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Integral Solutions of Ternary Quintic Diophantine Equation $x^2 + y^2 = 37z^5$

***R. Vanaja*[1], *R. Rajanila*[2]**

[1]Assistant professor, Department of Mathematics, Shrimati Indira Gandhi College, Affiliated to Bharathidasan University, Trichy-620 002, Tamil Nadu, India. Email: vanajvicky09@gmail.com
[2]M. Sc Scholar, Department of Mathematics, Shrimati Indira Gandhi College, Affiliated to Bharathidasan University, Trichy-620 002, Tamil Nadu, India. Email: rajanilarajasekar@gmail.com

ABSTRACT:

The non-zero distinct integral solutions of the Diophantine equation of degree five with three unknowns, represented by $x^2 + y^2 = 37z^5$ are examined. A few intriguing relationships between the values of x, y, z and special numbers are presented.

KEYWORDS: Polygonal numbers, integral solutions, and ternary quintic equations.

NOTATIONS:

$t_{m,n} = n\left[1 + \frac{(n-1)(m-2)}{2}\right] =$ Polygonal number of rank n with sides m

$P_n^r = \frac{1}{6}n(n+1)[(r-2)n + (5-r)] =$ Pyramidal number of rank n with sides r

$Pr_a = n(n+1)$ = Pronic number of rank n

$SO_n = n(2n^2 - 1) =$ Stella Octangula number

$Ct_{m,n} = \frac{mn(n+1)+2}{2} =$ Centered Polygonal number of rank n with sides m

$OH_n = \frac{1}{3}n(2n^2 + 1) =$ Octahedral number

$Pt_n = \frac{n(n+1)(n+2)(n+3)}{24} =$ Pentatope number of rank n

$gn_a = 2a - 1 =$ Gnomonic number of rank a

$HauyOH_n = \frac{1}{3}(2n-1)(2n^2 - 2n + 3) =$ Hauy Octahedral number

INTRODUCTION:

Number theory, a branch of mathematics focused on numerical properties that deals with numerical patterns and structures, especially integers. It explores topics like prime numbers, divisibility, congruences, and the solutions to equations involving whole numbers. Some well-known problems in number theory include finding prime numbers, understanding the distribution of primes, and solving Diophantine equations. It has applications in cryptography, computer science, and other fields. A ternary quintic Diophantine equation with non-homogeneous terms offers an unlimited field for research due to their variety [1-7]. Diophantine equations involve finding integer values for variables. Specifically, it involves finding integer values for variables that satisfy the equation. These equations can involve one or more variables, and the goal is to find integer solutions, as opposed to real or complex solutions. One may refer to [8-12] quintic Diophantine equations with three unknowns. Thus, in this communication a fifth degree non-homogeneous Diophantine equation with three unknowns represented by $x^2 + y^2 = 37z^5$ is analyzed for its non-zero integer solutions. A few interesting relations between the values of x, y, z and special numbers are presented.

METHOD OF ANALYSIS

PROBLEM:

The fifth degree Diophantine equation with three unknowns to be solved is

$$x^2 + y^2 = 37z^5 \quad (1)$$

Three different patterns of solutions of (1) are illustrated below:

Pattern 1:

Choosing z as $z = a^2 + b^2$ and write equation (1) in factorizable from as follows:

$$(x+iy)(x-iy) = (6+i)(6-i)(a+ib)^5(a-ib)^5 = S(constant)$$

$$\frac{(x+iy)}{(6+i)(a+ib)^5} = \frac{(6-i)(a-ib)^5}{(x-iy)} = S$$

Setting $(x+iy) = S(6+i)(a+ib)^5$

$$(x+iy) = S(6+i)(a^5 + (ib)^5 + 10a^2(ib)^3 + 10a^3(ib)^2 + 5a(ib)^4 + 5a^4(ib))$$

$$= S(6+i)(a^5 + ib^5 - i10a^2b^3 - 10a^3b^2 + 5ab^4 + i5a^4b)$$

$$= 6Sa^5 + i6Sb^5 - i60Sa^2b^3 - 60Sa^3b^2 + 30Sab^4 + i30Sa^4b + iSa^5 - Sb^5 + 10Sa^2b^3 - i10Sa^3b^2 + i5Sab^4 - 5Sa^4b$$

Equating real and imaginary parts

$$x(a,b) = 6Sa^5 - 60Sa^3b^2 + 30Sab^4 + 10Sa^2b^3 - 5Sa^4b - Sb^5$$

$$y(a,b) = Sa^5 - 60Sa^2b^3 + 30Sa^4b - 10Sa^3b^2 + 5Sab^4 + 6Sb^5$$

Let $S = 1$,

$$x(a,b) = 6a^5 - 60a^3b^2 + 30ab^4 + 10a^2b^3 - 5a^4b - b^5$$

$$y(a,b) = a^5 - 60a^2b^3 + 30a^4b - 10a^3b^2 + 5ab^4 + 6b^5$$

$$z = a^2 + b^2$$

The above solutions x, y and z satisfies the following properties

1) $x(a,1) - y(a,1) + 10Pr_a + 5gn_a^2 \equiv 0(mod\ 5)$
2) $7y(a,1) - 21OH_a + 5t_{a,2} + 2gn_a \equiv 6(mod\ 7)$
3) $x(a,1) + y(a,1) + t_{a,2} * Pr_a + 3Pr_a^2 - 3t_{a,2}^2 + 2 \equiv 0(mod\ 7)$
4) $x(a,1) - 24Pt_a + 9HauyOH_a - t_{a,2} + Pr_a \equiv 4(mod 6)$
5) $y(1,b) + Pr_a^2 - S0_b - t_{b,2} + 2 \equiv 0(mod 3)$
6) $2y(1,b) + 4t_{3,b} * Pr_b + 2Pr_b + gn_b + 3 \equiv 0(mod 4)$

Pattern 2:

Equation (1) can also be written in the factorable form as

$$(x+iy)(x-iy) = (1+6i)(1-6i)\ [(a+ib)(a-ib)]^5$$

Setting $(x+iy) = (1+6i)(a+ib)^5$

$$(x+iy) = (1+6i)(a^5 + (ib)^5 + 10a^2(ib)^3 + 10a^3(ib)^2 + 5a(ib)^4 + 5a^4(ib))$$

$$= (1+6i)(a^5 + ib^5 - i10a^2b^3 - 10a^3b^2 + 5ab^4 + i5a^4b)$$

$$= a^5 + ib^5 - i10a^2b^3 - 10a^3b^2 + 5ab^4 + i5a^4b + i6a^5 - 6b^5 + 60a^2b^3 - i60a^3b^2 + i30ab^4 - 30a^4b$$

Equating real and imaginary parts

$x(a,b) = a^5 - 10a^3b^2 + 5ab^4 + 60a^2b^3 - 30a^4b - 6b^5$

$y(a,b) = 6a^5 - 10a^2b^3 + 5a^4b - 60a^3b^2 + 30ab^4 - b^5$

$z(a,b) = a^2 + b^2$

The above solutions x, y and z satisfies the following properties:

1) $2x(b,1) + 2Ct_{b,1} - SO_b + 9OH_b \equiv 0(mod\ 2)$
2) $y(a,1) + Pr_a^2 - SO_a + 3t_{a,2} * t_{a,2} + 5Ct_{a,1} \equiv 0(mod\ 6)$
3) $y(a,1) + 6P_a^5 * 2t_{a,2}^2 - 3OH_a - 3Ct_{a,1}^2 + t_{a.2} - 11Pr_a^2 \equiv 0(mod 12)$
4) $y(a,1) - x(a,1) + 30P_a^5 + 5Ct_{a,1} \equiv 2(mod 5)$
5) $x(a,1) + y(a,1) - 3Pr_a^2 + 3SO_a + 3gn_a + 3t_{8,b} \equiv 0(mod 7)$
6) $x(a,1) + 12OH_a * t_{a,2}^2 - 12P_a^5 * t_{a,2} - 3SO_a + 3Pr_a - gn_a - 4 \equiv 0(mod 9)$
7) $y(a,1) - x(a,1) + 15HauyOH_a + 5gn_a + 30P_a^5 \equiv 3(mod 5)$

Pattern 3:

Introducing the transformation

$x = 37^3X$, $y = 37^3Y$ and $z = 37W$ (2)

We get $X^2 + Y^2 = W^5$ (3)

Choosing $W = A^2 + B^2$ and employing the method of factorization, (3) is expressed as a system of double equations given by

$(X + iY) = (A + iB)^5$

$(X - iY) = (A - iB)^5$

Equating real and imaginary in either of the above two equations the values of X and Y are given by

$X(a,b) = A^5 - 10A^3B^2 + 5AB^4$

$Y(a,b) = 5A^4B - 10A^2B^3 - B^5$

In view of (2), the non-zero integral solutions of (1) are given by

$x(A,B) = 37^3(A^5 - 10A^3B^2 + 5AB^4)$

$y(A,B) = 37^3(5A^4B - 10A^2B^3 - B^5)$

$Z(A,B) = 37(A^2 + B^2)$

For simplicity and clear understanding a few interesting properties satisfied by x, y and z are presented below:

1) $x(1,B) + y(1,B) + 37SO_B \equiv 0(mod\ 37)$
2) $x(1,B) - t_{B,2} + gn_B \equiv 0(mod\ 5)$
3) $y(A,1) + 30P_A^5 - 5Pr_A \equiv 3(mod\ 5)$
4) $x(A,1) - y(A,1) + 74Ct_{A,1} \equiv 0(mod\ 37)$
5) $2x(A,1) + 8Pr_A^2 + 6OH_A - gn_A + 1 \equiv 0(mod\ 2)$
6) $y(A,3) + 45HauyOH_A + 30P_A^5 * Pr_A \equiv 9(mod\ 15)$

Conclusion:

Three different approaches to obtaining non-zero distinct integer solutions of the non-homogeneous Diophantine equation $x^2 + y^2 = 37z^5$ have been presented in this paper. To conclude, one may search for other choices of non-homogeneous ternary quintic Diophantine equations along with solutions and their corresponding properties.

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