that the conditions (4.23), (4.26) and (4.29), are satisfied.*

By transvecting condition (4.3) by $g_{ir}$, and applying relations (1.1d), (1.2c), (1.12c) and (1.13f), we obtain

$$R_{rjkh|m|l|n} = d_{mln}R_{rjkh} + e_{mln}\left(g_{rk}g_{jh} - g_{rh}g_{jk}\right) + \frac{1}{4}v_{mln}\left(W_{rk}g_{jh} - W_{rh}g_{jk}\right)$$
$$+\frac{1}{4}c_{ml}\left(W_{rk}g_{jh} - W_{rh}g_{jk}\right)_{|n} + \frac{1}{4}\eta_{mn}\left(W_{rk}g_{jh} - W_{rh}g_{jk}\right)_{|l} + \frac{1}{4}\gamma_m\left(W_{rk}g_{jh} - W_{rh}g_{jk}\right)_{|l|n}$$
$$+\frac{1}{3}d_{mln}\left(g_{rk}R_{jh} - g_{jk}R_{rh}\right) + \frac{1}{3}\left(g_{rk}R_{jh} - g_{jk}R_{rh}\right)_{|m|l|n}. \quad (4.30)$$

This demonstrates that

$$R_{rjkh|m|l|n} = d_{mln}R_{rjkh} + e_{mln}\left(g_{rk}g_{jh} - g_{rh}g_{jk}\right) + \frac{1}{4}v_{mln}\left(W_{rk}g_{jh} - W_{rh}g_{jk}\right)$$
$$+\frac{1}{4}c_{ml}\left(W_{rk}g_{jh} - W_{rh}g_{jk}\right)_{|n} + \frac{1}{4}\eta_{mn}\left(W_{rk}g_{jh} - W_{rh}g_{jk}\right)_{|l}$$
$$+\frac{1}{4}\gamma_m\left(W_{rk}g_{jh} - W_{rh}g_{jk}\right)_{|l|n}. \quad (4.31)$$

If and only if

$$\left(g_{rk}R_{jh} - g_{jk}R_{rh}\right)_{|m|l|n} = d_{mln}\left(g_{rk}R_{jh} - g_{jk}R_{rh}\right). \quad (4.32)$$

Therefore, we can say

**Theorem 4.7.** *In the context of $G^{2nd}\,R_{|h}-BRF_n$, associate tensor $R_{rjkh}$ (Cartan's 3[th] curvature tensor $R^i_{jkh}$) represents a generalized trirecurrent Finsler space, if and only if the condition (4.32), is satisfied.*

It is known that Cartan's 3[th] curvature tensor $R^i_{jkh}$ and Cartan's 4[th] curvature tensor $K^i_{jkh}$ are connected by the formula

$$R^i_{jkh} = K^i_{jkh} + C^i_{jp}\,H^p_{kh}. \tag{4.33}$$

By taking the $h-$ covariant derivative of (4.33), with respcet to $x^m$, $x^l$ and $x^n$, we get

$$R^i_{jkh|m|l|n} = K^i_{jkh|m|l|n} + \left(C^i_{jp}\,H^p_{kh}\right)_{|m|l|n}. \tag{4.34}$$

By substituting equations (4.3) and (4.33) in to (4.34), we obtain:

$$\begin{aligned} K^i_{jkh|m|l|n} &= d_{mln}K^i_{jkh} + e_{mln}\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right) + \tfrac{1}{4}v_{mln}\left(W^i_h g_{jk} - W^i_k g_{jh}\right) \\ &+ \tfrac{1}{4}c_{ml}\left(W^i_h g_{jk} - W^i_k g_{jh}\right)_{|n} + \tfrac{1}{4}\eta_{mn}\left(W^i_h g_{jk} - W^i_k g_{jh}\right)_{|l} + \tfrac{1}{4}\gamma_m\left(W^i_h g_{jk} - W^i_k g_{jh}\right)_{|l|n} \\ &-\left(C^i_{jp}H^p_{kh}\right)_{|m|l|n} + d_{mln}(C^i_{jp}\,H^p_{kh}) + \tfrac{1}{3}d_{mln}\left(\delta^i_k\,R_{jh} - g_{jk}\,R^i_h\right) \\ &+ \tfrac{1}{3}\left(\delta^i_k\,R_{jh} - g_{jk}\,R^i_h\right)_{|m|l|n}. \end{aligned} \tag{4.35}$$

This demonstrates that

$$\begin{aligned} K^i_{jkh|m|l|n} &= d_{mln}K^i_{jkh} + e_{mln}\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right) + \tfrac{1}{4}v_{mln}\left(W^i_h g_{jk} - W^i_k g_{jh}\right) \\ &+ \tfrac{1}{4}c_{ml}\left(W^i_h g_{jk} - W^i_k g_{jh}\right)_{|n} + \tfrac{1}{4}\eta_{mn}\left(W^i_h g_{jk} - W^i_k g_{jh}\right)_{|l} + \tfrac{1}{4}\gamma_m\left(W^i_h g_{jk} - W^i_k g_{jh}\right)_{|l|n}. \end{aligned} \tag{4.36}$$

If and only if

$$\left(\delta^i_k\,R_{jh} - g_{jk}\,R^i_h\right)_{|m|l|n} = d_{mln}\left(\delta^i_k\,R_{jh} - g_{jk}\,R^i_h\right). \tag{4.37}$$

and

$$\left(C^i_{jp}\,H^p_{kh}\right)_{|m|l|n} = d_{mln}(\,C^i_{jp}\,H^p_{kh}). \tag{4.38}$$

Therefore, we can say

**Theorem 4.8.** *In the context of $G^{2nd}R_{|h}-TRF_n$, Cartan's 4[th] curvature tensor $K^i_{jkh}$ defines a generalized trirecurrent Finsler space if and only if the tensors $\left(\delta^i_k\,R_{jh} - g_{jk}\,R^i_h\right)$ and $\left(C^i_{jp}H^p_{kh}\right)$ are a generalized trirecurrent Finsler space.*

By contracting the indices i and h in equation (4.35) and utilizing equations (1.1d), (1.1b), (1.13e), (1.14e), (1.12c) and (1.12b), we obtain the following result:

$$\begin{aligned} K_{jk|m|l|n} &= d_{mln}K_{jk} + (1-n)e_{mln}g_{jk} + \tfrac{1}{4}v_{mln}W_{jk} + \tfrac{1}{4}c_{ml}W_{jk|n} + \tfrac{1}{4}\eta_{mn}W_{jk|l} + \tfrac{1}{4}\gamma_m W_{jk|l|n} \\ &-\left(C^i_{jp}\,H^p_{ki}\right)_{|m|l|n} + d_{mln}\,C^i_{jp}\,H^p_{ki} + \tfrac{1}{3}d_{mln}\left(R_{jk} - g_{jk}\,R\right) + \tfrac{1}{3}\left(R_{jk} - g_{jk}\,R\right)_{|m|l|n}. \end{aligned} \tag{4.39}$$

This demonstrates that

$$\begin{aligned} K_{jk|m|l|n} &= d_{mln}K_{jk} + (1-n)e_{mln}g_{jk} + \tfrac{1}{4}v_{mln}W_{jk} + \tfrac{1}{4}c_{ml}W_{jk|n} + \tfrac{1}{4}\eta_{mn}W_{jk|l} \\ &+ \tfrac{1}{4}\gamma_m W_{jk|l|n}. \end{aligned} \tag{4.40}$$

If and only if

$$\left(R_{jk} - g_{jk}\,R\right)_{|m|l|n} = d_{mln}\left(R_{jk} - g_{jk}\,R\right). \tag{4.41}$$

and

$$\left(C^i_{jp}\,H^p_{ki}\right)_{|m|l|n} = d_{mln}\,C^i_{jp}\,H^p_{ki}. \tag{4.42}$$

By transvecting condition (4.39) with $y^k$, and applying relations (1.1a), (1.1c), (1.4b), (1.12a), (1.12e), (1.14c) and (1.13d), we obtain the following result

$$\begin{aligned} K_{j|m|l|n} &= d_{mln}\mathrm{K}_j + (1-n)e_{mln}\,y_j + \tfrac{1}{3}\left(R_j - y_j\,R\right)_{|m|l|n} + \tfrac{1}{3}d_{mln}\left(R_j - y_j\,R\right) \\ &-\left(C^i_{jp}\,H^p_i\right)_{|m|l|n} + d_{mln}\,C^i_{jp}\,H^p_i. \end{aligned} \tag{4.43}$$

This demonstrates that

$$K_{j|m|l|n} = d_{mln}\mathrm{K}_j + (1-n)e_{mln}\,y_j. \tag{4.44}$$

If and only if

$$\left(R_j - y_j\,R\right)_{|m|l|n} = d_{mln}\left(R_j - y_j\,R\right). \tag{4.45}$$

and

$$\left(C^i_{jp}\,H^p_i\right)_{|m|l|n} = d_{mln}\,C^i_{jp}\,H^p_i. \tag{4.46}$$

Therefore, we conclude

**Theorem 4.9.** *In the context of $G^{2nd}$ $R_{|h} - TRF_n$ , the K−Ricci tensor $K_{jk}$ and curvature vector $K_j$ are defined in equations (4.40) and (4.44), respectively, provided that the conditions (4.41), (4.42), (4.45) and (4.46) are satisfied.*

By transvecting conditions (4.35) and (4.39) with $g^{jk}$, respectively, and applying relations (1.2a), (1.2b), (1.12d), (1.14d) and (1.14f), we obtain:

$$K^i_{h|m|l|n} = d_{mln}K^i_h + (n-1)e_{mln}\delta^i_h + \tfrac{1}{4}(n-1)v_{mln}W^i_h + \tfrac{1}{4}(n-1)c_{ml}W^i_{h|n}$$
$$+\tfrac{1}{4}(n-1)\eta_{mn}W^i_{h|l} + \tfrac{1}{4}(n-1)\gamma_m W^i_{h|l|n} - g^{jk}\big(C^i_{jp}H^p_{kh}\big)_{|m|l|n} + d_{mln}g^{jk}\big(C^i_{jp}H^p_{kh}\big)$$
$$+\tfrac{1}{3}(1-n)d_{mln}\big(R^i_h\big) + \tfrac{1}{3}(1-n)\big(R^i_h\big)_{|m|l|n} . \qquad (4.47)$$

This demonstrates that

$$K^i_{h|m|l|n} = d_{mln}K^i_h + (n-1)e_{mln}\delta^i_h + \tfrac{1}{4}(n-1)v_{mln}W^i_h + \tfrac{1}{4}(n-1)c_{ml}W^i_{h|n}$$
$$+\tfrac{1}{4}(n-1)\eta_{mn}W^i_{h|l} + \tfrac{1}{4}(n-1)\gamma_m W^i_{h|l|n} . \qquad (4.48)$$

If and only if

$$\big(C^i_{jp}\, H^p_{kh}\big)_{|m|l|n} = d_{mln}\, (C^i_{jp}\, H^p_{kh}), \text{ where } g^{jk} \neq 0 . \qquad (4.49)$$

$$R^i_{h|m|l|n} = d_{mln}\, R^i_h. \qquad (4.50)$$

$$K_{|m|l|n} = d_{mln}K + (1-n)ne_{mln} + \tfrac{1}{4}v_{mln}W + \tfrac{1}{4}c_{ml}W_{|n} + \tfrac{1}{4}\eta_{mn}W_{|l} + \tfrac{1}{4}\gamma_m W_{|l|n}$$
$$-g^{jk}\big(C^i_{jp}\, H^p_{ki}\big)_{|m|l|n} + d_{mln}\, g^{jk}(C^i_{jp}\, H^p_{kh}) + \tfrac{1}{3}(1-n)d_{mln}R + \tfrac{1}{3}(1-n)\, R_{|m|l|n} . \qquad (4.51)$$

This demonstrates that

$$K_{|m|l|n} = d_{mln}K + (1-n)ne_{mln} + \tfrac{1}{4}v_{mln}W + \tfrac{1}{4}c_{ml}W_{|n} + \tfrac{1}{4}\eta_{mn}W_{|l} + \tfrac{1}{4}\gamma_m W_{|l|n} . \qquad (4.52)$$

If and only if

$$\big(C^i_{jp}\, H^p_{kh}\big)_{|m|l|n} = d_{mln}\, \big(C^i_{jp}\, H^p_{kh}\big), \text{ where } g^{jk} \neq 0, \text{ and} \qquad (4.53)$$

$$R_{|m|l|n} = d_{mln}\, R. \qquad (4.54)$$

Therefore, the proof of theorem is completed, we conclude

**Theorem 4.10.** *In the context of $G^{2nd}$ $R_{|h} - TRF_n$ , the projective deviation tensor $K^i_h$ and scalar $K$ are defined in equations (4.48) and (4.52), respectively, provided that the conditions (4.49), (4.50), (4.53) and (4.54) are satisfied.*

By transvecting condition (4.35) by $g_{ir}$, applying relations (1.1d), (1.2c), (1.3c), (1.12c), (1.13f) and (1.14a), we obtain

$$K_{rjkh|m|l|n} = d_{mln}K_{rjkh} + e_{mln}\big(g_{rk}\, g_{jh} - g_{rh}\, g_{jk}\big) + \tfrac{1}{4}v_{mln}\big(W_{rk}g_{jh} - W_{rh}g_{jk}\big)$$
$$+\tfrac{1}{4}c_{ml}\big(W_{rk}g_{jh} - W_{rh}g_{jk}\big)_{|n} + \tfrac{1}{4}\eta_{mn}\big(W_{rk}g_{jh} - W_{rh}g_{jk}\big)_{|l} + \tfrac{1}{4}\gamma_m\big(W_{rk}g_{jh} - W_{rh}g_{jk}\big)_{|l|n}$$
$$-\big(C_{rjp}H^p_{kh}\big)_{|m|l|n} + d_{mln}(C_{rjp}H^p_{kh}) + \tfrac{1}{3}d_{mln}\big(g_{rk}\, R_{jh} - g_{jk}\, R_{rh}\big)$$
$$+\tfrac{1}{3}\big(g_{rk}\, R_{jh} - g_{jk}\, R_{rh}\big)_{|m|l|n}. \qquad (4.55)$$

This demonstrates that

$$K_{rjkh|m|l|n} = d_{mln}K_{rjkh} + e_{mln}\big(g_{rk}\, g_{jh} - g_{rh}\, g_{jk}\big) + \tfrac{1}{4}v_{mln}\big(W_{rk}g_{jh} - W_{rh}g_{jk}\big)$$
$$+\tfrac{1}{4}c_{ml}\big(W_{rk}g_{jh} - W_{rh}g_{jk}\big)_{|n} + \tfrac{1}{4}\eta_{mn}\big(W_{rk}g_{jh} - W_{rh}g_{jk}\big)_{|l}$$
$$+\tfrac{1}{4}\gamma_m\big(W_{rk}g_{jh} - W_{rh}g_{jk}\big)_{|l|n}. \qquad (4.56)$$

If and only if

$$\big(g_{rk}\, R_{jh} - g_{jk}\, R_{rh}\big)_{|m|l|n} = d_{mln}\big(g_{rk}\, R_{jh} - g_{jk}\, R_{rh}\big). \qquad (4.57)$$

$$\big(C_{rjp}H^p_{kh}\big)_{|m|l|n} = d_{mln}\big(C_{rjp}H^p_{kh}\big). \qquad (4.58)$$

Therefore, we can say

**Theorem 4.11.** *In the context of $G^{2nd}$ $R_{|h} - BRF_n$ , associate tensor $K_{rjkh}$ (Cartan's 4th curvature tensor $K^i_{jkh}$) represents a generalized trirecurrent Finsler space, if and only if the condition (4.57) and (4.58) are satisfied.*

It is known that Cartan's 4th curvature tensor $K^i_{jkh}$ and Berwald curvature tensor $H^i_{jkh}$ are connected by the formula

$$H^i_{jkh} = K^i_{jkh} + y^s\big(\dot\partial_j K^i_{skh}\big). \qquad (4.59)$$

By taking the $h$ −covariant derivative of (4.59), with respect to $x^m$, $x^l$ and $x^n$, using (1.4b), we get

$$H^i_{jkh|m|l|n} = K^i_{jkh|m|l|n} + y^s\big(\dot\partial_j K^i_{skh}\big)_{|m|l|n}. \qquad (4.60)$$

By substituting equations (4.35) and (4.59) in to (4.60), we obtain:

$$H^i_{jkh|m|l|n} = d_{mln}H^i_{jkh} + e_{mln}\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right) + \tfrac{1}{4}v_{mln}\left(W^i_h g_{jk} - W^i_k g_{jh}\right)$$
$$+\tfrac{1}{4}c_{ml}\left(W^i_h g_{jk} - W^i_k g_{jh}\right)_{|n} + \tfrac{1}{4}\eta_{mn}\left(W^i_h g_{jk} - W^i_k g_{jh}\right)_{|l} + \tfrac{1}{4}\gamma_m\left(W^i_h g_{jk} - W^i_k g_{jh}\right)_{|l|n}$$
$$-\left(C^i_{jp}\, H^p_{kh}\right)_{|m|l|n} + d_{mln}\, C^i_{jp}\, H^p_{kh} + \tfrac{1}{3}d_{mln}\left(\delta^i_k R_{jh} - g_{jk} R^i_h\right) + \tfrac{1}{3}\left(\delta^i_k R_{jh} - g_{jk} R^i_h\right)_{|m|l|n}$$
$$+y^s\left(\dot{\partial}_j \mathrm{K}^i_{skh}\right)_{|m|l|n} - d_{mln}y^s\left(\dot{\partial}_j \mathrm{K}^i_{skh}\right). \quad (4.61)$$

This demonstrates that

$$H^i_{jkh|m|l|n} = d_{mln}H^i_{jkh} + e_{mln}\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right) + \tfrac{1}{4}v_{mln}\left(W^i_h g_{jk} - W^i_k g_{jh}\right)$$
$$+\tfrac{1}{4}c_{ml}\left(W^i_h g_{jk} - W^i_k g_{jh}\right)_{|n} + \tfrac{1}{4}\eta_{mn}\left(W^i_h g_{jk} - W^i_k g_{jh}\right)_{|l} + \tfrac{1}{4}\gamma_m\left(W^i_h g_{jk} - W^i_k g_{jh}\right)_{|l|n}. \quad (4.62)$$

If and only if

$$\left(\delta^i_k R_{jh} - g_{jk} R^i_h\right)_{|m|l|n} = d_{mln}\left(\delta^i_k R_{jh} - g_{jk} R^i_h\right), \quad (4.63)$$

$$\left(C^i_{jp}\, H^p_{kh}\right)_{|m|l|n} = d_{mln}\, C^i_{jp}\, H^p_{kh}. \quad (4.64)$$

and

$$\left(\dot{\partial}_j \mathrm{K}^i_{skh}\right)_{|m|l|n} = d_{mln}\left(\dot{\partial}_j \mathrm{K}^i_{skh}\right), \text{ where } y^s \neq 0. \quad (4.65)$$

Therefore, we can say

**Theorem 4.12.** *In the context of $G^{2nd}R_{|h} - TRF_n$ , Berwald curvature tensor $H^i_{jkh}$ defines a generalized trirecurrent Finsler space if and only if the tensors $\left(\delta^i_k R_{jh} - g_{jk} R^i_h\right)$, $\left(C^i_{jp}H^p_{kh}\right)$ and $\left(\dot{\partial}_j K^i_{skh}\right)$ are a generalized trirecurrent Finsler space.*

By contracting the indices i and h in equation (4.61) and utilizing equations (1.1d), (1.1b), (1.13e), (1.13m), (1.14e), (1.12c) and (1.12b), we obtain the following result:

$$H_{jk|m|l|n} = d_{mln}H_{jk} + (1-n)e_{mln}g_{jk} + \tfrac{1}{4}v_{mln}W_{jk} + \tfrac{1}{4}c_{ml}W_{jk|n} + \tfrac{1}{4}\eta_{mn}W_{jk|l}$$
$$+\tfrac{1}{4}\gamma_m W_{jk|l|n} - \left(C^i_{jp}\, H^p_{ki}\right)_{|m|l|n} + d_{mln}\, C^i_{jp}\, H^p_{ki} + \tfrac{1}{3}d_{mln}\left(R_{jk} - g_{jk} R\right)$$
$$+\tfrac{1}{3}\left(R_{jk} - g_{jk} R\right)_{|m|l|n} + y^s\left(\dot{\partial}_j \mathrm{K}_{sk}\right)_{|m|l|n} - d_{mln}y^s\left(\dot{\partial}_j \mathrm{K}_{sk}\right). \quad (4.66)$$

This demonstrates that

$$H_{jk|m|l|n} = d_{mln}H_{jk} + (1-n)e_{mln}g_{jk} + \tfrac{1}{4}v_{mln}W_{jk} + \tfrac{1}{4}c_{ml}W_{jk|n} + \tfrac{1}{4}\eta_{mn}W_{jk|l}$$
$$+\tfrac{1}{4}\gamma_m W_{jk|l|n}. \quad (4.67)$$

If and only if

$$\left(R_{jk} - g_{jk} R\right)_{|m|l|n} = d_{mln}\left(R_{jk} - g_{jk} R\right), \quad (4.68)$$

$$\left(C^i_{jp}\, H^p_{ki}\right)_{|m|l|n} = d_{mln}\, C^i_{jp}\, H^p_{ki}, \quad (4.69)$$

$$\left(\dot{\partial}_j \mathrm{K}_{sk}\right)_{|m|l|n} = d_{mln}y^s\left(\dot{\partial}_j \mathrm{K}_{sk}\right), \text{ where } y^s \neq 0. \quad (4.70)$$

Therefore, we can say

**Theorem 4.13.** *In the context of $G^{2nd}R_{|h} - TRF_n$ , Berwald $H-$Ricci tensor $H_{jk}$ defines a generalized trirecurrent Finsler space if and only if the tensors $\left(R_{jk} - g_{jk} R\right)$,$\left(C^i_{jp}H^p_{ki}\right)$ and $\left(\dot{\partial}_j K_{sk}\right)$ are a generalized trirecurrent Finsler space.*

By contracting the indices i and h in equation (4.61) and utilizing equations (1.1d), (1.1b), (1.13e), (1.12c) and (1.13*l*), we obtain the following result:

$$(H_{hk} - H_{kh})_{|m|l|n} = d_{mln}(H_{hk} - H_{kh}) + \tfrac{1}{4}v_{mln}(W_{hk} - W_{kh}) + \tfrac{1}{4}c_{ml}(W_{hk} - W_{kh})_{|n}$$
$$+\tfrac{1}{4}\eta_{mn}(W_{hk} - W_{kh})_{|l} + \tfrac{1}{4}\gamma_m(W_{hk} - W_{kh})_{|l|n} - \left(C^i_{ip}H^p_{kh}\right)_{|m|l|n} + d_{mln}\left(C^i_{ip}H^p_{kh}\right)$$
$$+y^s\left(\dot{\partial}_i \mathrm{K}^i_{skh}\right)_{|m|l|n} - d_{mln}y^s\left(\dot{\partial}_i \mathrm{K}^i_{skh}\right). \quad (4.71)$$

This demonstrates that

$$(H_{hk} - H_{kh})_{|m|l|n} = d_{mln}(H_{hk} - H_{kh}) + \tfrac{1}{4}v_{mln}(W_{hk} - W_{kh}) + \tfrac{1}{4}c_{ml}(W_{hk} - W_{kh})_{|n}$$
$$+\tfrac{1}{4}\eta_{mn}(W_{hk} - W_{kh})_{|l} + \tfrac{1}{4}\gamma_m(W_{hk} - W_{kh})_{|l|n}. \quad (4.72)$$

If and only if

$$\left(C^i_{ip}\, H^p_{kh}\right)_{|m|l|n} = d_{mln}\left(C^i_{ip}\, H^p_{kh}\right). \quad (4.73)$$

$$\left(\dot{\partial}_i \mathrm{K}^i_{skh}\right)_{|m|l|n} = d_{mln}\left(\dot{\partial}_i \mathrm{K}^i_{skh}\right), \text{ where } y^s \neq 0. \quad (4.74)$$

Therefore, we can say

**Theorem 4.14.** *In the context of $G^{2nd}R_{|h} - TRF_n$, the tensor $(H_{hk} - H_{kh})$ defines a generalized trirecurrent Finsler space if and only if the tensors $(C^i_{ip}H^p_{kh})$ and $(\dot{\partial}_\iota K^i_{skh})$ are a generalized trirecurrent Finsler space.*

By taking the $h-$ covariant derivative of (1.14h), with respect to $x^m$, $x^l$ and $x^n$, using (1.4b), we get

$$H^i_{jkh|m|l|n} - K^i_{jkh|m|l|n} = \left(P^i_{jk|h} + P^r_{jk}P^i_{rh} - h/k\right)_{|m|l|n}. \quad (4.75)$$

From the equations (4.35) and (4.61), we get

$$H^i_{jkh|m|l|n} - K^i_{jkh|m|l|n} = d_{mln}(H^i_{jkh} - K^i_{jkh}) + y^s(\dot{\partial}_J K^i_{skh})_{|m|l|n} - d_{mln}y^s(\dot{\partial}_J K^i_{skh}). \quad (4.76)$$

By substituting equations (4.76) and (1.14h) in to (4.75), we obtain:

$$\left(P^i_{jk|h} + P^r_{jk}P^i_{rh} - \frac{h}{k}\right)_{|m|l|n} = d_{mln}\left(P^i_{jk|h} + P^r_{jk}P^i_{rh} - \frac{h}{k}\right) + y^s(\dot{\partial}_J K^i_{skh})_{|m|l|n}$$
$$-d_{mln}y^s(\dot{\partial}_J K^i_{skh}). \quad (4.77)$$

This demonstrates that

$$\left(P^i_{jk|h} + P^r_{jk}P^i_{rh} - \frac{h}{k}\right)_{|m|l|n} = d_{mln}\left(P^i_{jk|h} + P^r_{jk}P^i_{rh} - \frac{h}{k}\right). \quad (4.78)$$

If and only if

$$(\dot{\partial}_\iota K^i_{skh})_{|m|l|n} = d_{mln}(\dot{\partial}_\iota K^i_{skh}), \text{ where } y^s \neq 0. \quad (4.79)$$

Thus, we get

**Theorem 4.15.** *In $G^{2nd}\, W_{|h} - TRF_n$, the tensor $\left(P^i_{jk|h} + P^r_{jk}P^i_{rh} - \frac{h}{k}\right)$ is a generalized trirecurrent Finsler space if and only if the tensor $(\dot{\partial}_\iota K^i_{skh})$ is a generalized trirecurrent Finsler space.*

## 5. Conclusions

In this work, we introduced and studied a new class of Finsler spaces, referred to as generalized $W_{|h}$-trirecurrent Finsler spaces. This class extends the concept of trirecurrence in the context of Finsler geometry by exploring the third-order h-covariant derivatives of the Weyl's projective curvature tensor. We obtained expressions for the corresponding Ricci tensors, scalar curvatures, and higher-order derivatives. These expressions revealed that the scalar curvature $W$ in this context is non-vanishing. This paper focus on the conditions for some tensors which satisfy the trirecurrence property in the main space ($G^{2nd}R_{|h} - TRF_n$). These results can have potential implications in the geometric formulation of physical theories, particularly in gravitational and cosmological models where higher-order curvature interactions play a crucial role.

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