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Comprehensive Review of Chebyshev's Wavelet Methods for Solving Differential & Integral Equations

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ABSTRACT :

In this research, a comprehensive review of Chebyshev's wavelet methods has been presented for the numerical solutions of differential and integral equations. Comprehensive study of the research paper, illustrate the accuracy and efficiency of the Chebyshev's based wavelets methods for solving ordinary, partial, integral and integro-differential equations.

Keywords: Chebyshev's wavelets, Operational matrices, Differential equations, Integral equations.

Introduction :

In recent years, wavelets have been used in different fields of science and engineering. Analysis used in the wavelet is a new branch of mathematics that helps in numerical analysis, image processing and signal analysis. Many researchers have used many wavelets to solve mathematical and engineering problems and found many of such wavelets are complex and inefficient in solving numerical problems effectively. Chebyshev wavelets are one of the best ways to solve these numerical problems. These wavelets provide more accurate, reliable, and applicable solutions by reducing these problems to a system of algebraic equations. Several kinds of Chebyshev wavelets are introduced in the literature like Chebyshev wavelets of the first kind, the second kind, third kind, fourth kind, fifth kind, etc. Hence, a number of methods have been described for the numerical solution of different kind of Chebyshev wavelets.

In this work, we are trying to briefly explain the Chebyshev polynomials and Chebyshev wavelets of different kinds as how many of the researchers have proven this wavelet is more effective and easier to implement on various problems. We categorized this paper depending on different kinds of Chebyshev polynomials and Chebyshev wavelets. (Piessens & Branders, 1976) established a technique for the numerical solution of linear Fredholm integral equations of the second kind. The solution involves an approximation of the Chebyshev series, whose coefficients are the solution of a linear system of equations. Using recurrence relation, the coefficient of these equations is calculated that depends upon the kernel. The authors have given the recurrence relation for some important kernels. For instance, it is applied to an integral equation that appears in the theory of intrinsic viscosity of macromolecules.

(Horng & Chou, 1985) have suggested the shifted Chebyshev polynomials to solve Variational problems. This method is used to reduce a Variational problem to algebraic equations and the calculation is direct on a digital computer. (Mihaila & Mihaila, 2002) have suggested the numerical solution of differential equation with the expansion of an unknown function in Chebyshev polynomials and try directly to solve the system of linear equations for the values of the function at zeroes of Chebyshev polynomial of N-order (EI-gendi's method). At these extreme points, the solution obtained is exact, aside from computer errors being rounded off and the convergence of other numerical methods used in solving the linear system of equations. In time-dependent quantum field theory, initial value problems are applied and in fluid dynamics, second-order boundary value problems are applied. (Aysegul Akyuz Dascioglu, 2004) has presented a Chebyshev collocation method for solving systems of a linear integral equation with regard to Chebyshev polynomial. By using Chebyshev collocation points, the method changes the integral system into a matrix equation and the unknown including in this equation is a Chebyshev coefficient matrix. In order to solve a system of higher order linear integro-differential equation (IDE), (Dascioglu & Sezer, 2005) presented a Chebyshev collocation method as an expansion technique. This method converts the provided conditions and IDE system into matrix equations with the aid of collocation points. An operational matrix for integration and a product operation matrix for Chebyshev wavelets were proposed by (Babolian & Fattahzadeh, 2007). A straightforward technique for solving integral equations using the Chebyshev wavelet in Galerkin equations with continuous or discontinuous solutions has been given by (Babolian & Fattahzadeh, 2007). Additionally, the operational matrix of integration for Chebyshev wavelets has been utilized to solve Volterra type integral equations and reduce them to an algebraic system of equations. Maleknejad Sohrabi & Rostami, 2007) developed orthogonal Chebyshev polynomials to approximate the solution of linear and non-linear Volterra-integral equation. As an orthogonal basis has the advantage that it assures the stability of the matrix equation. (Liu, 2009) has solved the Fredholm integral equation of second kind, where chebyshev polynomials are enforced for the approximation to a solution for an unknown function in Fredholm integral equation and reduces the system of equations to linear equations.

(Kajani, Vencheh, & Ghasemi, 2009) focused mainly on Chebyshev wavelets, operational matrix of integration and product operation matrix. For the formation of different matrices, general procedure has been discussed. (Adibi & Assari, 2010) have proposed a numerical method for solving Fredholm integral equation of the first kind. Chebyshev wavelets have been utilized for this purpose. For making the wavelet coefficient matrices sparse, the properties of Chebyshev wavelets are based, which leads to the sparseness of the obtained coefficient matrix. The mth Chebyshev polynomial of a square matrix A is the monic polynomial that minimizes the matrix 2-norm of p(A) across all monic polynomials p(z) of degree m, according to (Faber, Liesen & Tichy, 2010). This polynomial is uniquely defined if the value of m is smaller than the degree of the smallest polynomial of A.

(Avazzadeh, Shaifee & Loghmani, 2011) have described a numerical method based on fractional calculus to solve Abel's integral equation. Chebyshev polynomials are used to apply fractional properties for the solution of first and second kind Abel's integral equation. (Zhu & Fan, 2011) have constructed the Chebyshev wavelet of the second kind and suggested a numerical method that depends upon Chebyshev wavelet of the second kind for solving a system of Fredholm integro-differential equations, which are non-linear and having fractional order. (Avazzadeh & Heydari, 2012) have presented a productive method to solve two-dimensional Fredholm and Volterra integral equations of the second kind. To find out solution for these equations, Chebyshev polynomials are applied. (Biazar & Ebrahimi, 2012) have described a new technique for solving non-linear systems of Volterra integral equations. The method is developed on the Chebyshev wavelets approximation on the interval [0,1]. (Wang and Fan, 2012) has proposed the second kind of Chebyshev wavelet method. Firstly, the author constructs the second kind of Chebyshev wavelet and derived operational matrices fractional-order integration. For the purpose of solving both linear and non-linear second order two-point boundary value problems, (Elhameed, Doha, & Youssri, 2013) presented a novel spectral technique assumed on shifted second kind Chebyshev wavelets operational matrices of derivatives.

For the solution of linear and non-linear multipoint boundary value problems, (Elhameed, Doha, & Youssri,2013) have introduced two wavelets collection methods. The main idea for acquiring numerical solutions to these equations is to include third and fourth kind Chebyshev wavelets with the collocation method to change the differential equation to a system of linear or non-linear algebraic equations with the use of boundary conditions in unknown expansion coefficient. (Doha, Elhameed & Youssri, 2013) have presented a new second kind of Chebyshev operational matrix of derivatives. With the help of second kind Chebyshev, the author described a technique for the numerical solution of linear and non-linear Lane-Emden type singular initial value problem. (Bahmanpour & Araghi, 2013) have suggested a technique to solve the Fredholm integral equation of the first kind, which is based on a wavelet basis. Also, the author considered Legendre, Chebyshev wavelets of the first kind, second kind, third kind and fourth kind on the interval [0,1] as basis functions in the modified Galerkin method to find the approximate solution of integral equations. (Ali, Iqbal & Mohyud-Din, 2013) finds the numerical solution of the Delay differential equation with the application of the Chebyshev wavelet method. (Shihab and Abdalrehman, 2013) presented a third kind Chebyshev wavelet operational matrix of derivative and which is applied for the solution of optimal control problems by using third kind Chebyshev wavelets expansions.

(Abdurrahman, 2014) first constructed a third kind of Chebyshev wavelet for the interval [0,1). Then author constructed a 2^k M.2^k Mmatrix named P, having almost a third kind of Chebyshev wavelet operational matrix of integration which is used for reducing optimal control problems to a system of algebraic equations with the use of the spectral method. Setia et al. (2014) have developed to solve the Bagley-Torvik equation. A fractional differential equation is said to be a Bagley-Torvik equation which occurs quite frequently in many branches of Mechanics and Applied Mathematics. (Zhou and Xu, 2014) have proposed an algorithm to solve the convection-diffusion equation. The method is based on Chebyshev wavelets of a second kind approximation. (Shihab & Sarhan, 2014) have described some operational matrices of fourth kind Chebyshev wavelets. Firstly, the author obtained a new formula for operational matrix of derivatives for the fourth kind of Chebyshev polynomials and then the derivation of both derivative and integration of the fourth Chevyshev wavelet was done. (Heydari Hooshmandasl & Ghaini, 2014) has developed an exact and systematic Chebyshev wavelets method for solving partial differential equation with boundary conditions of the telegraph type. (Gupta & Sahara, 2015) have proposed a new technique to solve the time-fractional fifth-order Sawada Kotera (SK) equation; the technique depends on Chebyshev wavelet expansion combined with operational matrices of fractional integration and derivative of wavelet functions. (Modammadi, 2015) has decribed a systematic wavelet Galerkin method depends on the stochastic operational matrix of the Chebyshev wavelet of the second kind for the solution of stochastic Ito-Volterra integral equations. Saeed, Rehman & Iqbal, 2015) have developed the shifted Chebyshev wavelet method to solve fractional delay differential equation, fractional delay Volterra integro-differential equation and fractional system of delay differential equation. An appropriate numerical approach to solving the space fractional order diffusion equation has been proposed by (Sweilam, Nagy & El-Sayed, 2015). The author seeks to determine the approximate solution, where fractional derivatives are stated in terms of Caputo type, using shifted Chebyshev polynomials of the second kind. The diffusion equation of fractional order in space is reduced to a system of ordinary differential equations by applying the Chebyshev collocation method and the properties of Chebyshev polynomials of the second kind. Using the Chebyshev series expansions, (Danaei Molaei &Khazili, 2015) studied the approximation of Fredholm integral equations of the second kind. Then these equations will be solved by m collocation points. Thus, the author will try to make the residual to zero at m points, which gives a system of m linear equations. (Nemati, Sedaghat, & Mohammadi, 2016) have proposed an algorithm of Chebyshev polynomials of the second kind on operational matrices to solve the fractional integro-differential equation with weakly singular kernels. (Darzi and Agheli, 2016) have proposed the Chebyshev wavelets product operation matrix and operational matrix of integration. To find the solution of Sturm Liouville problem, these matrices have been used. A third-kind Chebyshev wavelet-based numerical approach has been presented by (Zhou and Xu, 2016) to solve the class of time-fractional convection-diffusion equations with variable coefficients. For a third type of operational matrices of fractional and integer order integration, the author has obtained Chebyshev wavelets, respectively. In this, (Kumar and Vijesh, 2017) have found that radial basis function(RBF) is useful to solve coupled sine Gordan equations with initial and boundary conditions. This approach implies the production of moderate accuracy in a larger domain as it requires more grid points. The author of the present study developed an alternative numerical method to solve one-dimensional coupled sine Gordan equations for improving accuracy by reducing grid points. For the solution of some types of linear and non-linear fractional-order differential equations, (Elhamed & Yousrri, 2017) has presented and analyzed two new spectral algorithms. The suggested methods are obtained by using a certain kind of shifted Chebyshev polynomials termed as the shifted fifth kind Chebyshev polynomial as base function together with the application of the modified spectral tan method. A special class of fifth kind Chebyshev polynomial is a basic class of symmetric orthogonal polynomials that are established with an extended form of the Sturm-Liouville problem for symmetric functions.

(Azodi & Yoghouti, 2018) mainly deal with fourth kind Chebyshev wavelets (FKCW) to solve a model of HIV infection of CD4+T cells with Caputo fractional derivatives numerically. The problem has to be discussed as a system of a non-linear fractional differential equation. The solution is to be approximated in the form of FKCW truncated series. Several fractional-order Riccati-type differential equations have been numerically solved by (Polat & Nergis, 2019) using the third-kind Chebyshev wavelet operational matrix of fractional order integration. (Zhou & Xu, 2019) solved fractional Volterra Fredholm integro-differential equations with mixed boundary conditions using the fourth kind of Chebyshev wavelets collocation method. For a single Chebyshev wavelet in the Riemann-Liouville sense, the fractional integral formula has been constructed by the author using the fourth kind of shifted Chebyshev polynomials. The Chebyshev wavelet approach is a novel numerical technique for solving fractional delay differential equations (Farooq, Khan, Baleanu & Arif, 2019). The Caputo operator is applied to explain fractional derivatives. (Oruc, Bulut & Esen, 2019) have used Chebyshev wavelet along with the collocation method to find out the numerical solution of one-dimensional Coupled Berger's equation depends on time with suitable initial and boundary conditions. Then Coupled Berger's equation is converted into algebraic equations using the Chebyshev wavelet and their integrals can be solved. (Ali, Salam, Mohamed, Samet, Kumar & Osman, 2020) has discussed a numerical method for the solution of a general form of non-linear fractional order integral differential equation with linear functional arguments by using chebyshev series. The linear functional arguments with these recommended equations generate a general form of delay, proportional delay and advanced differential equation. (Atta Elhameed, Moatimid & Youssri, 2021) have implemented an algorithm to get an approximate solution of one dimensional linear hyperbolic partial differential equation. As a basis function, the author used a certain combination of shifted Chebyshev polynomial of the fifth kind. (Dhawan, Machado, Brzeziński, & Osman, 2021) have suggested different types of wavelet-based algorithms which provide a solution to a number of numerical problems. Chebyshev wavelet has been adopted to find the numerical solution to different models. By using fundamental properties for the selection of collocation points, a Chebyshev operational matrix is developed. Caputo fractional derivatives have been proposed as a novel and effective way to solve the general form of distributed order fractional differential equations in the time domain (Rashidinia, Eftekhari, & Maleknejad, 2021). This was the first time the author used the unit step function and the Laplace transforms approach to construct an exact expression for the operational vector of the Riemann-Liouville fractional integral operator. (Sadri and Aminikhah, 2021) have introduced the fifth kind of Chebyshev polynomials (based on the class of Chebyshev polynomials family) to solve the multi-term variable-order time-fractional diffusion wave equation (MVTFD-WE). In these equations, the fractional derivative operator appeared to be Caputo type. The collocation method coupled with fifth kind Chebyshev polynomials leads to reducing MVTFD-WE to a system of algebraic equations. The main aim of (Xu, Xiong & Zhou, 2021) is to construct a new orthonormal wavelet based on Chebyshev polynomials of the sixth kind for obtaining the solution to fractional optimal control problems with the inequality constraints. The relationship between second kind and sixth kind Chebyshev polynomials have been used to derive the exact formula for Riemann Liouville fractional integral operator of Chebyshev wavelets. In this article, the authors (Heydari, Avazzadeh, & Hosseinzadeh 2022), has been discribed Haar wavelet-based methodology for the solution of boundary values problems. The orthogonal Haar basis functions find extensive application in addressing initial value problems. (Singh & Singh, 2022) introduces a novel approach for obtaining the analytical solution of a 2D equation encountered in diverse fields of Physics and engineering, through the integration of Laplace transformation and a modified variational iteration method. (Xu & Zhou (2023) devised a highly effective technique combining to a novel type of Chebyshev wavelet with the Picard technique for the solution of fractional nonlinear differential equations under initial and boundary conditions. In this, the spectral collocation method by (Hammad, 2024), has garnered significant attention among researchers, primarily driven by the rapid advancements in achieving optimal solutions for real-world phenomena in our environment. In this study, the author adopts a distinct spectral collocation method (SCM) that differs from those employed in previous research to address the one- and two-dimensional Burger's equations.

Chebyshev Wavelet :

A single function known as the mother wavelet is translated and dilated to create a collection of functions called wavelets. A family of continuous wavelets is created as the translation parameter (ν) and dilation parameter (μ) vary continually. This family of continuous wavelets is a vital tool in many applications, including data compression, signal processing, and image analysis, since it enables the analysis of signals and images at various scales and resolutions. Thus, the following is the family of continuous wavelets:

$$\varphi_{\mu,\nu}(t) = |\mu|^{\frac{-1}{2}}\varphi\left(\frac{t-\nu}{\mu}\right), \mu,\nu \in R, \mu \neq 0.$$

If we restrict the parameters μ and ν to discrete values such as $\mu = \mu_0^{-j}, \nu = m\nu_0\mu_0^{-j}$, where $\mu_0 > 1, \nu_0 > 0$ and j, m denotes the positive integers, then we have the family of discrete wavelets as follows:

$$\psi_{j,m}(t) = |\mu_0|^{j/2}\psi(\mu_0^j t - m\nu_0),$$

which forms a wavelet basis for $L^2(R)$. Particularly, an orthonormal basis is formed as $\psi_{j,m}(t)$, when $\mu_0 = 2 \; and \; \nu_0 = 1$.
The wavelets have four arguments namely, p, q, r, t and can be written as $\psi_{q,r}(t) = \psi(p,q,r,t)$; q= $0, 1, 2, 3, \dots, 2^p - 1$, where any positive integer denoted by p, the degree of the Chebyshev polynomials denoted by r and normalized time denoted by t.
We denote

$$T_r(t), \qquad U_r(t), \qquad V_r(t), \qquad W_r(t)$$

as the first kind, second kind, third kind and fourth kind Chebyshev polynomials having degree r are defined as follows:

$$T_r(t) = \cos r\theta, U_r(t) = \frac{\sin(r+1)\theta}{\sin\theta}, V_r(t) = \frac{\cos\left(r+\frac{1}{2}\theta\right)}{\cos\left(\frac{1}{2}\theta\right)}, \qquad W_r(t) = \frac{\sin\left(r+\frac{1}{2}\theta\right)}{\sin\left(\frac{1}{2}\theta\right)}$$

where

$t = cos\theta$ for $-1 \leq \theta \leq 1$ and $0 \leq \theta \leq \pi$ and they are orthogonal to the weight functions

$$\omega(t) = \frac{1}{\sqrt{1-t^2}}, \text{for first kind}$$

$$\omega(t) = \sqrt{1-t^2}, \text{for second kind}$$
$$\omega(t) = \frac{\sqrt{1+t}}{\sqrt{1-t}}, \text{for third kind}$$
$$\omega(t) = \frac{\sqrt{1-t}}{\sqrt{1+t}}, \text{for fourth kind}$$

on the interval $[-1,1]$ respectively.

Conclusions :

The review above leads us to the conclusion that the Chebyshev wavelets collocation method is an effective way to solve a variety of differential and integral problems. It is simple to use and produces incredibly accurate and effective results. This field is still mostly uncharted, though. The application of this approach will be improved by additional study and development in this area, which will support the development and expansion of applied mathematics and other related fields.

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