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Fractional Pochhammer Symbols and Their Applications

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ABSTRACT :

A key concept in the study of special functions and hypergeometric series, the Pochhammer symbol is traditionally defined for integers and, via the gamma function, real or complex indices. With significant ramifications for fractional calculus, series summations, and applications in mathematical physics, the idea has recently been expanded to fractional and generalized orders. The definitions, characteristics, and applications of fractional Pochhammer symbols to fractional differential equations, generalized hypergeometric functions, and analytic expansions in the applied sciences are all covered in detail in this paper.

Key Words: fractional Pochhammer symbol, Fractional Generalizations, Fractional Differential Equations, Classical Pochhammer Symbol.

1. Introduction

The Pochhammer symbol, denoted $(a)_n$ and classically defined for integer n, appears throughout combinatorics, the theory of special functions, and solutions to hypergeometric-type differential equations [1][2]. Through its close association with the gamma function and factorials, the Pochhammer symbol is central to many summation formulas and series expansions. However, with the increasing importance of fractional difference and differential equations [3], and fractional versions of classic special functions [4], there is a natural mathematical and physical motivation to generalize the Pochhammer symbol to non-integer (fractional) values.

The fractional Pochhammer symbol opens doors to wider analytic continuations, new classes of hypergeometric functions, and the analytic machinery necessary for advanced applications in fractional calculus, quantum physics, and statistical mechanics [5][6]. This research aims to systematically present the definitions and properties of fractional Pochhammer symbols, and to showcase selected areas where their generalization is not only natural but essential.

2. Classical Pochhammer Symbol: Definition and Properties

The classical (rising) Pochhammer symbol is defined as:

$$(a)_n = a(a+1)\ldots(a+n-1)\ (n \in \mathbb{N}, n \geq 1), (a)_0 = 1$$

Alternatively, via the gamma function:

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$$

This extension instantly allows a and n to be real or complex, provided a is not a non-positive integer [1][2].

Key properties:

- **Recurrence:** $(a)_n = (a)_{n-1}(a+n-1)$
- **Relation to factorial:** $(1)_n = n!$
- **Relation to binomial coefficients:** $\binom{a}{n} = \frac{(a)_n}{n!}$

3. Fractional and Generalized Pochhammer Symbols

3.1 Definitions

Motivated by analytic continuation, the **fractional Pochhammer symbol**, for non-integer $\alpha > 0$, is defined as:

$$(a)_\alpha := \frac{\Gamma(a+\alpha)}{\Gamma(a)}$$

where $\Gamma(z)$ denotes Euler's gamma function and α may be any positive real (or complex) number, provided denominator is nonzero [3][4][7]. When $\alpha \in \mathbb{N}$, this definition recovers the classical integer case.

3.2 Properties

- **Recurrence relations:** $(a+1)_\alpha = (a+1)(a+2)\cdots(a+\alpha) = \frac{\Gamma(a+\alpha+1)}{\Gamma(a+1)}$
- **Relation to generalized factorial notation:** $(a)_\alpha = \prod_{j=0}^{\alpha-1}(a+j)$ This product is meaningful for integer α; for fractional orders, only the gamma ratio is generally valid.
- **Reduction to binomial theorem:** $(a)_\alpha = a(a+1)\cdots(a+\alpha-1) = \frac{\Gamma(a+\alpha)}{\Gamma(a)}$ This expression is analytic in both a and α except for poles of the gamma function.
- **Relation with falling factorial:** For integer n, $(a)_{-n} = \frac{1}{(a-n)_n}$

4. Fractional Hypergeometric Functions

4.1 Generalized Hypergeometric Series

The classical hypergeometric series ${}_pF_q$ is:

$${}_pF_q\left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}; x\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k \dots (a_p)_k}{(b_1)_k \dots (b_q)_k} \frac{x^k}{k!}$$

By replacing k with a continuous parameter (or replacing the Pochhammer symbol with its fractional counterpart), one obtains *fractional hypergeometric series* or their analytic continuations [4][6][8].

4.2 Fractional Generalizations

A fractional hypergeometric function can be defined as

$${}_pF_q^{(\alpha)}\left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}; x\right) := \sum_{k=0}^{\infty} \frac{(a_1)_{k\alpha} \dots (a_p)_{k\alpha}}{(b_1)_{k\alpha} \dots (b_q)_{k\alpha}} \frac{x^{k\alpha}}{\Gamma(1+k\alpha)}$$

where $(a)_{k\alpha}$ denotes the fractional Pochhammer symbol with parameter increment α [6][9]. This series converges for small $|x|$ under suitable conditions on parameters.

4.3 Connections to Fractional Calculus

Fractional-order differentiation and integration, such as those defined by the Riemann-Liouville or Caputo operators, have series solutions involving fractional generalizations of the Pochhammer symbol. For example, consider the fractional differential equation:

$$D_{0^+}^{\alpha} y(x) = \lambda y(x), 0 < \alpha < 1,$$

where $D_{0^+}^{\alpha}$ is the Riemann-Liouville fractional derivative of order α, and λ is a constant.
The general solution is expressed in terms of the Mittag-Leffler function:

$$y(x) = y_0 E_\alpha(\lambda x^\alpha) = y_0 \sum_{k=0}^{\infty} \frac{(\lambda x^\alpha)^k}{\Gamma(\alpha k+1)},$$

which bears a strong resemblance to a generalized hypergeometric expansion, with the denominator essentially a fractional Pochhammer symbol:

$$\frac{1}{\Gamma(\alpha k+1)} = \frac{1}{(\Gamma(1))_\alpha^k}$$

particularly the Riemann-Liouville and Caputo derivatives, often gives rise to solutions involving generalized hypergeometric or Mittag-Leffler functions where the fractional Pochhammer symbol appears naturally in their series representations [3][10].

5. Applications

5.1 Fractional Differential Equations

Certain linear and nonlinear fractional differential equations (FDEs) possess solutions in terms of fractional hypergeometric or Mittag-Leffler type functions. For example, the solution to the linear FDE:

$$D^{\alpha} y(x) = \lambda y(x), D^{\alpha} \text{ is the Riemann-Liouville (or Caputo) derivative}$$

is given by

$$y(x) = E_{\alpha}(\lambda x^{\alpha}) = \sum_{k=0}^{\infty} \frac{(\lambda x^{\alpha})^k}{\Gamma(1+\alpha k)}$$

Here, the denominator involves a generalized factorial, which can be interpreted in terms of the fractional Pochhammer symbol [10][11].
Consider a more general linear fractional ordinary differential equation (FODE):

$$\left(D_{0^+}^{\alpha} + a_1 D_{0^+}^{\alpha-1} + \cdots + a_n D_{0^+}^{\alpha-n}\right) y(x) = f(x).$$

Applying Laplace transforms and assuming zero initial conditions, the solution often takes the form:

$$y(x) = \sum_{k=0}^{\infty} \frac{F^{(k)}(0)}{k!} \frac{x^{k\alpha}}{\Gamma(k\alpha+1)},$$

where $F^{(k)}$ is the k-th derivative of the Laplace-transformed forcing function evaluated at 0.
The coefficients $\frac{1}{\Gamma(k\alpha+1)}$ can alternatively be thought of as inverse fractional Pochhammer symbols:

$$\frac{1}{(1)_{k\alpha}} = \frac{1}{\Gamma(1+k\alpha)}$$

for the case $(a)_{k\alpha} = \Gamma(a+k\alpha)/\Gamma(a)$.
For nonlinear or variable-coefficient equations, similar series expansions involve $(a)_{k\alpha}$ for general values of a and k.

5.2 Series Summation and Analytical Continuation

Fractional Pochhammer symbols enable analytic continuation of series expansions—especially those arising in mathematical physics and engineering—where non-integer increments or operator orders arise [4][7]. They provide a uniform notation for generalizing identities and transformations that, in the standard setting, depend crucially on integer indices.
Analytic continuation of hypergeometric series frequently uses the gamma form of the Pochhammer symbol. For non-integer increments, we may express the generalized (fractional) hypergeometric function as:

$${}_pF_q^{(\alpha)}\left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}; x\right) := \sum_{k=0}^{\infty} \frac{(a_1)_{k\alpha} \dots (a_p)_{k\alpha}}{(b_1)_{k\alpha} \dots (b_q)_{k\alpha}} \frac{x^{k\alpha}}{\Gamma(1+k\alpha)}$$

where

$$(a)_{k\alpha} = \frac{\Gamma(a+k\alpha)}{\Gamma(a)}$$

The use of the fractional Pochhammer symbol allows for the extension of summation techniques to scenarios with arbitrary step-size, yielding analytic constructions in fractional operator theory, combinatorial analysis, and continuum approximations.

5.3 Statistical Mechanics and Quantum Theory

Partition functions in models with fractional statistics (e.g., anyons, generalized spin models), and solutions to certain quantum-field theoretic equations, frequently require summation formulas with generalized factorials [12][13]. The fractional Pochhammer symbol is an efficient tool in such computations.
In quantum mechanics and statistical mechanics, the partition function Z for certain systems can be expressed as:

$$Z = \sum_{n=0}^{\infty} \frac{g(n)\, e^{-\beta E_n}}{n!}$$

If fractional energy levels or generalized statistics are considered, the summation may involve generalized factorials and thus fractional Pochhammer symbols:

$$Z = \sum_{k=0}^{\infty} \frac{g(k\alpha)\, e^{-\beta E_{k\alpha}}}{\Gamma(1+k\alpha)} = \sum_{k=0}^{\infty} \frac{g(k\alpha)\, e^{-\beta E_{k\alpha}}}{(1)_{k\alpha}}$$

where $g(k\alpha)$ is the degeneracy and $E_{k\alpha}$ is the fractional energy.

Fractional Pochhammer symbols generalize statistical weights and allow analytic expressions for systems interpolating between integer quantum numbers or in contexts with fractional excitations.

5.4 Fractional Operator Calculus

The application of fractional difference and sum operators in discrete analysis leads naturally to series where coefficients are given by fractional Pochhammer ratios, extending the usefulness of the symbol to discrete-continuous interpolations [5][6].
Discrete fractional calculus (fractional difference operators) employs fractional binomial coefficients, which can be written using the fractional Pochhammer symbol:

$$\Delta^{\alpha} y(n) = \sum_{k=0}^{n} \binom{\alpha}{k} (-1)^k y(n-k)$$

with

$$\binom{\alpha}{k} = \frac{(\alpha)_k}{k!}$$

and

$$(\alpha)_k = \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)}$$

When α is non-integer, these coefficients interpolate between classical finite differences and fractional differencing.
Similarly, fractional sums and products have expansion coefficients expressed in terms of $(a)_{k\alpha}$, enabling the discrete analysis of fractional order systems.
Fractional Pochhammer symbols form the backbone of discrete fractional calculus, directly extending Newton's forward and backward difference theorems and facilitating analyses of systems and signals with intrinsic memory or non-integer scaling.

Theorem 1: Analytic Continuation and Multiplicative Property of the Fractional Pochhammer Symbol
Statement:
Let $a \notin \{0, -1, -2, \ldots\}$ and let $\alpha, \beta \in \mathbb{C}$. The fractional Pochhammer symbol satisfies the multiplicative relation

$$(a)_{\alpha+\beta} = (a)_{\alpha} \cdot (a+\alpha)_{\beta}$$

where

$$(a)_s := \frac{\Gamma(a+s)}{\Gamma(a)}$$

for arbitrary $s \in \mathbb{C}$.

Solution/Proof:
Using the definition,

$$(a)_{\alpha} = \frac{\Gamma(a+\alpha)}{\Gamma(a)}.$$

Similarly,

$$(a+\alpha)_{\beta} = \frac{\Gamma(a+\alpha+\beta)}{\Gamma(a+\alpha)}.$$

Multiplying the two gives:

$$(a)_{\alpha} \cdot (a+\alpha)_{\beta} = \frac{\Gamma(a+\alpha)}{\Gamma(a)} \cdot \frac{\Gamma(a+\alpha+\beta)}{\Gamma(a+\alpha)} = \frac{\Gamma(a+\alpha+\beta)}{\Gamma(a)},$$

which matches exactly the definition for $(a)_{\alpha+\beta}$ by the gamma function:

$$(a)_{\alpha+\beta} = \frac{\Gamma(a+(\alpha+\beta))}{\Gamma(a)}.$$

Hence, the identity is proved.

Theorem 2: Fractional Hypergeometric Series with Fractional Pochhammer Symbols
Statement:
Let $|x| < 1$, $a, b \notin \mathbb{Z}_{\leq 0}$, and $0 < \alpha \leq 1$. Consider the fractional hypergeometric series

$$F_{\alpha}(a, b; c; x) = \sum_{k=0}^{\infty} \frac{(a)_{k\alpha}(b)_{k\alpha}}{(c)_{k\alpha}\, \Gamma(1+k\alpha)} x^{k\alpha}$$

where $(q)_{k\alpha} = \Gamma(q+k\alpha)/\Gamma(q)$. Then F_{α} converges for $|x| < 1$, and when $\alpha = 1$, it coincides with the classical Gaussian hypergeometric function

$$ {}_2F_1(a,b;c;x) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k \, k!} x^k. $$

Solution/Proof:

Convergence:

First, for $|x| < 1$ and bounded parameters, the growth of $\Gamma(1 + k\alpha)$ in the denominator (as $k \to \infty$) dominates the numerator for fixed a, b, c, ensuring convergence, by the root or ratio test:

$$ \lim_{k\to\infty} \left| \frac{x^{(k+1)\alpha}}{x^{k\alpha}} \cdot \frac{\Gamma(a+(k+1)\alpha)}{\Gamma(a+k\alpha)} \frac{\Gamma(b+(k+1)\alpha)}{\Gamma(b+k\alpha)} \frac{\Gamma(c+k\alpha)}{\Gamma(c+(k+1)\alpha)} \cdot \frac{\Gamma(1+k\alpha)}{\Gamma(1+(k+1)\alpha)} \right| $$

As $k \to \infty$, the ratio of gamma functions can be estimated using Stirling's approximation:

$$ \Gamma(z+s)/\Gamma(z) \sim (z)^s \text{ for large } z $$

showing that the terms decay and the series converges for $|x| < 1$.

Reduction to classical case:

When $\alpha = 1$, we have

$$ (a)_{k\alpha} = (a)_k = \frac{\Gamma(a+k)}{\Gamma(a)}, \Gamma(1+k\alpha) = \Gamma(1+k) = k! $$

Thus,

$$ F_1(a,b;c;x) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k \, k!} x^k = {}_2F_1(a,b;c;x) $$

which is the classical hypergeometric function.

Key notes:

- The analytic generalization of hypergeometric-type series is supported by this theorem, which makes fractional Pochhammer notation useful for studying fractional differential and difference equations.
- The fractional Pochhammer symbol allows for the extension of several characteristics of classical hypergeometric functions to their fractional equivalents, such as recursion and contiguous relations.

6. Open Problems and Research Prospects

Although fractional Pochhammer symbols' characteristics and analytical tendencies are largely recognized, there are still a number of unresolved issues: Analytic continuation and asymptotic expansions of generalized hypergeometric functions with fractional steps are studied.

• The addition of connections with fractional Appell functions and multivariate settings.

• Thorough development of probability distributions and fractional combinatorics.

• Additional investigation on applications in variable-order fractional partial differential equations.

7. Conclusion

The fractional Pochhammer symbol emerges as a natural and powerful generalization of classical factorial-based notation, seamlessly extending the reach of special function theory, series expansions, and fractional calculus. Its applicability to hypergeometric representations, analytic continuation, and the solutions of fractional differential equations exemplifies its foundational role in modern applied mathematics. As the interaction between classical analysis and fractional operators intensifies across research disciplines, the systematic study and application of the fractional Pochhammer symbol promises to deepen our understanding of both analytic theory and its practical implementations.

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