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Homotopy Perturbation vs Variational Iteration: Solving the "Boussinesq-Kortewege-de Vries (BKDV) Equation

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ABSTRACT

In this abstract, two semi-empirical techniques, HPM and VIM stand for Homotopy Perturbation Method and Variational Iteration Method, respectively, in the solution of the Boussinesq-Korteweg-de Vries (B-KdV) nonlinear partial differential equation (PDE). Numerous ordinary and partial differential equations, both linear and nonlinear, can be solved using the flexible Homotopy Perturbation Method (HPM). It allows for the efficient handling of a variety of mathematical problems by building a series solution that becomes more accurate the more terms are added. However, the Variational Iteration Method (VIM) focuses on accurately solving nonlinear partial differential equations by iteratively addressing exact solutions using a correction functional. In this study, the approximate and exact solutions of the B-KdV equation are first derived using the Homotopy Perturbation Method (HPM), which holds.The results from both methods are meticulously compared and evaluated, focusing particularly on their computational efficiency and accuracy in simulating the behavior of the B-KdV equation across different boundary conditions. This comparative analysis offers valuable insights into the practical utility of HPM and VIM for tackling complex nonlinear PDEs within the field of mathematical physics. Furthermore, the study discusses the discrepancies between the numerical solutions derived from HPM and VIM and the exact solutions, underscoring the efficacy of these methods in real-world applications. By conducting comparisons under specific boundary conditions, the research assesses the discrepancies in errors and gauges the robustness of HPM and VIM in generating dependable approximations for nonlinear PDEs.

Keywords: Partial differential equation; Tricomi equation; Nonlinear partial differential equation; Homotopy perturbation method, Homotopy analysis method, BKDVE, and variational iteration method.

1 Introduction

An equation known as a partial differential equation (PDE) explains how a quantity varies in relation to several factors, like time and space. It's an essential idea in physics, engineering, mathematics, and other disciplines. PDEs (partial differential equations) with nonlinear systems may have multiple alternative solutions. Midway through the 1940s, John von Neumann discovered that numerical methods could be a useful tool for solving PDEs. Nowadays, scientific computation is understood as the "greater whole." Numerical techniques for solving partial differential equations PDEs are at the heart of scientific computation, which over the past 60 years has shown to be the most efficient way to accomplish complex scientific calculations. Wall Street financial models to Main Street traffic models were all influenced by numerical solutions. Specifically, we focus on nonlinear PDEs and give an overview of the evolution of these numerical techniques. Examples and applications from real life are creating heat sink designs for electronic equipment. Estimating the behavior of tsunami waves. Simulating ocean circulation and weather patterns. Recognizing the progression and management of cancer. Modeling epidemiology and population dynamics. Financial derivatives pricing and management. Estimating the spread of pollution in cities. Modeling population growth.[1] Ion-acoustic wave nonlinear propagation and bifurcation analysis in a negative ion plasma with quartic nonlinearity present.[2] A fresh approach to liquid time-constant network-based nonlinear partial differential equation solution.[3] When employing Newton's Method to solve nonlinear time-dependent PDEs, a comparison is made between RBF and FDM.[4] A nonlinear stochastic PDE model of cancer progression with optimal control.[5] Current engineering and mathematical physics are investigating new analytical characteristics for certain nonlinear PDEs.[6] Nonlinear Schrödinger model in pure cubic form featuring optical multipeaks, homoclinic breathers, periodic-cross-kink, and M-shaped solitons.[7] An approach to computational PDEs in higher dimensions based on extreme learning machines.[8] Optical solitons, multiwave, breather and M-shaped solitons for a nonlinear Schrödinger equation in (2+ 1)-dimensions with cubic–quintic–septic law.[9] In the thermophoretic motion equation via graphene sheets: bifurcation solitons, Y-type, separate lumps, and generalized breather.[10] Time-dependent pdes: implicit brain spatial representations.[11] A neural network-based deep Fourier residual approach to PDE solutions.[12] Adaptive trajectory sampling using deep learning techniques to solve PDEs.[13]

Motion using breathers, rogue waves, and rational solitons are examples of optical devices.[14] An extremely accurate PDE solution technique based on neural networks.

The homotopy and perturbation techniques were integrated to develop the homotopy perturbation approach, suitable for addressing both initial and boundary value problems as well as nonlinear issues. Scientists have been using the homotopy perturbation approach to solve mathematical problems more and more in recent years. This method is esteemed for its capability to convert a straightforward, readily solvable problem into a more intricate one under investigation through continuous deformation. [15] Mathematical analysis on novel coronavirus model using HPM.[16] A least squares HPM critical analysis for nonlinear oscillations.[17] Dynamic nonlinear oscillators with an optimum and modified homotopy perturbation approach.[18] investigation of an autonomous conservative oscillator with an enhanced perturbation technique. [19] A remark regarding the use of homotopy perturbation in solving the nonlinear coagulation problem to enhance series solutions over extended periods. [20] Wave Propagation and Soliton Behaviors Using the Expansion Technique and Sub-ODE Method for the Strain Equation.[21] Heat transfer problems are studied with Python's homotopy perturbation approach. [22] Ordinary differential equations: the least square homotopy perturbation approach. [23] Optical soliton solution control for higher order applying bilinear form to the Boussinesq equation.[24] By using the polynomial law of nonlinearity and Chupin Liu's theorem, envelope solitons, multi-peak solitons, and breathers in optical fibers can be identified.[25] Investigation of lump solutions, rogue waves, and breathers for nonlinear atomic chains.In recent work in this area, the variational iteration method has become more significant. First introduced by the Chinese mathematician He as an adaptation of the generic Lagrange multiplier method, this approach has proven to be extremely effective in resolving a wide range of problems. It is useful for many equations, including Burger's and coupled Burger's equations, shallow water equations, delay differential equations, and coupled Schrodinger-KdV equations.[26] A novel analytical method for obtaining the single solutions of a fractionally-order nonlinear evolution equation.[27] Multiple solutions for an exponentially diminishing surface with non-linear radiative mixed convective hybrid nanofluid flow.[28] Dynamics study of the Korteweg-de Vries equation for (1+ 1)-dimensional time-fractional potential.[29] A macroscopic quasilinear approach to parametric analysis of heat flux inhibition in the solar wind.[30] Interactions between solitary wave solutions for the nonlinear Schrödinger-Poisson equation and dark, brilliant, parabolic optical solitons using the Hirota technique.[31] hints about the Homann-Agrawal hybrid nanofluid flow's thermodynamic efficiency.[32] The Fornberg Whitham equation has several solitary wave solutions, including the Weierstrass and Jacobi elliptic solutions, multiwave, homoclinic breather, kink-periodic-cross rational, and others.

2 Homotopy Perturbation Method

By examining the various kinds of pathways that can be drawn within a region, a geometric region can be categorized using homotopy. If a path may be continually deformed into another, maintaining its defined region and leaving the end points unaltered, then two paths with common endpoints are said to be homotopic. In mathematics, perturbation theory describes approximation techniques that start with the solution of a related problem and work their way toward a problem's approximate solution. To facilitate problem solving, this is accomplished by decomposing the problem into manageable or perturbative components. Resolving a large range of coupled linear and nonlinear equations successfully is possible with the homotopy perturbation method (HPM), a semi-analytical solution for linear and nonlinear differential equations [16]. We have investigated the HPM and VIM in this study in order to describe the many kinds of nonlinear partial differential equations. In order to present the fundamental concepts of the approach, [34] examined the subsequent nonlinear partial differential equation.

$$A(u) - f(r) = 0, r \in \Omega \tag{2.1}$$

$$A(u) = f(r) \tag{2.2}$$

The term "analytic function" which is $f(r)$ refers to the situation where the generic differential operator is A.

$$B(u, \frac{\partial u}{\partial n}) = 0, r \in \Gamma \tag{2.3}$$

The boundary operator is denoted by B and the domain boundary by Γ. $r \in \Omega \in \Gamma$ Indicates that the domain cannot go above.

$$L(u) + N(u) - f(r) = 0 \tag{2.4}$$

$$\therefore v(r, p) : \Omega \times [0,1] \to R$$

From Eq. (2.4), we can conclude that

$$L(u) + N(u) = f(r) \tag{2.5}$$

Let

$$f(r) = A(u) \tag{2.6}$$

$$A(u) = L(u) + N(u) \tag{2.7}$$

$$A(u) - L(u) = N(u) \tag{2.8}$$

By Homotopy Method

$$H(u, p) = (1-p)[L(v) - L(u_o)] + p[A(v) - f(r)] = 0, p \in [0,1], r \in \Omega \tag{2.8*}$$

or

$$H(v, p) = L(v) - L(u_o) - pL(v) + pL(u_o) + p[A(v) - f(r)] = 0 \tag{2.9}$$

$$H(v, p) = L(v) - L(u_o) + pL(u_o) + p[A(v) - f(r)] - pL(v) = 0 \tag{2.10}$$

$$H(v, p) = L(v) - L(u_o) + pL(u_o) + p[A(v) - L(v) - f(r)] = 0 \tag{2.11}$$

$$H(v, p) = L(v) - L(u_o) + pL(u_o) + p[N(v) - f(r)] = 0 \tag{2.12}$$

Where $p \in [0,1]$

$$H(v,0) = L(v) - L(u_o) = 0 \tag{2.13}$$

$$H(v,1) = L(v) - L(u_o) + L(u_o) + [N(v) - f(r)] = 0 \tag{2.14}$$

$$H(v,1) = L(v) + [N(v) - f(r)] = 0 \tag{2.15}$$

$$H(v,1) = L(v) + [N(v) - f(r)] = 0 \tag{2.16}$$

Assume

$$v = v_o + pv_1 + p^2 v_2 + p^3 v_3 + \ldots \tag{2.17}$$

Setting $p = 1$

$$u = \lim_{p \to 1} \left(v = v_o + v_1 + v_2 + v_3 + \ldots \right) \tag{2.18}$$

2.1 Utilizing the Homotopy Perturbation Approach for the Boussinesq Korteweg de Vries:

$$u_{xt} - \alpha u u_x + \beta u_{xxx} - \gamma u_{xxxxx} = 0, \tag{2.19}$$

Subject to the following initial condition,

$$u(x,0) = f(x), \frac{\partial u(x,0)}{\partial t} = g(x) \tag{2.20}$$

$$u(0,t) = f(t), \frac{\partial u(0,t)}{\partial x} = g(t) \tag{2.21}$$

To solve (2.19) with the initial condition (2.20):

$$(1-p)\left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial x^2}\right) + p\left(\frac{\partial^2 u}{\partial x \partial t} - \alpha u \frac{\partial u}{\partial x} + \beta \frac{\partial^3 u}{\partial x^3} - \gamma \frac{\partial^5 u}{\partial x^5}\right) = 0 \tag{2.22}$$

Simplifying:

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u_o}{\partial x^2} - p\frac{\partial^2 u}{\partial x^2} + p\frac{\partial^2 u_o}{\partial x^2} + p\frac{\partial^2 u}{\partial x \partial t} - p\alpha u \frac{\partial u}{\partial x} + p\beta \frac{\partial^3 u}{\partial x^3} - p\gamma \frac{\partial^5 u}{\partial x^5} = 0 \tag{2.23}$$

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u_o}{\partial x^2} = p\frac{\partial^2 u}{\partial x^2} - p\frac{\partial^2 u_o}{\partial x^2} - p\frac{\partial^2 u}{\partial x \partial t} + p\alpha u \frac{\partial u}{\partial x} - p\beta \frac{\partial^3 u}{\partial x^3} + p\gamma \frac{\partial^5 u}{\partial x^5} \tag{2.24}$$

With initial approximation:

$$u_o(x,t) = f(x) + g(x) \cdot t \tag{2.25}$$

Let the solution of eq. (2.19) has the form:

$$v = v_o + p v_1 + p^2 v_2 + p^3 v_3 + \ldots \qquad (2.26)$$

Put eq (2.26) in (2.24)

$$\frac{\partial^2 u_o}{\partial x^2} + p\frac{\partial^2 u_1}{\partial x^2} + p^2\frac{\partial^2 u_2}{\partial x^2} + p^3\frac{\partial^2 u_3}{\partial x^2} - \frac{\partial^2 u_o}{\partial x^2} = [\frac{\partial^2 u_o}{\partial x^2} + p\frac{\partial^2 u_1}{\partial x^2} + p^2\frac{\partial^2 u_2}{\partial x^2} + p^3\frac{\partial^2 u_3}{\partial x^2} -$$

$$\frac{\partial^2 u_o}{\partial x^2} - \frac{\partial^2 u_o}{\partial x\partial t} - p\frac{\partial^2 u_1}{\partial x\partial t} - p^2\frac{\partial^2 u_2}{\partial x\partial t} - p^3\frac{\partial^2 u_3}{\partial x\partial t} + \alpha u_o\frac{\partial u_o}{\partial x} + p\alpha u_1\frac{\partial u_1}{\partial x} + p^2\alpha u_2\frac{\partial u_2}{\partial x} + p^3\alpha u_3\frac{\partial u_3}{\partial x} -$$

$$\beta\frac{\partial^3 u_o}{\partial x^3} - p\beta\frac{\partial^3 u_1}{\partial x^3} - p^2\beta\frac{\partial^3 u_2}{\partial x^3} - p^3\beta\frac{\partial^3 u_3}{\partial x^3} + \gamma\frac{\partial^5 u_o}{\partial x^5} + p\gamma\frac{\partial^5 u_1}{\partial x^5} + p^2\gamma\frac{\partial^5 u_2}{\partial x^5} + p^3\gamma\frac{\partial^5 u_3}{\partial x^5}] \qquad (2.27)$$

Comparing co-efficient of identical degrees of p:

$$p^0 : \frac{\partial^2 u_o}{\partial x^2} - \frac{\partial^2 u_o}{\partial x^2} = 0 \qquad (2.28)$$

$$p^1 : \frac{\partial^2 u_1}{\partial x^2} = -\frac{\partial^2 u_o}{\partial x\partial t} + \alpha u_o\frac{\partial u_o}{\partial x} - \beta\frac{\partial^3 u_o}{\partial x^3} + \gamma\frac{\partial^5 u_o}{\partial x^5} \qquad (2.29)$$

$$p^2 : \frac{\partial^2 u_2}{\partial x^2} = \frac{\partial^2 u_1}{\partial x^2} - \frac{\partial^2 u_1}{\partial x\partial t} + \alpha u_1\frac{\partial u_1}{\partial x} - \beta\frac{\partial^3 u_1}{\partial x^3} + \gamma\frac{\partial^5 u_1}{\partial x^5} \qquad (2.30)$$

$$p^3 : \frac{\partial^2 u_3}{\partial x^2} = \frac{\partial^2 u_2}{\partial x^2} - \frac{\partial^2 u_2}{\partial x\partial t} + \alpha u_2\frac{\partial u_2}{\partial x} - \beta\frac{\partial^3 u_2}{\partial x^3} + \gamma\frac{\partial^5 u_3}{\partial x^5} \qquad (2.31)$$

And so on…

For simplicity, we'll take:

$$v_o = u_o = f(x) + g(x)\cdot t \qquad (2.32)$$

respectively, we have:

$$v_1 = \int_0^t\int_0^t \left[\frac{\partial^2 u_o(x,\tau)}{\partial x\partial \tau} + \alpha u_o(x,\tau)\frac{\partial u_o(x,\tau)}{\partial x} - \beta\frac{\partial^3 u_o(x,\tau)}{\partial x^3} + \gamma\frac{\partial^5 u_o(x,\tau)}{\partial x^5} \right] d\tau dt \qquad (2.33)$$

$$v_2 = \int_0^t\int_0^t \left[\frac{\partial^2 u_1(x,\tau)}{\partial x^2} - \frac{\partial^2 u_1(x,\tau)}{\partial x\partial \tau} + \alpha u_1(x,\tau)\frac{\partial u_1(x,\tau)}{\partial x} - \beta\frac{\partial^3 u_1(x,\tau)}{\partial x^3} + \gamma\frac{\partial^5 u_1(x,\tau)}{\partial x^5} \right] d\tau dt \qquad (2.34)$$

$$v_3 = \int_0^t\int_0^t \left[\frac{\partial^2 u_2(x,\tau)}{\partial x^2} - \frac{\partial^2 u_2(x,\tau)}{\partial x\partial \tau} + \alpha u_2(x,\tau)\frac{\partial u_2(x,\tau)}{\partial x} - \beta\frac{\partial^3 u_2(x,\tau)}{\partial x^3} + \gamma\frac{\partial^5 u_2(x,\tau)}{\partial x^5} \right] d\tau dt \qquad (2.35)$$

Thus, one can derive the approximate solution of equation (2.19) by setting p=1,

$$u = v_o + v_1 + v_2 + v_3 + \ldots \qquad (2.36)$$

Boussinesq Korteweg de Vries 5th Order Equation by Homotopy Perturbation method:

Example 01:

$$u_{xt} - 3uu_x + 5u_{xxx} - 2u_{xxxxx} = 0, \qquad (2.37)$$

Connected to the subsequent initial state

$$u(x,0) = x^2, \frac{\partial u(x,0)}{\partial t} = 0 \qquad (2.38)$$

For the unique case where the partial derivative and u are not dependent on the equation (2.37), we have utilized homotopy perturbation. We build the following homotopy in accordance with the homotopy perturbation in order to solve equation (2.37) given initial conditions.

$$(1-p)\left(\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u_0}{\partial t^2}\right) + p\left(\frac{\partial^2 u}{\partial x\partial t} - 3u\frac{\partial u}{\partial x} + 5\frac{\partial^3 u}{\partial x^3} - 2\frac{\partial^5 u}{\partial x^5}\right) = 0 \qquad (2.39)$$

or

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u_0}{\partial t^2} - p\frac{\partial^2 u}{\partial t^2} + p\frac{\partial^2 u_0}{\partial t^2} + p\frac{\partial^2 u}{\partial x\partial t} - 3pu\frac{\partial u}{\partial x} + 5p\frac{\partial^3 u}{\partial x^3} - 2p\frac{\partial^5 u}{\partial x^5} = 0 \qquad (2.40)$$

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u_0}{\partial t^2} = p\frac{\partial^2 u}{\partial t^2} - p\frac{\partial^2 u_0}{\partial t^2} - p\frac{\partial^2 u}{\partial x\partial t} + 3pu\frac{\partial u}{\partial x} - 5p\frac{\partial^3 u}{\partial x^3} + 2p\frac{\partial^5 u}{\partial x^5} = 0 \qquad (2.41)$$

With initial approximation $u_0 = f(x) + g(x)t$,

$$u_o(x,t) = x^2 \tag{2.42}$$

Suppose that the solution of (4.1) has of the form

$$u = u_0 + Pu_1 + P^2u_2 + P^3u_3 + P^4u_4 + ... \tag{2.43}$$

Put (2.43) in (2.41) we get comparisons coefficient of identical degree of

For simplicity we take

$$u_0 = f(x) + g(x)t \, . \tag{2.44}$$

Accordingly we have

$$u_o(x,t) = x^2, \quad \frac{\partial u_o(x,\tau)}{\partial x} = 2x, \quad \frac{\partial^2 u_o(x,\tau)}{\partial x \partial \tau} = 0, \quad \frac{\partial^2 u_o(x,\tau)}{\partial x^2} = 2, \quad \frac{\partial^3 u_0(x,\tau)}{\partial x^3} = 0, \quad \frac{\partial^4 u_0(x,\tau)}{\partial x^4} = 0, \quad \frac{\partial^5 u_0(x,\tau)}{\partial x^5} = 0 \tag{2.45}$$

By (2.45) putting these values in (2.33) then get first approximation

$$u_1(x,t) = 3x^3t^2 \tag{2.46}$$

Similarly use this process to find u_2 …

$$u_2(x,t) = 3x^3t^2 + 3x^2t^3 + \frac{27}{10}x^5t^6 - \frac{15}{2}t^4 \tag{2.47}$$

$$u_3(x,t) = 3x^3t^2 + \frac{27}{5}x^5t^6 - 15t^4 - \frac{3}{2}xt^4 + \frac{9}{5}x^4t^6 + \frac{162}{15}x^7t^{10} + \frac{27}{28}x^3t^8 + \frac{1701}{1100}x^6t^{11} + \frac{729}{3640}x^9t^{14} - \frac{1755}{112}x^2t^8 - \frac{5}{8}xt^9 - \frac{405}{528}x^4t^{12} + \frac{81}{7}t^8 \tag{2.48}$$

So the 3rd order approximation solution

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) \tag{2.49}$$

Put (2.42),(2.46),(2.47),(2.48) in equation (2.49)

$$u(x,t) = x^2 + 12xt^6 + \frac{81}{10}xt^{30} - \frac{45}{2}t^4 - \frac{681430373}{134217728} + \frac{387}{140}xt^{24} + \frac{54}{5}xt^{70} + \frac{1701}{1100}xt^{66} + \frac{229}{3640}xt^{126} - \frac{1755}{112}xt^{16} - \frac{135}{176}xt^{48} + \frac{81}{7}t^8 \tag{2.50}$$

This is an exact solution**:**

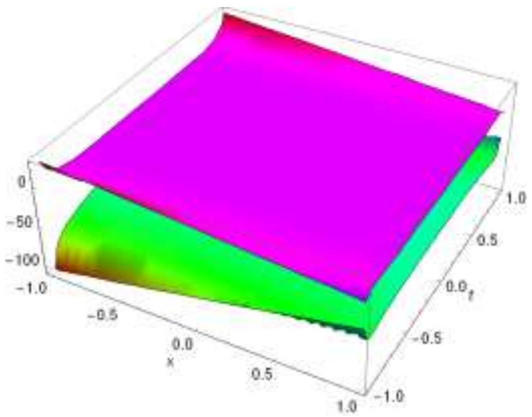

Figure 3.1: The 3D wave profile of u(x,t) with particular intervals.

To determine 3D wave profile of u(x,t) with particular intervals we use equation (2.50) for the function u(x,t).Figure 3.1 above shown 3D wave profile of u(x,t) with particular intervals for -1 ≤ x,t ≥ 1

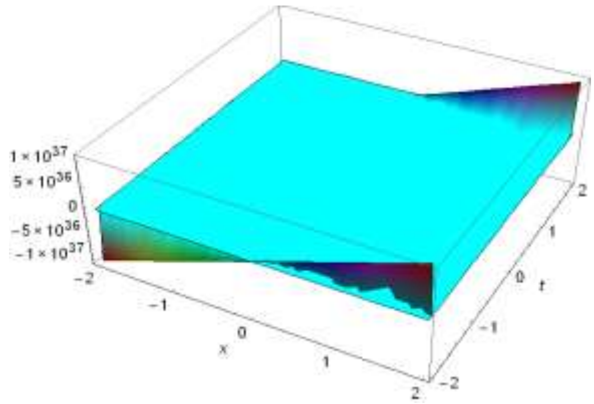

Figure 3.2: The 3D wave profile of u(x,t) with particular intervals.

To determine 3D wave profile of u(x,t) with particular intervals we use equation (2.50) for the function u(x,t).Figure 3.2 above shown 3D wave profile of u(x,t) with particular intervals for -2 ≤ x,t ≥ 2

Boussinesq Korteweg de Vries 5th Order Equation by Homotopy Perturbation method:

Example 02:

$$u_{xt} - 4uu_x + \frac{1}{5}u_{xxx} - 13u_{xxxxx} = 0 \tag{2.51}$$

With the initial

$$u(0,t) = t+1, \frac{\partial u(0,t)}{\partial x} = 2 \tag{2.52}$$

For particular circumstances where the coefficients in (2.51) do not depend on the partial derivative and u, we have used the homotopy perturbation approach.

Using the homotopy perturbation approach, we construct the following homotopy to solve equation (2.51) given the initial condition:

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u_o}{\partial x^2} = p\frac{\partial^2 u}{\partial x^2} - p\frac{\partial^2 u_o}{\partial x^2} - p\frac{\partial^2 u}{\partial x \partial t} + 4pu\frac{\partial u}{\partial x} - p\frac{1}{5}\frac{\partial^3 u}{\partial x^3} + 13p\frac{\partial^5 u}{\partial x^5} \tag{2.53}$$

With initial approximation

$$u_0 = f(x) + g(x)t \tag{2.54}$$

$$u_o(x,t) = t + 1 + 2x \tag{2.55}$$

Assume that the form of equation (4.5)'s solution

$$u = p^0 u_0 + p^1 u_1 + p^2 u_2 + p^3 u_3 + p^4 u_4 + ... \tag{2.56}$$

For simplicity we take

$$u_0 = f(x) + g(x)t. \tag{2.57}$$

Accordingly we have

$$u_o(x,\tau) = \tau + 1 + 2x \, , \; \frac{\partial u_o(x,\tau)}{\partial x} = 0 + 0 + 2x \, , \; \frac{\partial u_o(x,\tau)}{\partial x} = 2 \, , \; \frac{\partial^2 u_o(x,\tau)}{\partial x \partial t} = 0 \, , \; \frac{\partial^2 u_o(x,\tau)}{\partial x^2} = 0 \, ,$$

$$\frac{\partial^5 u_o(x,\tau)}{\partial x^5} = 0 \, , \; \frac{\partial^3 u_o(x,\tau)}{\partial x^3} = 0 \tag{2.58}$$

By using equation (2.58) and (2.33) the solution reads as

$$u_1 = \int_0^t \int_0^t \left[-\frac{\partial^2 u_o(x,\tau)}{\partial x \partial t} + 4u_o(x,\tau)\frac{\partial u_o(x,\tau)}{\partial x} - \frac{1}{5}\frac{\partial^3 u_o(x,\tau)}{\partial x^3} + 13\frac{\partial^5 u_o(x,\tau)}{\partial x^5} \right] d\tau dt$$

$$u_1(x,t) = \frac{20x^3}{3} + 4x^2 \tag{2.59}$$

Similarly process used this to get

$$u_2(x,\tau) = \frac{40x^3}{3} + \frac{1600x^7}{21} + \frac{860x^6}{9} + \frac{128x^5}{5} \tag{2.60}$$

$$u_3(x,t) = 277360x^3 + \frac{16000x^7}{21} - \frac{800x^6}{9} - \frac{4864x^5}{15} + \frac{422392x^{11}}{3465} + \frac{3200x^{10}}{3} + \frac{8192x^9}{27} + \frac{152000x^{15}}{189} + \frac{8320000x^{14}}{1323} + \frac{13383680x^{13}}{2457} + \frac{56320x^{12}}{27} + \frac{3119616x^4}{5} \tag{2.61}$$

So the 3rd approximation solution is

$$u(x,t) = u_o(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) \tag{2.62}$$

Put (2.55),(2.58),(2.59),(2.60) in equation (2.62)

$$u(x,t) = t + 1 + 2x + 277380x^3 + 4x^2 + \frac{17600x^7}{21} + \frac{20x^6}{3} - \frac{896x^5}{3} + \frac{422392x^{11}}{3465} + \frac{3200x^{10}}{3} + \frac{8192x^9}{27} + \frac{152000x^{15}}{189} + \frac{8320000x^{14}}{1323} + \frac{13383680x^{13}}{2475} + \frac{56320x^{12}}{27} + \frac{3119616x^4}{5} \tag{2.63}$$

This is an exact solution.

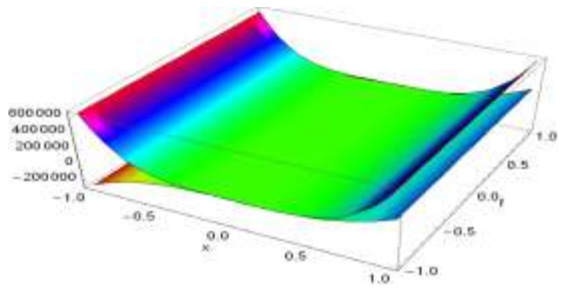

Figure 2.3: The 3D wave profile of u(x,t) with particular intervals.

To determine 3D wave profile of u(x,t) with particular intervals we use equation (2.63) for the function u(x,t).Figure 2.3 above shown 3D wave profile of u(x,t) with particular intervals for -1 ≤ x,t ≥ 1

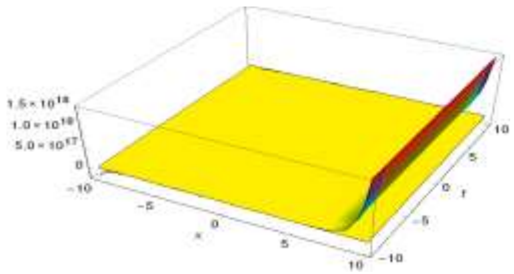

Figure 2.4: The 3D wave profile of u(x,t) with particular intervals.

To determine 3D wave profile of u(x,t) with particular intervals we use equation (2.63) for the function u(x,t).Figure 2.4 above shown 3D wave profile of u(x,t) with particular intervals for -10 ≤ x,t ≥ 10

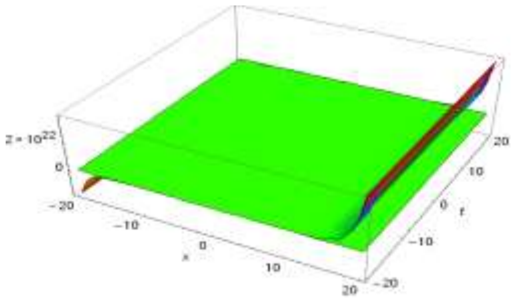

Figure 2.5: The 3D wave profile of u(x,t) with particular intervals.

To determine 3D wave profile of u(x,t) with particular intervals we use equation (2.63) for the function u(x,t).Figure 2.5 above shown 3D wave profile of u(x,t) with particular intervals for -20 ≤ x,t ≥ 20

3 Variational Iteration Method

He was a Chinese mathematician who developed the VIM, a computational technique that solves many linear and nonlinear problems with ease. It delivers analytical answers without discretization and requires little memory or processing capacity (Inc, 2008).

The 2differential equation that follows should be considered.

$$Lu + Fu = g(t) \tag{3.1}$$

In this case, the heterogeneous term is $g(t)$, the nonlinear operator is F, and the linear operator is L.

Note: A heterogeneous phrase has different sorts and is made up of components with vastly disparate elements or constituents.

The Variational Iteration Method states that a correction functional has the following form:

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda\left(Lu_n(\tau) + Fu_n(\tau) - g(\tau)\right)d\tau \tag{3.2}$$

Note: λ is a general Lagrange multiplier and the variational theory allow for its optimal identification. The subscript n represents an n^{th} approximation and u_n is regarded as a limited variant, i.e. $\delta u_n = 0$

$$\delta \tilde{u}_{k+1} = 0 \tag{3.3}$$

From (3.3) we find out Lagrange Multiplier

$$\delta \tilde{u}_{k+1} = \delta \tilde{u}_k(t) + \delta \int_0^t \lambda(\tau)\left[L\tilde{u}_k(\tau) + F\tilde{u}_k(\tau) - g(\tau)\right]d\tau,$$

$$\delta \tilde{u}_{k+1} = \delta \tilde{u}_k(t) + \int_0^t \lambda(\tau)\left[\delta L\tilde{u}_k(\tau) + \delta F\tilde{u}_k(\tau) - g(\tau)\right]d\tau,$$

$$\delta \tilde{u}_{k+1} = \delta \tilde{u}_k(t) + \int_0^t \lambda(\tau)\left[L\left(\delta \tilde{u}_k(\tau)\right) + F\left(\delta \tilde{u}_k(\tau)\right) - g(\tau)\right]d\tau. \tag{3.4}$$

Equation (3.4) cleared that the value of λ ,

If $\mathrm{m} = 1$ then $\lambda = -1$

If $\mathrm{m} = 2$ then $\lambda = -\tau$

$$\lambda = \frac{(-1)^m}{(m-1)!}(\tau - t)^{m-1} \quad \text{for, } m \geq 1. \tag{3.5}$$

Putting the (3.5) into (3.4) then we can have

$$\tilde{u}_{k+1} = \tilde{u}_k(t) + \int_0^t \frac{(-1)^m}{(m-1)!}(\tau - t)^{m-1}\left[L\tilde{u}_k(\tau) + F\tilde{u}_k(\tau) - g(\tau)\right]d\tau \tag{3.6}$$

Now, if we define the operator

$$G(\tilde{u}) = \int_0^t \frac{(-1)^m}{(m-1)!}(\tau - t)^{m-1}\left[L\tilde{u}_k(\tau) + F\tilde{u}_k(\tau) - g(\tau)\right]d\tau. \tag{3.7}$$

Additionally, specify the elements $\tilde{v}_k$, K=0,1, 2,3… as

$$\tilde{v}_0 = \tilde{u}_0 \tag{3.8}$$

$$\tilde{v}_1 = G(\tilde{u}_0) \tag{3.9}$$

$$\tilde{v}_{k+1} = G(\tilde{u}_0 + \tilde{u}_1 + \tilde{u}_2 + \ldots + \tilde{u}_k) \tag{3.10}$$

Consequently, the answer to (3.2) can be found as

$$\tilde{u}(x, y) = \lim_{k\to\infty} \tilde{u}_k(x, y) = \sum_{k=0}^{\infty} \tilde{v}_k(x, y) \tag{3.11}$$

The truncated series / The abbreviated sequence

$$u(x,y)=\sum_{k=0}^{\infty} u_k(x,y) \tag{3.12}$$

can be applied to roughly determine the precise solution of (3.2):

The problem's initial guess (2.20) should be selected to satisfy the provided initial and boundary requirements.

Boussinesq Korteweg de Vries 5th Order Equation by Variational Iteration Method:

Example 03:

$$u_{xt}-3uu_x+5u_{xxx}-2u_{xxxxx}=0 \tag{3.13}$$

$$u(x,0)=x^2,\ \frac{\partial u(x,0)}{\partial t}=0 \tag{3.14}$$

Solution: Here we have

$$Lu=\frac{\partial u}{\partial x},\quad Nu=u^2,\quad g(t)=0 \tag{3.15}$$

To determine the Lagrange multiplier, we insert these expressions and obtained

$$u_{n+1}(x,t)=u_n(x,t)+\int_0^t \lambda(\tau)[\frac{\partial^2 u}{\partial x\partial t}u_n(x,\tau)-3u_n(x,\tau)\frac{\partial u}{\partial x}u_n(x,\tau)+5\frac{\partial^3 u}{\partial x^3}u_n(x,\tau)-2\frac{\partial^5 u}{\partial x^5}u_n(x,\tau)]d\tau \tag{3.16}$$

The Lagrangian multiplier can be identified by the formula:

$$\lambda(\tau)=\frac{(-1)^n}{(n-1)!}(\tau-t)^{n-1} \tag{3.17}$$

As Boussinesq Korteweg de vries is an fifth order nonlinear partial differential equation, so put $n=5$ in above equation to get:

$$\lambda(\tau)=\frac{-1}{24}(\tau-t)^4 \tag{3.18}$$

From equation (5.28) and Put n=0 in equation (5.26) and we get

$$u_1(x,t)=u_o(x,t)+\left(\frac{-1}{24}\right)\int_0^t(\tau-t)^4[\frac{\partial^2 u_o(x,\tau)}{\partial x\partial t}-3u_o(x,\tau)\frac{\partial u_o(x,\tau)}{\partial x}+5\frac{\partial^3 u_o(x,\tau)}{\partial x^3}-2\frac{\partial^5 u_o(x,\tau)}{\partial x^5}]d\tau \tag{3.19}$$

We have to choose a starting function, $u_0(x,t)$:

which satisfies the given initial condition:

$$u_o(x,t)=x^2 \tag{3.20}$$

we compute the following successive approximations:

$$u_1(x,t)=u_o(x,t)+\left(\frac{-1}{24}\right)\int_0^t(\tau-t)^4[\frac{\partial^2 u_o(x,\tau)}{\partial x\partial t}-3u_o(x,\tau)\frac{\partial u_o(x,\tau)}{\partial x}+5\frac{\partial^3 u_o(x,\tau)}{\partial x^3}-2\frac{\partial^5 u_o(x,\tau)}{\partial x^5}]d\tau \tag{3.21}$$

$$u_o(x,t)=x^2,\ \frac{\partial u_o(x,\tau)}{\partial x}=2x,\ \frac{\partial^2 u_o(x,\tau)}{\partial x^2}=2,\ \frac{\partial^3 u_o(x,\tau)}{\partial x^3}=0,\ \frac{\partial^5 u_o(x,\tau)}{\partial x^5}=0,\ \frac{\partial^2 u_o(x,\tau)}{\partial x\partial t}=0 \tag{3.22}$$

Put (3.22) of all values in equation (3.21) we get

$$u_1(x,t)=x^2-\frac{1}{24}\int_0^t(\tau-t)^4\left[-3x^2(2x)\right]d\tau \tag{3.23}$$

$$u_1(x,t) = x^2 - \frac{1}{24}\int_0^t (\tau - t)^4 \left[-6x^3\right] d\tau \tag{3.24}$$

$$u_1(x,t) = x^2 + \frac{1}{20}t^5x^3 \tag{3.25}$$

Similarly

$$u_2(x,t) = u_1(x,t) - \frac{1}{24}\int_0^t (\tau - t)^4 [\frac{\partial^2 u_1(x,\tau)}{\partial x \partial t} - 3u_1(x,\tau)\frac{\partial u_1(x,\tau)}{\partial x} + 5\frac{\partial^3 u_1(x,\tau)}{\partial x^3} - 2\frac{\partial^5 u_1(x,\tau)}{\partial x^5}] d\tau$$

$$u_2(x,t) = x^2 + \frac{1}{20}t^5x^3 - \frac{1}{24}\int_0^t (\tau - t)^4 [\frac{15}{20}t^4x^2 - 3\left(x^2 + \frac{1}{20}t^5x^3\right)\left(2x + \frac{3}{20}t^5x^2\right) + 5\left(\frac{6}{20}t^5\right) - 2(0)] d\tau$$

$$u_2(x,t) = x^2 + \frac{1}{10}t^5x^3 + \frac{1}{16016000}t^{15}x^5 + \frac{1}{40320}t^{10}x^4 - \frac{1}{20160}t^9x^2 \tag{3.26}$$

$$\begin{aligned} u_3(x,t) = {} & x^2 + \frac{1}{20}t^5x^3 + \frac{23}{73382400}t^{15}x^5 + \frac{1}{13440}t^{10}x^4 - \frac{1}{6720}t^9x^2 - \frac{1}{219108105216000000}t^{29}x^6 \\ & + \frac{1}{13440}t^{13}x - \frac{1}{40320}t^{15}x + \frac{76151}{3087855074304000000}t^{25}x^7 - \frac{1}{230329516032000}t^{24}x^5 \\ & + \frac{18}{27351630288000}t^{23}x^3 + \frac{1}{124156032000}t^{20} + \frac{733}{25540669440000}t^{20}x^6 \\ & - \frac{1}{99324825600}t^{20}x^2 - \frac{1}{17643225600}t^{19}x^4 - \frac{1}{151351200}t^{14}x^3 - \frac{1}{10080}t^{10} \\ & + \frac{1}{6661766935183360000000}t^{35}x^9 + \frac{1}{409001796403200000}t^{30}x^8 \end{aligned} \tag{3.27}$$

Hence similarly:

$$u(x,t) = u_o(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) \tag{3.28}$$

Put (3.20),(3.25),(3.26),(3.27) in equation (3.28)

$$\begin{aligned} u(x,t) = {} & 4x^2 + \frac{1}{5}t^{15}x + \frac{1517}{403603200}t^{75}x + \frac{29560957}{29560957}t^{40}x - \frac{1}{5040}t^{18}x - \frac{1}{219108105216000000}t^{174}x \\ & + \frac{2591894727475201}{121006263724814531738616550339649511312886218512714956800000000000000000} \\ & + \frac{76151}{3087855074304000000}t^{175}x - \frac{69286669}{2417884094545920000}t^{120}x + \frac{1}{1519535016000}t^{69}x \\ & + \frac{1}{124156032000}t^{20} - \frac{1}{17643225600}t^{76}x - \frac{1}{151351200}t^{42}x - \frac{1}{10080}t^{10} \\ & + \frac{1}{6661766935183360000000}t^{315}x + \frac{1}{409001796403200000}t^{240}x \end{aligned} \tag{3.29}$$

This an exact solution.

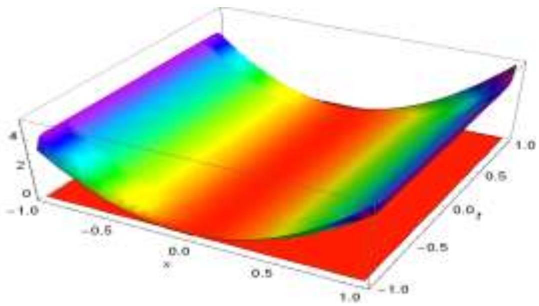

Figure 3.1: The 3D wave profile of u(x,t) with particular intervals.

To determine 3D wave profile of u(x,t) with particular intervals we use equation (3.29) for the function u(x,t).Figure 3.1 above shown 3D wave profile of u(x,t) with particular intervals for $-1 \leq x,t \geq 1$

4. Results and Discussion

This chapter includes solved numerical examples of nonlinear fifth-order BKdVEPDEs with constant coefficients. The iterative methods employed are the variational iteration approach and the HPM.

Rather than using discretization or linearization, these techniques produce the solution as an infinite series.These methods provide a series or iterative solution that can be refined to achieve the desired accuracy. carefully examined the iterative formulations of the aforementioned methods and confirmed that the resulting series of solutions converges.

Both linear and nonlinear PDEs can be analytically solved using the HPM approach.By creating a progressive translation from a known linear operator to the target differential equation, the Homotopy Perturbation Method (HPM) produces a series solution for differential equations. It involves breaking down the problem into simpler parts and perturbing the solution iteratively. The method is based on the concept of homotopy, where a continuous transformation is applied to map the given problem to a simpler one for which a solution is known. HPM can accurately provide both approximate and exact solutions for a broad spectrum of linear and nonlinear PDEs, particularly demonstrating high precision in cases involving small perturbation parameters. HPM is particularly useful for handling weakly nonlinear problems, as it avoids linearization and provides accurate results.

A computer technique called VIM is used to solve a wide range of differential equations and PDEs, including both linear and nonlinear forms. It is based on the concept of variance iteration, which seeks an approximate solution by gradually refining an initial guess through an iterative process. VIM follows a Lagrange multiplier to impose boundary conditions, resulting in an accurate solution within a few iterations. VIM is versatile and widely able to solve various differential equations. In summary, HPM and VIM are analytical methods that construct approximate solutions through different approaches, while VIM is a numerical method that iteratively refines an initial guess to obtain an accurate solution. Each method has its advantages and is suitable for different types of differential equations and problem scenarios. They effectively produced approximate and correct solutions to several nonlinear problems from the literature without the use of any arbitrary functions. Ultimately, they solved the well-known nonlinear problem known as the PDE problem by applying every aforementioned iterative strategies. Comparative study has been done to determine the semi-analytical solution of the partial differential problem given beginning circumstances. An infinite series that quickly converges inside its designated domain is the partial differential equation's solution. In terms of propagation coordinates and time, this sort of solution provides an exact description of the beam's acoustic pressure. This kind of solution has been providing new insights into how the confined beam passes through the nonlinear material since it hasn't been found before. This study looked into a nonlinear partial differential problem using the HPM approach.

The HPM, in contrast to previous approaches, offers precise numerical solutions for nonlinear problems without requiring large amounts of computer memory or the discretization of variables t and x. The outcomes demonstrate how well the HPM approach works for resolving nonlinear PDE. Using Maple software, computational analyses were carried out results $u_3(x,t)$ and u(x,t) give negative answer on different values of x and t from table 01.Similarly when we apply VIM and results $u_3(x,t)$ and u(x,t) are positive answer same values(from table 03) of x and t already used in table 01 . Moreover, nonlinear PDE were solved using the VIM without the need for tiny parameters or linearization strategies. By using variational iteration theory, the solution process is quite straightforward, and a small number of iterations result in highly accurate solutions within a defined zone of convergence . The Variational Iteration method solution cannot be relied upon when the solution is required outside of the convergence zone. The use of the approach in this study results in a treatment that widens the zone of convergence. The HPM and VIM have proven highly useful for solving PDEs, yielding remarkable results in various applications, thereby suggesting their potential as valuable methods for diverse practical uses. All of the examples demonstrated the great agreement between the findings of the current approaches and the precise answers. In certain circumstances, it can reach the desired outcomes with the least iteration, or even only one.

Table 1.1: ***The solution of equation (2.48) and (2.50) for different value of x and t.***

x	t	**Approximate Solution** $u_3(x,t)$ **(HPM)**	**Exact solution** $u(x,t)$**(HPM)**	**Error**
0.01	0.02	−5.077054315	−5.076955515	0.00009880
0.001	0.002	−5.077051915	−5.077050915	0.00000100
0.03	0.04	−5.077090313	−5.076209506	0.00088080
0.003	0.004	−5.077051919	−5.077042921	0.00000899

Table 1.2: ***The solution of equation (2.61) and (2.63) for different value of x and t.***

x	t	**Approximate Solution** $u_3(x,t)$ **(HPM)**	**Exact solution** $u(x,t)$**(HPM)**	**Error**
0.009	0.008	1.2062889810	1.232627562	0.02633851
0.20	0.10	3217.057698	3217.892982	0.835284
0.23	0.12	5122.437196	5122.505353	0.068157
0.22	0.11	4416.771812	4416.731892	0.03992

Table 1.3:The solution of equation (3.25) and (3.27) for different value of x and t.

x	t	**Approximate Solution** $u_3(x,t)$ **(VIM)**	**Exact solution** $u(x,t)$**(VIM)**	**Error**
0.01	0.02	0.0001	0.0004	0.0003
0.001	0.002	0.000001	0.000004	0.000003
0.03	0.04	0.0009	0.0036	0.0027
0.003	0.004	0.000009	0.000036	0.000027

5. Conclusions

In this work, we have achieved an approximation for solution of hyperbolic partial differential equation by applying homotopy perturbation and variational iteration method. VIM is a numerical method that iteratively refines an initial guess to reach an accurate answer, whereas HPM and VIM are analytical methods that generate approximate solutions using distinct methodologies. Every approach has benefits and works well with various kinds of differential equations and circumstances. They successfully created approximate and accurate solutions to several nonlinear problems from the literature, without the use of any arbitrary functions. In the end, they solved the well-known nonlinear problem known as the PDE problem by applying each of the aforementioned iterative strategies.

The partial differential problem's semi-analytical solution under initial conditions is found by a comparative investigation to be an infinite series with quick convergence within the domain. This type of solution offers a more precise expression of the beam's acoustic pressure in terms of time and propagation coordinates. Since this type of solution has not been identified previously, it has been offering new insights into the way the confined beam flows through the nonlinear material. According to the results, the HPM yielded exceptionally high-quality results. This work developed the capacity to approximate a partial differential equation solution through the application of the homotopy perturbation technique. Because of its rapid convergence and low computational burden, the homotopy perturbation method emerges as a more reliable solution that greatly accelerates the PDE solution process.

It is significant to remember that the results in Examples 1-3 are identical to those of the homotopy perturbation approach. In Chapter numerical results discusses many examples of nonlinear partial differential equation i.e., Boussinesq Korteweg de vries equation, using the HPM and the VIM to find the approximate and exact solution. Examples 1 to 02 have been solved by using the Homotopy perturbation method with some initial condition. Then HPM technique has been applied. To precede the solution, identical degrees of "p" has been compared and found out the equation for integration with the limit 0 to t. the initial approximation have been used and replaced the variable "t" with "τ". After found out the relevant partial derivatives, integral has been solved for first iteration as $u_1(x,t)$. Proceed the solution similarly for $u_2(x,t)$ and $u_3(x,t)$ where the approximate solution is $u_3(x,t)$ and $u(x,t)$ is the exact solution.

Similarly,example 03 have been solved by using the Variational Iteration method with some initial condition. Then HPM technique has been applied. To precede the solution, identical degrees of "p" has been compared and found out the equation for integration with the limit 0 to t. the initial approximation have been used and replaced the variable "t" with "τ". After found out the relevant partial derivatives, integral has been solved for first iteration as $u_1(x,t)$. Proceed the solution similarly for $u_2(x,t)$ and $u_3(x,t)$ where the approximate solution is $u_3(x,t)$ and $u(x,t)$ is the exact solution.

References

[1] Hafez, M. G., Akter, P., Chettri, K., Sakthivel, R., & Saha, A. (2024). Nonlinear propagation of ionacoustic waves along with their bifurcation analysis in a negative ion plasma in the presence of quartic nonlinearity. *Physics of Fluids*, *36*(7).

[2] Sun, J., Dong, H., & Fang, Y. (2024). A new method for solving nonlinear partial differential equations based on liquid time-constant networks. *Journal of Systems Science and Complexity*, *37*(2), 480-493.

[3] Murphy, S., Ogunleye, O., Rubasinghe, K., & Yao, G. (2024, May). Comparison between RBF and FDM for solving nonlinear time-dependent PDEs using Newton's Method. In *Journal of Physics: Conference Series* (Vol. 2766, No. 1, p. 012166). IOP Publishing.

[4] Esmaili, S., Eslahchi, M. R., & Torres, D. F. (2024). Optimal control for a nonlinear stochastic PDE model of cancer growth. *Optimization*, *73*(9), 2745-2789.

[5] Ganie, A. H., Sadek, L. H., Tharwat, M. M., Iqbal, M. A., Miah, M. M., Rasid, M. M....& Osman, M. S. (2024). New investigation of the analytical behaviors for some nonlinear PDEs in mathematical physics and modern engineering. *Partial Differential Equations in Applied Mathematics*, *9*, 100608.

[6] Seadawy, A. R., Rizvi, S. T., Ahmed, S., & Khaliq, A. (2022). Pure-cubic nonlinear Schrödinger model with optical multi peak, homoclinic breathers, periodic-cross-kink and M-shaped solitons. *Optical and Quantum Electronics*, *54*(11), 771.

[7] Wang, Y., & Dong, S. (2024). An extreme learning machine-based method for computational PDEs in higher dimensions. *Computer Methods in Applied Mechanics and Engineering,418,116578.*

[8] Ahmed, S., Seadawy, A. R., & Rizvi, S. T. (2024). Optical solitons, multiwave, breather and M-shaped solitons for an nonlinear Schrödinger equation in (2+ 1)-dimensions with cubic–quintic–septic law. *Journal of Nonlinear Optical Physics & Materials*, *33*(03), 2350029.

[9] Seadawy, A. R., Ahmad, A., Rizvi, S. T., & Ahmed, S. (2024). Bifurcation solitons, Y-type, distinct lumps and generalized breather in the thermophoretic motion equation via graphene sheets. *Alexandria Engineering Journal*, *87*, 374-388.

[10] Chen, H., Wu, R., Grinspun, E., Zheng, C., & Chen, P. Y. (2023, July). Implicit neural spatial representations for time-dependent pdes. In *International Conference on Machine Learning* (pp. 5162-5177). PMLR.

[11] Taylor, J. M., Pardo, D., & Muga, I. (2023). A deep Fourier residual method for solving PDEs using neural networks. *Computer Methods in Applied Mechanics and Engineering*, *405*, 115850.

[12] Chen, X., Cen, J., & Zou, Q. (2024). Adaptive trajectories sampling for solving PDEs with deep learning methods. AppliChen, X., Cen, J., & Zou, Q. (2024). Adaptive trajectories sampling for solving PDEs with deep learning methods. Applied Mathematics and Computation, 481, 128928.ed Mathematics and Computation, 481, 128928.

[13] Rehman, U., Ahmed, S., & Mubaraki, A. M. (2024). Optical devices: motion via breathers, rogue waves and rational solitons. *Optical and Quantum Electronics*, *56*(5), 851.

[14] Jiang, Z., Jiang, J., Yao, Q., & Yang, G. (2023). A neural network-based PDE solving algorithm with high precision. *Scientific Reports*, *13*(1), 4479. Jiang, Z., Jiang, J., Yao, Q., & Yang, G. (2023). A neural network-based PDE solving algorithm with high precision. Scientific Reports, 13(1), 4479.

[15] Anitha, S., Selvi, K. T., & Senthamarai, R. (2024). Mathematical analysis on novel coronavirus model using HPM. In *E3S Web of Conferences* (Vol. 531, p. 03013). EDP Sciences.

[16] Rafiq, M., Kamran, M., Ahmad, H., & Saliu, A. (2024). Critical analysis for nonlinear oscillations by least square HPM. *Scientific Reports*, *14*(1), 1456.

[17] Roy, T., Rath, B., Asad, J., Maiti, D. K., Mallick, P., & Jarrar, R. (2024). Nonlinear oscillators dynamics using optimal and modified homotopy perturbation method. *Journal of Low Frequency Noise, Vibration and Active Control*, 14613484241272253.

[18] Zephania, C. S., & Sil, T. (2021). Study of autonomous conservative oscillator using an improved perturbation method. *Journal of Vibration Engineering & Technologies*, *9*, 409-419.

[19] Yadav, N., Singh, M., Singh, S., Singh, R., & Kumar, J. (2023). A note on homotopy perturbation approach for nonlinear coagulation equation to improve series solutions for longer times. *Chaos, Solitons & Fractals*, *173*, 113628.

[20] Ahmed, S., Alkahtani, B. S. T., & Alzaid, S. S. (2024). Wave Propagation and Soliton Behaviors for the Strain Equation by Using the Sub-ODE Method and Expansion Technique. *International Journal of Applied and Computational Mathematics*, *10*(4), 122.

[21] Jalili, P., Jalili, B., Ahmad, I., Hendy, A. S., Ali, M. R., & Ganji, D. D. (2024). Python approach for using homotopy perturbation method to investigate heat transfer problems. *Case Studies in Thermal Engineering*, *54*, 104049.

[22] Qayyum, M., & Oscar, I. (2021). Least square homotopy perturbation method for ordinary differential equations. *Journal of Mathematics*, *2021*(1), 7059194.

[23] Rizvi, S. T., Seadawy, A. R., Farah, N., & Ahmad, S. (2023). Controlling optical soliton solutions for higher order Boussinesq equation using bilinear form. *Optical and Quantum Electronics*, *55*(10), 865.

[24] Ahmed, S., Seadawy, A. R., & Rizvi, S. T. (2023). Envelope solitons, multi-peak solitons and breathers in optical fibers via Chupin Liu's theorem and polynomial law of nonlinearity. *Optical and Quantum Electronics*, *55*(7), 632.

[25] Ahmed, S., Seadawy, A. R., & Rizvi, S. T. (2022). Study of breathers, rogue waves and lump solutions for the nonlinear chains of atoms. *Optical and Quantum Electronics*, *54*(5), 320.

[26] Ghaffar, A., Ali, A., Ahmed, S., Akram, S., Junjua, M. U. D., Baleanu, D., & Nisar, K. S. (2020). A novel analytical technique to obtain the solitary solutions for nonlinear evolution equation of fractional order. *Advances in Difference Equations*, *2020*(1), 308.

[27] Sarfraz, M., Yasir, M., & Khan, M. (2023). Multiple solutions for non-linear radiative mixed convective hybrid nanofluid flow over an exponentially shrinking surface. *Scientific Reports*, *13*(1), 3443.

[28] Akram, G., Sadaf, M., Sarfraz, M., & Anum, N. (2022). Dynamics investigation of (1+ 1)-dimensional time-fractional potential Korteweg-de Vries equation. *Alexandria Engineering Journal*, *61*(1), 501-509.

[29] Ahmed, S., Sarfraz, M., & Aziz, T. (2023). Parametric analysis of heat flux inhibition in the solar wind: a macroscopic quasilinear approach. *Zeitschrift für Naturforschung A*, *78*(5), 463-479.

[30] Rizvi, S. T., Seadawy, A. R., Farah, N., Ahmad, S., & Althobaiti, A. (2024). The interactions of dark, bright, parabolic optical solitons with solitary wave solutions for nonlinear Schrödinger–Poisson equation by Hirota method. *Optical and Quantum Electronics*, *56*(7), 1162.

[31] Sarfraz, M., Khan, M., Al-Zubaidi, A., & Saleem, S. (2023). Insights into the thermodynamic efficiency of Homann-Agrawal hybrid nanofluid flow. *Alexandria Engineering Journal*, *82*, 178-185.

[32] Bashir, A., Seadawy, A. R., Ahmed, S., & Rizvi, S. T. (2022). The Weierstrass and Jacobi elliptic solutions along with multiwave, homoclinic breather, kink-periodic-cross rational and other solitary wave solutions to Fornberg Whitham equation. *Chaos, Solitons & Fractals*, *163*, 112538.