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Projective Motions in Generalized Fifth Recurrent Finsler Space via Lie Derivative of Berwald Covariant Tensors

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ABSTRACT:

In this paper, we investigate the nature of projective motion within a generalized fifth recurrent Finsler space, denoted as $G\mathcal{B}K - 5RF_n$. Utilizing Berwald's connection and higher-order covariant derivatives of Cartan's fourth curvature tensor, we analyze the behavior of projective transformations through the application of the Lie-derivative. A comprehensive system of tensorial identities involving the $h(v)$ −torsion tensor, deviation tensor, and scalar curvature is employed to establish a series of theorems that characterize the projective motion in this setting. Our findings reveal that the Lie-derivatives of higher-order derivatives of the Cartan curvature tensor, as well as associated torsion and curvature tensors, exhibit invariant projective behavior. Additionally, we demonstrate the vanishing of specific curvature-related parameters, including the Berwald connection components and the partial derivatives of the scalar curvature vector.

Keywords: Finsler geometry, projective motion, $G\mathcal{B}K - 5RF_n$, Berwald connection, curvature tensors, Lie-derivative.

1. Introduction

Finsler geometry has attracted significant interest in recent years due to its wide-ranging applications in differential geometry, theoretical physics, and cosmology. Within this framework, various classes of recurrent Finsler spaces have been studied extensively, especially in relation to projective motions, curvature tensors, and Lie derivatives. In particular, the concept of generalized recurrent spaces notably those involving higher-order curvature tensors has emerged as a fruitful area of exploration.

Abdallah [1] discussed the relationship between two curvature tensors in Finsler different spaces. Abdallah and Hardan [2] laid foundational work on P-third order generalized Finsler spaces in the Berwald sense, while the authors in [3, 4] provided a detailed examination of the structure and properties of P-generalized Finsler spaces with third-order Berwald connections. Parallel developments have occurred in the study of higher-order recurrences, such as those in $K^{(h)}$-generalized recurrent Finsler spaces [5], and in the analysis of Lie derivatives of tensors within generalized BK-fifth recurrent structures [6, 7].

Recent works by Al-Qashbari et al. [8–11] have significantly advanced our understanding of generalized $\mathcal{B}K$-fifth recurrent Finsler spaces ($G\mathcal{B}K - 5RF_n$), especially through the lens of projective transformations, Lie derivatives, and the inheritance properties of various curvature tensors, including the Kulkarni–Nomizu product and M-projective tensors. These contributions have provided critical insights into the geometric behavior of such spaces and their underlying tensorial symmetries.

Additional studies have explored related curvature structures such as the R-projective curvature tensor in recurrent Finsler spaces [12], W-generalized birecurrent spaces [14], and Cartan's curvature decompositions [16]. The theoretical framework for projective changes and flatness in Finsler metrics was initially outlined in Matsumoto's seminal work [17], and has since been expanded upon in doctoral dissertations [18, 23] and modern monographs such as Ohta's Comparison Finsler Geometry [19].

Moreover, the role of the Lie derivative in tensor analysis, particularly within the context of Finsler and fluid mechanics, has been emphasized in [15, 21, 22], highlighting its utility in describing geometric invariants and recurrence relations.

These studies collectively underscore the rich structural variety of generalized recurrent Finsler spaces and provide a robust mathematical foundation for the present investigation into projective motions in $G\mathcal{B}K - 5RF_n$, particularly under the action of higher-order Lie derivatives. The aim of this paper is to study projective motions on certain types of curvature, torsion and deviation tensors in generalized fifth recurrent Finsler space by Lie-derivative.

2. Preliminaries

In this section, we explore a generalized $\mathcal{B}K$-fifth recurrent Finsler space, denoted as $G\mathcal{B}K - 5RF_n$, that satisfies the following set of identities [10]:

(2.1) $\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m K^i_{jkh} = a_{sqlnm}K^i_{jkh} + b_{sqlnm}(\delta^i_h g_{jk} - \delta^i_k\, g_{jh})$

$-c_{sqlnm}(\delta^i_h C_{jkn} - \delta^i_k\, C_{jhn}) - d_{sqlnm}(\delta^i_h C_{jkl} - \delta^i_k\, C_{jhl})$

$-e_{sqlnm}(\delta^i_h C_{jkq} - \delta^i_k\, C_{jhq}) - 2b_{qlnm}y^r\mathcal{B}_r(\delta^i_h C_{jks} - \delta^i_k\, C_{jhs})$.

Additional definitions and tensorial properties such as those of the metric tensor g_{ij} , the tangent vector y^i , Cartan's fourth curvature tensor K^i_{jkh} , h(v)-torsion tensor H^i_{kh} , deviation tensor H^i_h, scalar curvature H, (v)hv-torsion tensor C^i_{jk} , and Berwald connection parameters are stated in equations (2.2) to (2.15). These serve as foundational elements in the subsequent analysis of projective motions within $G\mathcal{B}K - 5RF_n$

(2.2) $\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m H^i_{kh} = a_{sqlnm}H^i_{kh} + b_{sqlnm}(\delta^i_h y_k - \delta^i_k y_h)$.

(2.3) $\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m H^i_h = a_{sqlnm}H^i_h + b_{sqlnm}(\delta^i_h F^2 - y^i y_h)$.

(2.4) $\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m H = a_{sqlnm}H + b_{sqlnm}F^2$.

The metric tensor g_{ij}, the non-zero vector y^i and the Kronecker delta δ^i_h are satisfying the relations [13]

(2.5) a) $\dot{\partial}_j y^j = 1$, b) $y_j\, y^j = F^2$ and c) $g_{ij}\, g^{ik} = \delta^k_j = \begin{cases} 1 & if\ j = k \ , \\ 0 & if\ j \neq k\ . \end{cases}$

The Berwald's covariant derivative of the vectors y^i , y_i are vanishing, i.e.

(2.6) a) $\mathcal{B}_k\, y^i = 0$ and b) $\mathcal{B}_k\, y_i = 0$.

The Cartan's fourth curvature tensor K^i_{jkh} , $h(v)$ −torsion tensor H^i_{kh}, deviation tensor H^i_h and the curvature scalar H satisfy the following relations

(2.7) a) $K^i_{jkh}\, y^j = H^i_{kh}$ b) $K^i_{jkh} = R^i_{jkh} - C^i_{js}H^s_{kh}$ c) $H^i_{kh}y_i = 0$

d) $H^r_r = (n-1)H$ e) $H^i_{kh}y^k = -H^i_{hk}y^k = H^i_h$.

The $(v)hv$ −torsion tensor C^i_{jk} satisfies the following relation [20]

(2.8) $C^i_{jk}y_i = 0$.

Berwald's connection parameter G^i_{kh} is positively homogeneous of degree zero in $y^i$and satisfies

(2.9) a) $G^i_{kh}y^k = G^i_h$ b) $G^i_h y^h = 2G^i$ c) $\dot{\partial}_j G^i_{kh} = G^i_{jkh}$ d) $G^i_{jkh}y^j = 0$.

The Lie-derivative of a general mixed tensor field T^i_{jkh} is given by[16]

(2.10) $L_v T^i_{jkh} = v^m\, \mathcal{B}_m\, T^i_{jkh} - T^m_{jkh}\, \mathcal{B}_m\, v^i + T^i_{mkh}\, \mathcal{B}_j v^m + T^i_{jmh}\, \mathcal{B}_k v^m$

$+ T^i_{jkm}\, \mathcal{B}_h v^m + \dot{\partial}_m\, T^i_{jkh}\, \mathcal{B}_r v^m y^r$,

where $v^m \neq 0$.

The Berwald covariant derivative of the contravariant vector field v^m vanish identically [6]

(2.11) $\mathcal{B}_j v^m = 0$.

The Lie-derivative of the vectors y^i , y_i are vanishing, i.e.

(2.12) a) $L_v y^i = 0$ b) $L_v\, y_i = 0$.

The projective motion characterized by the condition [23]

(2.13) $L_v G^i_{kh} = \delta^i_k\, P_h + \delta^i_h\, P_k + y^i P_{kh}$.

The Ricci tensor P_{jk} satisfies the following relation

(2.14) $P_{jk}y^k = 0$.

The curvature vector P_k ,and the curvature scalar P satisfy the following relations

(2.15) a) $P_{kh} = \dot{\partial}_h P_k$ and b) $P_h = \dot{\partial}_h P$.

3. Main Results

In this section, we examine varoius relationships of projective motions in generalized fifth recurrent Finsler space by using Lie-derivative. Consider the generalized fifth recurrent relation for Cartan's fourth curvature tensor under Berwald's covariant derivative as given in (2.1). Using [(2.5)c] in (2.1), we get

$$\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m K^i_{jkh} = a_{sqlnm} K^i_{jkh}\,.$$

Which by using [(2.9)d] can be written as

$$\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m K^i_{jkh} = a_{sqlnm} K^i_{jkh} + G^i_{jkh} y^j.$$

Using [(2.9)c] and [(2.5)a] in above equation, we get

$$\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m K^i_{jkh} = a_{sqlnm} K^i_{jkh} + G^i_{kh}.$$

Taking the Lie-derivative of both sides of above equation, we get

$$L_v(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m K^i_{jkh}) = (L_v a_{sqlnm}) K^i_{jkh} + a_{sqlnm} L_v K^i_{jkh} + L_v G^i_{kh}.$$

Using (2.13) in above equation, we get

$$L_v(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m K^i_{jkh}) = (L_v a_{sqlnm}) K^i_{jkh} + a_{sqlnm} L_v K^i_{jkh} + \delta^i_k P_h + \delta^i_h P_k + y^i P_{kh}.$$

Using [(2.5)c], (2.10) and (2.11) in right side of above equation, we get

(3.1) $$L_v(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m K^i_{jkh}) = K^i_{jkh}\, v^m\, \mathcal{B}_m a_{sqlnm} + a_{sqlnm} v^m\, \mathcal{B}_m K^i_{jkh} + y^i P_{kh}.$$

Thus, we conclude

Theorem 3.1: *In $GBK\text{-}5RF_n$, the Lie-derivative of the fifth-order Berwald covariant derivative of Cartan's fourth curvature tensor K^i_{jkh} represents projective motion.*

Now, we have two corollaries related to the previous theorem. Multiplying (3.1) by y^j, using [(2.6)a], [(2.7)a] and [(2.12) a)], we get

(3.2) $$L_v(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m H^i_{kh}) = H^i_{kh}\, v^m\, \mathcal{B}_m a_{sqlnm} + a_{sqlnm} v^m\, \mathcal{B}_m H^i_{kh} + y^j y^i P_{kh}\,.$$

Thus, we conclude

Corollary 3.1: *The Lie-derivative of the fifth-order Berwald covariant derivative of the h(v)-torsion tensor H^i_{kh} also represents projective motion.*

Using [(2.7)b] in (3.1), we get

(3.3) $$L_v[\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m(R^i_{jkh} - C^i_{js}H^s_{kh})] = (R^i_{jkh} - C^i_{js}H^s_{kh}) v^m\, \mathcal{B}_m a_{sqlnm}$$
$$+ a_{sqlnm} v^m\, \mathcal{B}_m(R^i_{jkh} - C^i_{js}H^s_{kh}) + y^i P_{kh}.$$

Multiplying above equation by y_i, using [(2.6)b], (2.8) and [(2.12)b], we get

(3.4) $$L_v(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m R^i_{jkh}) = R^i_{jkh} v^m\, \mathcal{B}_m a_{sqlnm} + a_{sqlnm} v^m\, \mathcal{B}_m R^i_{jkh} + y^i P_{kh}\,.$$

Thus, we conclude

Corollary 3.2: *The same holds for the Cartan's third curvature tensor R^i_{jkh}, under assumption (3.4).*

Let as assume the tensor behaves as fifth recurrent, i.e.

(3.5) $$\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\left(C^i_{js}H^s_{kh}\right) = a_{sqlnm}\left(C^i_{js}H^s_{kh}\right).$$

Using (3.5) in (3.3), we get

(3.6) $$L_v(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m R^i_{jkh}) = R^i_{jkh} v^m\, \mathcal{B}_m a_{sqlnm} + a_{sqlnm} v^m\, \mathcal{B}_m R^i_{jkh} + y^i P_{kh}.$$

From equation (3.1), we get

$$y^i P_{kh} = L_v(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m K^i_{jkh}) - K^i_{jkh}\, v^m\, \mathcal{B}_m a_{sqlnm} - a_{sqlnm} v^m\, \mathcal{B}_m K^i_{jkh}\,.$$

Using above equation in (3.6), we get

$L_v(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m R^i_{jkh}) = L_v(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m K^i_{jkh})$, if

(3.7) $$L_v\left(a_{sqlnm} R^i_{jkh}\right) = L_v(a_{sqlnm} K^i_{jkh})\,.$$

Using identity (3.7), we further establish:

Theorem 3.2: *If the Lie-derivatives $L_v(a_{sqlnm}R^i_{jkh})$ and $L_v(a_{sqlnm}K^i_{jkh})$ are equal, then R^i_{jkh} and K^i_{jkh} generate the same projective motion.*

Multiplying (3.2) by y_i, using[(2.5)b], [(2.6)b], [(2.7)c] and [(2.12)b], we get

$$y^j F^2 P_{kh} = 0.$$

Which by using [(2.9)d] can be written as

(3.8) $$y^j F^2 P_{kh} = G^i_{jkh} y^j.$$

Using [(2.9)c] and [(2.5)a] in above equation, then taking Lie-derivative of both sides of result equation, using[(2.5)b], (2.10), (2.11) and [(2.12)a,b] we get

$$y^j F^2 v^m \mathcal{B}_m P_{kh} = L_v G^i_{kh}$$

Using [(2.5)c] and (2.13) in above equation, we get

$$y^j F^2 v^m \mathcal{B}_m P_{kh} = y^i P_{kh}$$

Using[(2.5)b] in above equation where $y^i \neq 0$, then contracting the indices i and j in result equation, we get

(3.9) $$P_{kh} = F^2 v^m \mathcal{B}_m P_{kh} .$$

Proceeding, through scalar contraction and application of Berwald connection properties, we deduce:

Theorem 3.3: *The Ricci tensor P_{kh} satisfies the identity: $P_{kh} = F^2 v^m \mathcal{B}_m P_{kh}$ implying its role in characterizing projective motion.*

Multiplying (3.8) by y^h and using (2.14), we get

$$G^i_{jkh} y^j = 0.$$

Using [(2.9)c] and [(2.5)a] in above equation, we get

$$G^i_{kh} = 0.$$

Multiplying above equation by y^k and using [(2.9)a], we get

$$G^i_h = 0 .$$

Again, multiplying above equation by y^h and using [(2.9)b], we get

$$G^i = 0 .$$

Thus, we conclude

Theorem 3.4: *The Berwald connection parameters G^i_{kh} , G^i_h and G^i all vanish in $G\mathcal{B}K - 5RF_n$.*

Taking the Lie-derivative of both sides of (2.4) using [(2.5)b], (2.10), (2.11) and [(2.12)a,b], we get

(3.10) $$L_v(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m H) = H v^m \mathcal{B}_m a_{sqlnm} + a_{sqlnm} v^m \mathcal{B}_m H + F^2 L_v b_{sqlnm}.$$

Multiplying (3.2) by y^h, using [(2.7)e],[(2.6)a],[(2.12)a] and (2.14), we get

$$L_v(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m H^i_k) = H^i_k v^m \mathcal{B}_m a_{sqlnm} + a_{sqlnm} v^m \mathcal{B}_m H^i_k .$$

Contracting the indices i and k in above equation and using [(2.7)d], we get

$$L_v(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m H) = H v^m \mathcal{B}_m a_{sqlnm} + a_{sqlnm} v^m \mathcal{B}_m H .$$

In view of above equation and (3.10), we get

$$L_v b_{sqlnm} = 0 .$$

Thus, we conclude

Theorem 3.5: *The Lie-derivative of the vector field b_{sqlnm} vanishes identically.*

Multiplying (3.2) by y^k, using [(2.7)e], [(2.6)a] and [(2.12)a], we get

(3.11) $$L_v(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m H^i_h) = H^i_h v^m \mathcal{B}_m a_{sqlnm} + a_{sqlnm} v^m \mathcal{B}_m H^i_h + y^j y^i y^k P_{kh} .$$

Taking the Lie-derivative of both sides of (2.3) using [(2.5)c], (2.10), (2.11) and [(2.12)a,b], we get

$$L_v(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m H^i_h) = H^i_h v^m \mathcal{B}_m a_{sqlnm} + a_{sqlnm} v^m \mathcal{B}_m H^i_h - y^i y_h v^m \mathcal{B}_m b_{sqlnm}.$$

In view of above equation and (3.11), we get

(3.12) $\mathcal{B}_m b_{sqlnm} = -\frac{y^j y^k P_{kh}}{y_h v^m}$.

Thus, we conclude

Theorem 3.6: *The Berwald covariant derivative of b_{sqlnm} satisfies a condition that further confirms its involvement in projective motion:* $\mathcal{B}_m b_{sqlnm} = -\frac{y^j y^k P_{kh}}{y_h v^m}$.

Similarly, taking the Lie-derivative of both sides of (2.2) using [(2.5)c], (2.10), (2.11), we get

$$L_v(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m H^i_{kh}) = H^i_{kh}\, v^m\, \mathcal{B}_m a_{sqlnm} + a_{sqlnm} v^m\, \mathcal{B}_m H^i_{kh} .$$

Using [(2.15)a] in (3.2) and using the result equation in above equation, we get

$$\dot{\partial}_h P_k = 0 .$$

Using [(2.15)b] in above equation, we get

$$\dot{\partial}_h \dot{\partial}_k P = 0.$$

Lastly, through further application of higher-order identities and derivatives:

Theorem 3.7: *The partial derivative of the curvature vector P_k with respect to y^h, and the second-order partial derivatives of the curvature scalar P with respect to y^h and y^k, vanish identically in $G\mathcal{B}K - 5RF_n$.*

4. Conclusions

This study has provided a comprehensive examination of projective motion in the framework of generalized fifth recurrent Finsler spaces ($G\mathcal{B}K - 5RF_n$). We obtained new projective motions on certain types of some tensors. And we established the same projective motion between the Cartan's fourth curvature tensor K^i_{jkh} and Cartan's third curvature tensor R^i_{jkh} under certain condition. Furthermore, the results reveal that the projective motion preserves the structure of several fundamental tensors, including the deviation tensor, scalar curvature vector and $h(v)$ −torsion tensor, highlighting their invariance under specific conditions. The vanishing of the Lie-derivatives of certain curvature quantities underscores the restrictive nature of projective symmetry in such higher-order Finsler spaces.

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