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Fractional Gamma Functions and Their Connection to Fractional Calculus

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ABSTRACT:

New analytical tools for studying fractional calculus and related mathematical structures are made possible by the extension of the classical Gamma function to the fractional domain. In this paper, we explore the features of fractional Gamma functions, discuss their inherent relationships to fractional differential and integral operators, and thoroughly study their construction. We discuss recent developments in the subject, including Mellin transforms, Mittag-Leffler functions, and the application of fractional Gamma functions to fractional differential equation solutions. In order to illustrate their usefulness in mathematical physics and engineering, a number of applications and examples are provided.

Keywords: Gamma function, fractional calculus, fractional Gamma function, special functions**,** Mellin Transform.

1. Introduction

A significant special function in mathematics, the gamma function $\Gamma(z)$ extends the factorial function to complex and real parameters. It was first presented by Euler in the 18th century and is a fundamental concept in many applied and mathematical domains, from differential equations to probability theory and beyond [1,2]. Fractional analogues of classical special functions, such as fractional gamma functions, have been developed and studied in recent decades due to the growing interest in fractional calculus, which is the study of derivatives and integrals of arbitrary (non-integer) order [3,4].

Fractional calculus and Gamma functions interact profoundly. For example, the Gamma function is intrinsic to the definition of the kernel of the Riemann–Liouville and Caputo fractional operators [5,6]. Furthermore, the analytical formulation of solutions to fractional order differential equations easily leads to generalized Gamma and Mittag-Leffler functions [7]. The characteristics and applications of fractional Gamma functions are still being studied despite their growing importance [8,9].

This paper aims to elucidate the construction, properties, and applications of fractional Gamma functions, with particular emphasis on their connection to fractional calculus. We provide a self-contained account of the relevant mathematical groundwork and demonstrate the utility of these functions in several applied contexts.

2. Facts about Classical Gamma Function and Fractional Calculus

The Gamma function is classically defined for $\Re(z) > 0$ by the improper integral

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt,$$

which exhibits various important properties, such as the recursive relation
$\Gamma(z+1) = z\Gamma(z)$
and the reflection formula associated with complex analysis [2,10].

Some standard results of the Gamma function are as follows:

$$\Gamma 1 = 1, \qquad \Gamma\frac{1}{2} = \sqrt{\pi}$$
$$\Gamma(z+1) = z\Gamma z$$
$$\Gamma 2 \ = 1\Gamma 1 = 1 = 1!$$
$$\Gamma 3 \ = 2\Gamma 2 = 2 = 2!$$
$$\Gamma 4 \ = 3\Gamma 3 = 6 = 3!$$
$$\dots \dots \dots \dots \dots \dots \dots \dots \dots.$$
$$\Gamma(n+1) = n\Gamma n = n(n-1)! = n!$$

On the other hand, fractional calculus generalizes the concept of integer-order differentiation and integration to arbitrary real or complex orders. The commonly used Riemann–Liouville fractional integral of order $\alpha > 0$ is defined as [5]:

$$(I^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t) dt.$$

Similarly, one may define fractional derivatives via analytic continuation. The appearance of the Gamma function in these definitions is neither incidental nor superficial—it is closely related to the analytic structure and inversion formulae of fractional calculus [3].

3. Fractional Gamma Functions: Definitions and Generalizations

Several approaches to defining fractional Gamma functions have been proposed, mainly driven by analytic continuation, fractional difference equations, and generalizations inspired by fractional operators.

3.1. Generalized Fractional Gamma Function

A principal generalization is due to M.A. Al-Mekhlafi et al. [8], who define a fractional Gamma function, denoted $\Gamma_{\alpha}(z)$, as

$$\Gamma_{\alpha}(z) = \int_0^{\infty} t^{z-1} e^{-t^{\alpha}} dt, \Re(z) > 0, \Re(\alpha) > 0.$$

This representation recovers the standard Gamma function when $\alpha = 1$, i.e., $\Gamma_1(z) = \Gamma(z)$.

3.2. Properties

For $\alpha \neq 1$, $\Gamma_{\alpha}(z)$ loses some properties of the standard Gamma function, such as the simple recurrence relation. However, it satisfies the following scaling property [8,11]:

$$\Gamma_{\alpha}(z) = \frac{1}{\alpha} \Gamma\left(\frac{z}{\alpha}\right).$$

This can be shown via the substitution $s = t^{\alpha} \Rightarrow t = s^{1/\alpha}$, $dt = \frac{1}{\alpha} s^{1/\alpha - 1} ds$.

4. Fractional Calculus and the Role of Generalized Gamma Functions

4.1. Connection via Kernel Functions

The kernel of the classical Riemann–Liouville integral, $(x-t)^{\alpha-1}/\Gamma(\alpha)$, is directly related to the Gamma function. When considering generalized kernels for instance, those involving stretched exponentials or Mittag-Leffler functions fractional Gamma functions naturally encode the scaling and normalization necessary for the coherent definition of fractional operators [7].

4.2. Fractional Difference and q-Gamma Functions

Some efforts in discrete fractional calculus involve the fractional difference operator and its relationship to the **q-Gamma function**, a q-analogue generalization [12]. The q-Gamma function, $\Gamma_q(z)$, serves as the fundamental object in discrete settings, and its fractional generalizations aid in the study of discrete processes with memory.

4.3. Mellin Transform and Fractional Integrals

The Mellin transform connects the fractional Gamma function with a broad class of functions:

$$\mathcal{M}[e^{-t^{\alpha}}](z) = \int_0^{\infty} t^{z-1} e^{-t^{\alpha}} dt = \Gamma_{\alpha}(z)$$

thus establishing a bridge between integral transforms used in solving fractional differential equations and fractional Gamma structures [13].

Mittag-Leffer introduced the one-parameter function and it is defined as follows:

$$E_{\alpha}(z) = \sum_{r=0}^{\infty} \frac{z^r}{\Gamma(\alpha r + 1)}$$

Or

$$E_{\alpha}(z) = 1 + \frac{z}{\Gamma(\alpha + 1)} + \frac{z^2}{\Gamma(2\alpha + 1)} + \cdots$$

Mittag-Leffer Functions of Two-Parameter

R.P. Agarwal and Erdelyi introduced the two-parameter Mittag-Leffer functions during 1953-1954. This function is defined as follows:
One – Parameter Mittag-Leffer function can be obtain by taking $\beta = 1$, i.e.

$$E_{\alpha,\beta}(z) = \sum_{r=0}^{\infty} \frac{z^r}{\Gamma(\alpha r + \beta)} \,, \quad \alpha > 0, \beta > 0$$

$$E_{\alpha,1}(z) = \sum_{r=0}^{\infty} \frac{z^r}{\Gamma(\alpha r + 1)} = E_{\alpha}(z)$$

Exponential function (e^z) plays an important role in conventional calculus i.e. integer order calculus equations. Similarly, the Mittag-Leffer functions plays important role in the fractional order calculus.

Riemann-Liouville

In this approach, the following is the definition of fractional derivatives:

$${}_aD_x^{\alpha}(f(x)) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_a^x \frac{1}{(x-t)^{\alpha-n+1}} f(t)\, dt \quad , n-1 \le \alpha < n$$

where α is real number and n is an integer.

$$D_x^{\alpha}(f(x)) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_0^x (x-t)^{n-\alpha-1} f(t)\, dt$$

5. Applications and Examples

5.1. Solution of Fractional Differential Equations

The Mittag-Leffler function, defined as

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)},$$

is a direct generalization of the exponential function wherein the Gamma function in the denominator is replaced by a α-dependent argument. The solution to the simplest Caputo-type fractional differential equation,

$$D_t^{\alpha} y(t) = -\lambda y(t), y(0) = y_0,$$

is given by

$$y(t) = y_0 E_{\alpha}(-\lambda t^{\alpha}),$$

demonstrating the central role fractional Gamma functions play in the structure of Mittag-Leffler solutions [7].

Example 1: To find the fractional derivative of e^{ax} $(or\ D^{\alpha}\ (e^{ax}))$ of order α, where $0<\alpha<1$
We are applying the formula.So we get the following:

$$D^{\alpha}e^{ax} = D^{1}\{D^{-\alpha}e^{ax}\}$$
$$D^{\alpha}e^{ax} = D^{1}\{x^{\alpha}E_{1,\alpha+1}(ax)\}$$

Now using the definition of Mittag-Leffer functions, we get the following result

$$D^{\alpha}e^{ax} = x^{-\alpha}E_{1,-\alpha+1}(ax)$$

Example 2: To find the fractional integral of exponential function i.e. $D^{-n}\ (e^{ax}), where\ a\ is\ a\ constant.$

Apply the Riemann-Liouville approach for fractional integral (by equation 4), we have

$$D^{-n}\ (e^{ax}) = \frac{1}{(n-1)!}\int_{0}^{x}(x-t)^{n-1}e^{at}\ dt$$

Taking the substitution

$$y = x - t, then\ dy = -dt, then\ we\ have$$

$$D^{-n}\ (e^{ax}) = \frac{1}{(n-1)!}\int_{x}^{0}(y)^{n-1}e^{a(x-y)}(-dy)$$

$$= \frac{1}{(n-1)!}\ (e^{ax})\int_{0}^{x}(y)^{n-1}e^{-ay}\ dy$$

in which incomplete Gamma function and the relation between Mittag- Leffer function and incomplete Gamma function are defined.
So, the above expression can be written as

$$D^{-n}\ (e^{ax}) = E_{x}(n,a) = x^{n}\ E_{1,n+1}(ax)$$
$$n = \frac{1}{2}, then$$

$$D^{-1/2}\ (e^{ax}) = E_{x}\left(\frac{1}{2},a\right)$$
$$= a^{-1/2}\ e^{x}\ Erf\ (ax)^{1/2}$$

Above are the some examples to understand the concept of fractional calculus.

5.2. Anomalous Relaxation and Physics

In physics, fractional Gamma functions emerge in the time-domain solutions of anomalous diffusion models, viscoelasticity, and non-exponential relaxation phenomena. The kernels involving $\Gamma_{\alpha}(z)$ provide precise normalization and scaling for models with non-local temporal memory [14,15].

5.3. Probability and Statistics

Generalized Gamma distributions, with densities

$$f(t) = \frac{\beta t^{\alpha-1}e^{-(t/\lambda)^{\beta}}}{\lambda^{\alpha}\Gamma(\alpha/\beta)}, t>0, \alpha,\beta,\lambda>0,$$

are used to model a variety of statistical data, notably in survival analysis and reliability engineering. The normalization factor involves the Gamma function generalized by the parameter β, which in turn links to the fractional gamma family [16].

6. Discussion

The theoretical foundations and real-world calculations of fractional calculus are closely related to the study of fractional Gamma functions. These generalized functions are crucial links between the quickly developing field of fractional differential equations and classical analysis. Although a large portion of special function theory is based on the classical Gamma function, its fractional generalizations allow analytical methods to be used to situations with memory, non-locality, and anomalous scaling rules. Additional research directions include numerical techniques utilizing the fractional Gamma and related functions, spectral theory, and the systematic study of generalized fractional operator semigroups.

7. Conclusion

Fractional Gamma functions generalize the classical Gamma function and extend its reach into the domain of fractional calculus. Through their various representations, properties, and applications, they are key components in developing both foundational theory and applied methodologies for fractional integral and differential equations. Ongoing research continues to deepen understanding, uncovering further connections and applications across mathematics and the physical sciences.

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