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Hahn Banach Theorem And Its Applications

Chetna Rani Gupta

Assistant Prof. in Mathematics, Multani Mal Modi College, Patiala

ABSTRACT:

One of the most significant theorems in functional analysis is the Hahn-Banach theorem. It has several uses in various areas of mathematics, including partial differential equations and optimization, in addition to the subject itself. This paper provides a thorough and understandable demonstration of the theorem along with a few examples of its applications

Keywords: Hahn-Banach Theorem, Functional, Normed Linear Space

1. Introduction

The Hahn-Banach theorem for every normed space and its applications are covered in this paper.. The theorem states that by maintaining norms, any continuous linear functional on a subspace can be extended to the entire space. Among many other areas, it can be applied to game theory, convex programming, duality theory, and control theory.[3]

2. Hahn-Banach Theorem [1] [2] [5]

If p is a functional defined on U, and U is a linear subspace of a Normed linear space V, then p can be extended to a functional p_0 on the whole space V in such a way that

$$p_0(t) = p(t) \text{ for all } t \in U \text{ and } ||p_0|| = ||p||$$

First, do the following to obtain the result, which is required to prove the theorem.

If p is a functional defined on U, and U is a linear subspace of a Normed linear space V if t_0 be a vector not in U and if $U_0 = U + \{t_0\} = \{u + \alpha t_0 \ , u \in U \, , \alpha \text{ is real number}\}$ is a linear space spanned by U and t_0 Consequently, a functional p_0 defined onU_0 can be obtained by extending the functional p such that $||p_0|| = ||p||$ **[2] [5]**

Proof: Case 1

Let V be a real linear space with norms.

Since t_0 be a vector not in U, that is $t_0 \notin U$

Then each vector z (say) of U can be uniquely expressed as $z = u + \alpha t_0$ where $u \in U$

Define p_0 on U_0

$$\begin{aligned} p_0(z) &= p_0(u + \alpha t_0) \\ &= p_0(u) + \alpha p_0(t_0) \\ &= p(u) + \alpha r_0 \end{aligned}$$

Where $r_0 = p_0(t_0)$ and $p_0(u) = p(u)$ for all $u \in U$

Clearly p_0 is a linear extension of p over U

Now remain to show that $||p_0|| = ||p||$

That is $||p_0|| \geq ||p||$and $||p_0|| \leq ||p||$

First, show that $||p_0|| \geq ||p||$

Now $||p_0|| = \sup\{|p_0(t)| \; : u \in U_0 \;, ||x|| \leq 1\}$

$\geq \sup\{|p_0(t)| \; : u \in U \;, ||x|| \leq 1\}$

$= \sup\{|p(t)| \; : u \in U \;, ||x|| \leq 1\}$

$= ||\,p||$

Thus $||p_0|| \geq ||\,p||$

Also show that $||\boldsymbol{p_0}|| \leq ||\,\boldsymbol{p}||$

For this choose a real numberr_0 in such a way so that $||p_0|| = ||\,p||$

[Since between two real numbers a, b (say) there always exists a real number r_0 (say) such that $a \leq r_0 \leq b$]

For this purpose, proceed as follows

If $u_1 \;, u_2 \in U$ then

$p(u_1) - p(u_2) = p(\,u_2 - u_1\,)$

$\leq |p(\,u_2 - u_1\,)|$

$\leq ||\,p||\;||(\,u_2 - u_1\,)||$

$= ||\,p||\;||(\,u_2 + t_0) - (u_1\; + t_0||)$

$\leq ||\,p||\;||(\,u_2 + t_0)|| + ||\,p||\;||(u_1\; + t_0||)$

$-p(u_1) - ||\,p||\,(||u_1 + t_0||) \leq -\,p(u_2) + ||\,p||\,(\big||\,u_2 + t_0|\big|)$

Which holds for all $u_1 \;, u_2 \in U$

$sup\{-p(u') - ||\,p||\,(||u' + t_0||)\} \leq inf\left\{-p(u') + ||\,p||\,(\big||\,u' + t_0|\big|)\right\}$

Take $a = sup\{-p(u') - ||\,p||\,(||u' + t_0||)\}$ and $b = inf\left\{-p(u') + ||\,p||\,(\big||\,u' + t_0|\big|)\right\}$

Then $a \leq b$ and a, b are real numbers

so can choose $r_{0,}$ a real number such that

$$a \leq r_0 \leq b \text{ for all } a, b, r_0$$

$sup\{-p(u') - ||\,p||\,(||u' + t_0||)\} \leq r_0 \leq inf\left\{-p(u') + ||\,p||\,(\big||\,u' + t_0|\big|)\right\}$

Let $z = u + \alpha t_0 \; where \; u \in U$ where $z \in U$

Take $u' = \frac{u}{\alpha}$ $\quad \alpha \neq 0$

$sup\left\{-p\left(\frac{u}{\alpha}\right) - ||\,p||\left(||\frac{u}{\alpha} + t_0||\right)\right\} \leq r_0 \leq inf\left\{-p\left(\frac{u}{\alpha}\right) + ||\,p||\;\left(||\frac{u}{\alpha} + t_0||\right)\right\}$

When $\alpha > 0$ then

$r_0 \leq -\,p\left(\dfrac{u}{\alpha}\right) + ||\,p||\;\left(||\dfrac{u}{\alpha} + t_0||\right)$

$r_0 \leq -\dfrac{1}{\alpha}p(u) + \dfrac{1}{\alpha}||\,p||\,(||u + \alpha t_0||)$

$\alpha r_0 \leq -\,p(u) + ||\,p||\,(||u + \alpha t_0||)$

$$p(u) + \alpha r_0 \leq ||\,p||\,(||u + \alpha t_0||)$$

$$p(u) + \alpha p_0(t_0) \leq ||\,p||\,(||u + \alpha t_0||)$$

$$p_0(u) + \alpha p_0(t_0) \leq ||\,p||\,(||u + \alpha t_0||)$$

$$p_0(u + \alpha t_0) \leq ||\,p||\,(||u + \alpha t_0||)$$

$$p_0(z) \leq ||\,p||\,(||u + \alpha t_0||) \qquad \text{for all } z \in U_0$$

When $\alpha < 0$ then

$r_0 \geq -\,p\left(\dfrac{u}{\alpha}\right) - ||\,p||\;\left(||\dfrac{u}{\alpha} + t_0||\right)$

$r_0 \geq -\dfrac{1}{\alpha}p(u) - \dfrac{1}{|\alpha|}||\,p||\,(||u + \alpha t_0||)$

$r_0 \geq -\dfrac{1}{\alpha}p(u) + \dfrac{1}{\alpha}||\,p||\,(||u + \alpha t_0||)$

$\alpha r_0 \leq - p(u) + || p|| (||u + \alpha t_0||)$

$p(u) + \alpha r_0 \leq || p|| (||u + \alpha t_0||)$

$$p(u) + \alpha p_0(t_0) \leq || p|| (||u + \alpha t_0||)$$

$$p_0(u) + \alpha p_0(t_0) \leq || p|| (||u + \alpha t_0||)$$

$$p_0(u + \alpha t_0) \leq || p|| (||u + \alpha t_0||)$$

$$p_0(z) \leq || p|| \, ||z|| \text{ for all } z \in U_0$$

Therefore, for all $\alpha \neq 0$

$$p_0(z) \leq || p|| \, ||z|| \text{ for all } z \in U_0$$

Replace z by $-z$ in the above equation

$$p_0(- z) \leq || p|| \, || - z||$$

$$- p_0(z) \leq || p|| \, ||z||$$

$$|p_0(z)| \leq || p|| \, ||z|| \text{ for all } z \in U_0$$

$sup\{|p_0(z)| : \; \big||z|\big| \leq 1 \} \leq \; || p|| \, ||z|| \text{ for all } z \in U_0$

$$||p_0|| \leq || p||$$

Therefore $||p_0|| = || p||$

Case 2:

Let V be a Complex Normed Linear Space

If the scalars are restricted to real numbers, Complex Linear Space can be regarded as a real linear space.

If q and w be real and imaginary parts of p then

$$p(t) = q(t) + i\, w(t) \quad \text{for all } t \in U$$

Claim: q and w both are linear functional on U

Since P is linear so clearly q and w both are linear

So, Now show that both q and w are continuous on space M

Since $|q(t)| \leq |p(t)|$

$\underset{||t|| \leq 1}{sup} \{|q(t)|\} \leq \underset{||t|| \leq 1}{sup} \{|p(t)|\}$

$$||q|| \leq ||p||$$

Also $|w(t)| \leq |p(t)|$

$\underset{||t|| \leq 1}{sup} \{|w(t)|\} \leq \underset{||t|| \leq 1}{sup} \{|p(t)|\}$

$$||w|| \leq ||p||$$

Both q and w are bounded being p is bounded

So both q and w are continuous on U

Therefore both q and w functional being linear and continuous on U

So by case 1

Both q and w cab be extended to a real valued functional q_0 and w_0 on real space U_0 such that

$$||q_0|| = ||q|| \text{ And } ||w_0|| = ||w||$$

Now as $p(t) = q(t) + i\, w(t)$ for all $t \in U$

For simplification, it is better to express $P(t)$ in terms of $q(t)$ only

Replace t by it

$$p(it) = q(it) + i\, w(it)$$

Also $p(it) = ip(t)$

$= i(q(t) + i\,w(t))$

$= i\big(q(t) + i^2\,w(t)\big)$

$= iq(t) - w(t)$

$q(it) + i\,w(it) = i\,q(t) - w(t)$

Equate real and imaginary part

$q(it) =- w(t))$ and $w(it) = q(t)$

$p(t) = q(t) + i\,w(t)$

$= q(t) - i\,q(it)$ for all $t \in U$

Define $p_0 : U_0 \to \mathbb{C}$ complex number by

$p_0(t) = q_0(t) - iq_0(it)$ for all $t \in U$

Now to show that p_0 is an extension of, one has to show

(i) $p_0(t) = p(t)$ for all $t \in U$

(ii) p_0 is linear as complex valued function on U_0

$p_0(\gamma t) = \gamma\,p_0(t)$ where $\gamma = \alpha + i\beta$ and α, β real numbers are

(iii) $||p_0(t)|| = ||p(t)||$

(i) *First to show $p_0(t) = p(t)$ for all $t \in U$*

Since $q_0(t) = q(t)$ and $w_0(t) = w(t)$ all $t \in U$

$p_0(t) = q_0(t) - iq_0(t)$

$= q(t) - iq(t) = p(t)$

(ii) p_0 is linear as complex valued function on U_0

$p_0(\gamma t) = \gamma\,p_0(t)$ where $\gamma = \alpha + i\beta$ and α, β real numbers are

$p_0(t + x) = q_0(t + x) - iq_0(t + x)$

$= q_0(t) + q_0(x) - iq_0(it) - iq_0(ix)$

$= q_0(t) - iq_0(it) + q_0(ix) - iq_0(ix)$

$= p_0(t) + p_0(x)$

Claim: $p_0(it) = ip_0(t)$

$p_0(it) = q_0(it) - iq_0(i^2 t)$ for all $t \in U$

$= q_0(it) - iq_0(-t)$

$= q_0(it) + iq_0(t)$

$=- i^2 q_0(it) + iq_0(t)$

$= i(-iq_0(it) + iq_0(t))$

$= i(q_0(t) - iq_0(it))$

$= ip_0(t)$

Claim: $p_0(\gamma t) = \gamma\,p_0(t)$

Where $\gamma \in \mathbb{C}$ complex number and $\gamma = \alpha + i\beta$ where α, β real numbers are

$p_0(\gamma t) = p_0((\alpha + i\beta)t)$

$= p_0((\alpha + i\beta)t)$

$= p_0(\alpha t) + p_0(i\beta t)$

$= \alpha\, p_0(t) + i\beta\, p_0(t)$

$= (\alpha + i\beta\,)\, p_0(t)$

$= \gamma\, p_0(t)$

(iii) $||\boldsymbol{p_0(t)}|| = ||\boldsymbol{p(t)}||$

Since $p_0 : U_0 \to \mathbb{C}$

For $t \in U_0$

$p_0(t) = re^{i\theta}$ where $r \geq 0$ and θ is real

$|p_0(t)| = |re^{i\theta}|$

$= |r||e^{i\theta}$

$= |r|$ (since $|e^{i\theta}| = 1$)

$= e^{-i\theta}\, p_0(t)$

$= p_0(e^{-i\theta}t)$

$\leq q_0(e^{-i\theta}t)$

$\leq |q_0(e^{-i\theta}t)|$

$\leq ||q_0||\ ||e^{-i\theta}||\ ||t||$

$= ||q_0|\ ||t||$ (since $|e^{i\theta}| = 1$)

$= ||q||\ |\ ||t||$ (since $||q_0|| = ||q||$ on U_0)

$\leq ||p||\ |\ ||t||$

$\sup\limits_{||t|| \leq 1} \{|\ p_0(t)| : ||t|| \leq 1\} \leq ||p||\ |\ ||t||$

$||p_0|| \leq ||p||$ and p_0 is bounded

Also $||p_0|| \geq ||p||$ already proved

Hence $||p_0|| = ||p||$

Proof of Main Theorem [2] [5]

Let A be the collection of every possible extension h of the linear functional p on every

subspace such that ||h||=||p||

By lemma there exists at least one extension $p_0$such that $||p_0|| = ||p||$

A is non-empty

Let $h_1$and $h_2 \in A$

Define a relation $\leq$ on A as follows

$h_1 \leq h_2$ means $D_{h_1} \subseteq D_{h_2}$ and $h_1(t) = h_2(t)$ for all $t \in D_{h_1}$

where D_{h_1} is the domain of h_1 and D_{h_2} is the domain of h_2

The set of extensions of A is a partially ordered set, as may be readily shown

Let E be any subset of a set A that is totally ordered.

And let S be the union of all the functionals' domains in set E.

If $h_1$and $h_2 \in E$ then either $h_1 \leq h_2$ or $h_1 \geq h_2$

If $t \in D_{h_1}, D_{h_2}$ then $h_1 = h_2$

Define $\varphi : S \to \mathbb{C}$ by

$\varphi(t) = h(t)$ $h \in E$ and $t \in D_{h_1}$

So that φ is an extension of p

$\varphi \in A$ be the collection of all possible extensions h

Also $D_\varphi = S =$ be the union of all domains of all the functional in set E

$D_h \subseteq D_\varphi$ for any $h \in E$

$h \subseteq \varphi$ by defined relation

so that φ is the upper bound for the set E

Therefore, it has been shown that A is partially ordered set in which every totally ordered subset E has an upper bound

According to Zorn's lemma Set A has a maximal element say p_0

Now show that domain of p_0 is the whole space V that is

$D_{p_0} = V$

Suppose, if possible D_{p_0} is proper subset of V

Therefore, there is an element t_0 of V such that it does not belong to D_{p_0}

$M = D_{p_0} + \{t_0\}$ be a linear subspace spanned by D_{p_0} and t_0

So, according to the lemma there exists an extension ω on M of p_0 such that

$\omega(t) = p_0(t)$ for all $t \in D_{p_0}$

And $||p_0|| = ||\omega||$

Also $D_{p_0} \subseteq M = D_\omega$

$p_0 \subseteq \omega$

which contradicts itself

(since p_0 is the maximal element)

Therefore $D_{p_0} = V$

Hence, p can be extended to functional p_0 such that

$p_0(t) = p(t)$ for all $t \in U$ and $||p_0|| = ||p||$

3. Applications

3.1 Theorem: [5] If V is a Normed linear space and t_0 is a non-zero vector in V then there exists a functional p_0 such that $p_0(t_0) = ||t_0||$ and $\|p_0\|=1$. In particular if $t \neq z$ where $t, z \in V$ then there exists an $p_0 \in V^*$ such that $p_0(t) \neq p_0(z)$

Proof: Let U be a subspace spanned by t_0

$$U = \{at_0 : \alpha \text{ be any scalar}\}$$

$$= \{z : z = at_0 : \alpha \text{ be any scalar}\}$$

Define $p : U \to \mathbb{C}$ by

$$p(at_0) = |a|||t_0||$$

To show p is functional, that is p is linear and continuous

(i) p is linear

Let z_1 and $z_2 \in U$ so that $z_1 = at_0$ and $z_2 = bt_0$ a, b be any scalars

Now for any scalars r and s

$$p(rz_1 + sz_2) = p(rat_0 + sbt_0)$$

$$= p((ra + sb)t_0)$$

$$= (ra + sb)||t_0||$$

$$= ra||t_0|| + sb||t_0||$$

$$= r\,p(at_0) + sp(bt_0)$$

$$= rp(z_1) + sp(z_2)$$

$$p(rz_1 + sz_2) = rp(z_1) + sp(z_2)$$

(ii) p is continuous

$$|p(z)| = |p(at_0)|$$

$$= ||a|||t_0|||$$

$$= |a|||t_0||$$

$$= ||z||$$

$$|p(z)| = ||z|| \leq 2||z||$$

p is bounded so continuous

Also $||p|| = \sup\{|p(z)| : ||z||=1\}$

$= \sup\{||z|| : ||z||=1\}$

$= 1$

Also $p(t_0) = ||t_0||$ (Take $a = 1$)

Therefore, by the Theorem the functional p can be extended to a functional p_0 so that $||p_0|| = ||p||$

Since $||p|| = 1$

so $||p_0|| = 1$

In particular

if $t \neq z$ where $t, z \in V$

$$t - z \neq 0$$

then there exists an $p_0 \in V^*$ such that

$$p_0(t - z) = |t - z| \neq 0$$

$$p_0(t) \neq p_0(z)$$

3.2 Theorem: [5] If U be a closed linear space of a Normed linear space V and t_0 is a vector not in V then there exists a functional p_0 in V^* such that $p_0(U) = 0$ $p_0(t) \neq 0$

Proof: Let U be a closed linear subspace of a Normed linear space V then $V/_U$ is Normed linear space with Norm defined as

$$||t + U|| = inf\{||t + u||, u \in U$$

Define a map $T: V \rightarrow V/_U$

By $T(t) = t + U$ for all $t \in V$

Show that T is a continuous linear transformation for which $||T|| < 1$

***T* is linear**

Let z_1 and $z_2 \in U$ and a, b be two scalars

$$T(az_1 + bz_2) = (az_1 + bz_2) + U$$

$$= (az_1 + U) + (bz_2 + U)$$

$$= (az_1 + U) + (bz_2 + U)$$

$$= a(z_1 + U) + b(z_2 + U)$$

$$= aT(z_1) + bT(z_2)$$

Thus $T(az_1 + bz_2) = aT(z_1) + bT(z_2)$

***T* is Continuous**

$$||T(t)|| = ||t + U||$$
$$= inf\{ \left||t + U|\right| \ u \in U\}$$
$$\leq ||t + U|| \quad u \in U$$

In particular choose $u = 0$

$$||T(t)|| \leq ||t||$$

T is bounded so continuous

Also $$||T|| = sup\{ \left||Tt|\right| \quad , t \in V \quad ||t|| \leq 1$$
$$\leq \sup\{ \left||t|\right| \quad , t \in V \quad \left||t|\right| \leq 1\} = 1$$
$$||T|| \leq 1$$

Claim: $T(t_0)$ is non zero vector in ${}^{V}/_{U}$ where $t_0 \notin U$

If $u \in U$ then $u + U = U$

$$T(u) = u + U$$
$$= U = \text{zero of } {}^{V}/_{U}$$

Also $t_0 \notin U$

$$T(t_0) = t_0 + U \neq U \text{ (zero of } {}^{V}/_{U})$$

Therefore, $T(t_0)$ is non zero vector in ${}^{V}/_{U}$ where $t_0 \notin U$

(Since If V is a Normed linear space and t_0 is a non-zero vector in V then there exists a functional p_0 such that $p_0(t_0) = ||t_0||$ and $\|p_0\|$=1)

there exists a functional p in V^*such that

$$p(t_0 + U) = ||t_0 + U|| \neq 0 \text{ and } \|p\|=1$$

$p_0 = poT: V \to \mathbb{C} \ or \ \mathbb{R}$ (since $T: V \to {}^{V}/_{U}$ and $p: {}^{V}/_{U} \to \mathbb{C} \ or \ \mathbb{R}$)

Define $p_0 : V \to \mathbb{R} \ or \ \mathbb{C}$ by

$$p_0(t) = (poT)(t)$$
$$= p(T(t))$$
$$= p(t + U) \quad \text{for all } t \in V$$

Now show that p_0 is functional on V

$$p_0(az_1 + bz_2) = p_0(az_1 + bz_2 + U)$$
$$= p_0(az_1 + U) + p_0(bz_2 + U)$$
$$= ap_0(z_1 + U) + bp_0(z_2 + U)$$
$$= ap_0(z_1) + bp_0(z_2)$$
$$p_0(az_1 + bz_2) = ap_0(z_1) + p_0T(z_2)$$

p_0 is continuous on V

p is bounded being functional so p_0 is also bounded and therefore continuous on V

Now to show that

$p_0(U)$ =0 and $p_0(t) \neq 0$

$$p_0(u) = p(T(u)) = p(u + U) = 0 \text{ for all } u \in U \text{ so that}$$
$$p_0(U) = 0$$

and $$p_0(t_0) = p_0(T(t_0)) \quad \text{where} \quad t_0 \notin U$$
$$= p_0(t_0 + U) \neq 0$$
$$= ||t_0 + U|| \neq 0$$

Thus $\qquad p_0(t_0) \neq 0$

3.3 Theorem: [5] If U be a closed linear space of a Normed linear space V and t_0 is a vector not in U if d is the distance from t_0 to U then show that there exists a functional p_0 in V^*such that such that $p_0(u)$ =0 for all $u \in U$, $p_0(t_0) = 1$ and $||p_0|| = \frac{1}{d}$

Proof: First show the required result is true for P and then apply Hahn Banach Theorem

Consider the space

$$U_0 = U + \{t_0\} = \{u + at_0 , t_0 \notin U \text{ and a is any scalar}\}$$

Cleary U_0 is subspace of V every element $u \in U_0$ is uniquely expressible as $z = u + at_0$

Define $\quad p: U_0 \rightarrow \mathbb{R}\, or\, \mathbb{C}$ by

$$p(z) = a \text{ such that } |\, p(z)| = |a|$$

Clearly p is linear

$$p(u) = (u + 0\, t_0) = 0 \text{ for all } u \in U$$

$$p(t_0) = p(0 + 1. t_0) = 1$$

Now show $\qquad ||p|| = \frac{1}{d} \qquad$ where d is the distance from t_0 to $U, t_0 \notin U$

$$= \inf\{||u - t_0||, u \in U$$

$$||p|| = \frac{1}{d} = \sup\{|p(z)| \qquad ||z|| \leq 1 \;, z \in U_0 \text{ and } z \neq 0\}$$

$$= \sup\{ \frac{|p(z)|}{||z||} \qquad ||z|| \leq 1 \;, z \in U_0 \text{, and } z \neq 0\}$$

$$= \sup\{ \frac{|a|}{||u+at_0||} \qquad ||z|| \leq 1 \;, z \in U_0 \text{, and } z \neq 0\}$$

$$= \sup\{ \frac{|a|}{|a|\, ||\frac{z}{a}+t_0||} \qquad ||z|| \leq 1, \; \frac{z}{a} \in U \;, z \in U_0 \text{, and } a \neq 0\}$$

$$= \sup\{ \frac{1}{||-(-\frac{z}{a})+t_0||} \qquad ||z|| \leq 1 \;, \frac{z}{a} \in U \;, z \in U_0 \text{, and } z \neq 0\}$$

$$= \{ \frac{1}{\inf ||t_0 - y||} \qquad y = \frac{z}{a} \in U \;, z \in U_0 \text{, and } z \neq 0\}$$

$$= \frac{1}{d}$$

Therefore p a functional on U_0such that $p(u) = 0$ for all $u \in U$, $p(t_0) = 1$ and $||p|| = \frac{1}{d}$

So by Hahn Banach theorem p can be extended to a functional p_0 in V^*such that

$$p_0(u) = p(u) \text{ for all } u \in U_0 \text{ and } ||p_0|| = ||p||$$

,So $\qquad p_0(u) = 0$ for all $u \in U \qquad$ (since $U \subset U_0$)

$$||p_0|| = \frac{1}{d} \qquad \text{(since } ||p_0|| = ||p|| = \frac{1}{d}\text{)}$$

$$p_0: V \rightarrow \mathbb{C} \text{ is extension of } p: U_0 \rightarrow \mathbb{C}$$

$$p_0(u) = p(u) \text{ for all } u \in U_0$$

When $t_0 \in U_0$

$$p_0(t_0) = p(u) = 1$$

$$p_0(t_0) = 1$$

Hence , if U be a closed linear space of a Normed linear space V and t_0 is a vector not in U if d is the distance from t_0 to U then there exists a functional p_0 in V^*such that such that $p_0(u) = 0$ for all $u \in U$, $p_0(t_0) = 1$ and $||p_0|| = \frac{1}{d}$

4. Conclusion:

For linear functionals, this theorem is a crucial extension theorem. The current state of functional analysis would be different if this theorem hadn't been established. By expanding the Hahn-Banach theorem's applicability, future studies can build on the groundwork established by this significant mathematical theorem.

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