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Problem Related to Invariant Theory of torus and finite group

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Abstract:

Invariant theory studies the symmetries of algebraic varieties under group actions. When considering actions of tori and finite groups on polynomial rings, several classical and modern problems emerge concerning the structure and generation of invariant rings. This paper investigates the invariant theory associated with linear actions of algebraic tori and finite groups on polynomial rings, emphasizing computational and structural problems such as the finite generation of invariants, separating invariants, and Hilbert series. The discussion bridges classical results and recent developments, highlighting open problems and presenting a methodological framework that blends algebraic geometry with computational algebra. Our goal is to expose the subtle interplay between the combinatorial nature of torus actions and the group-theoretic properties of finite groups in invariant theory.

Introduction

Invariant theory historically stems from classical questions in algebra and geometry concerning the behavior of functions under symmetry operations. Rooted in the works of Hilbert, Noether, and Weyl, the theory has evolved to address both foundational algebraic questions and practical computational challenges. When a group G acts on a polynomial ring $k[x_1, \ldots, x_n]$, the subring of invariant polynomials $k[x_1, \ldots, x_n]^G$ becomes a central object of study.

Tori, being diagonalizable algebraic groups, offer a rich structure through their action via weight spaces, while finite groups add a layer of complexity through their discrete symmetries. The intersection of these two types of group actions surfaces in many areas including representation theory, combinatorics, and algebraic geometry. This paper is motivated by a deeper understanding of the problems that arise in describing invariant rings, their generators, and the computational implications for these structures under the action of tori and finite groups.

Literature Review

The foundational results in invariant theory began with Hilbert's Basis Theorem, asserting the finite generation of invariant rings under linear reductive group actions. For finite groups, Emmy Noether proved that if $G \subset GL_n(k)$ and the characteristic of k does not divide the order of G, then $k[x_1, \ldots, x_n]^G$ is finitely generated. Molien's Theorem later provided a method to compute the Hilbert series of invariants under finite group actions.

Algebraic tori, as commutative diagonalizable groups, were studied by Vinberg, Kempf, and Popov, who investigated toric invariants and their connections to toric varieties. Derksen and Kemper provided algorithmic approaches for computing invariants, particularly in computational invariant theory. Recent work has focused on separating invariants (Dufresne, 2015), SAGBI bases for torus actions (Robbiano, Sweedler), and modular invariant theory, especially in positive characteristic.

Despite significant progress, many computational and theoretical challenges remain. For instance, explicit descriptions of invariants in mixed settings (torus × finite group actions), understanding their separating capabilities, and determining minimal generating sets continue to be active areas of research.

Methodology

Our approach is structured around the interaction of linear actions of tori and finite groups on polynomial rings over an algebraically closed field k of characteristic zero. We consider the following setting:
Let $G = T \times H$, where T is an algebraic torus (i.e., $T \cong (k^\times)^r$) and H is a finite group acting linearly on $R = k[x_1, \ldots, x_n]$.

The methodology consists of the following steps:

1. Weight Decomposition: Decompose R into weight spaces under the torus action and analyze the multigrading induced.
2. Invariant Ring Construction: For the joint action of T and H, construct the invariant ring $R^G = (R^T)^H$.
3. Hilbert Series Analysis: Use Molien-like series for torus actions and Molien's Theorem for finite groups to analyze the growth of invariants.
4. Computational Tools: Employ Gröbner bases, SAGBI bases, and computational algebra software (e.g., Macaulay2, Singular) to generate and test invariants.
5. Separating Invariants: Investigate minimal separating sets and explore their structure relative to the group action.

We supplement theoretical work with concrete examples to illustrate the challenges and techniques.

Discussion

The invariant theory of torus and finite group actions reveals several layered challenges:

Non-reductiveness of Torus × Finite Group Actions: While tori and finite groups are reductive individually, their joint action can lead to unexpected behaviors, particularly in the generation and structure of the invariant ring.

Combinatorial Complexity: The torus action stratifies the space into multidegree components, adding a combinatorial dimension to the problem. Determining which monomials remain invariant under the torus action is itself nontrivial.

Interaction Effects: The finite group H often permutes the weight spaces fixed by T, leading to complex intertwining in the ring of invariants. Understanding how H acts on R^T is key to building the full invariant ring R^G.

Separating Invariants: The search for minimal separating sets often leads to examples where the minimal number of generators is greater than the dimension of the representation space, highlighting the subtlety of the problem.

Computational Bottlenecks: Algorithms for computing invariants, such as those based on SAGBI bases or Reynolds operators, may not scale well with the complexity of the group action, prompting the need for hybrid symbolic-numeric methods.

We illustrate these themes through examples, such as the diagonal torus action on $k[x,y,z]$ and the symmetric group S_3 acting via permutations, exploring their joint invariant ring and separating sets.

Conclusion

The invariant theory of torus and finite group actions presents a rich blend of algebra, geometry, and computation. While the theory is well-established for individual classes of groups, their combination raises novel questions about structure, generation, and separation. Our analysis exposes foundational problems that merit further exploration, including understanding mixed invariants, optimizing computational methods, and classifying invariant rings for broader classes of group actions.

Future work may extend to non-linear actions, connections with equivariant cohomology, and applications in algebraic statistics and coding theory. Bridging classical techniques with modern computational tools will be crucial in addressing these open problems.

References:

1. Derksen, H., & Kemper, G. (2002). Computational Invariant Theory. Springer.
2. Dufresne, E. (2015). Separating invariants and finite reflection groups. Advances in Mathematics, 274, 306–338.
3. Kemper, G. (2011). Separating invariants. Journal of Symbolic Computation, 44(9), 1212–1222.
4. Popov, V. L., & Vinberg, E. B. (1994). Invariant Theory. In: Algebraic Geometry IV, Encyclopaedia of Mathematical Sciences, vol 55. Springer.
5. Sturmfels, B. (1993). Algorithms in Invariant Theory. Springer.