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Study of Modified Exponential-Type Imputation Scheme of Population Mean Using Robust Outliers Free Measures Under Two-Phase Sampling

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ABSTRACT

The efficiency of some modified exponential-type imputation estimators of population mean using robust outliers free measure under two-phase sampling focused on the modification of some mean imputation and estimation that utilizes function of auxiliary variables using simple random sample without replacement with presence of outliers were examined. This research work aimed at developing efficient estimators of population mean in the presence of non-response using imputation schemes. Method of mean squared error (MSE) was used in comparing both the conventional and proposed estimators for both Real-life and simulated data for both positive and negative for both cases that are used under different super population. The results revealed that the estimators of modified scheme demonstrated high level of efficiency over many of the existing estimators considered in this study. Moreover, among the modified estimators, members of $\hat{T}_1^d$outperformed in number than others.

Keywords: Outliers, Estimation, Population Mean, Mean Squared Errors

1.0 Introduction

In survey sampling, the utilization of supplementary information furnishes a better improvement over the efficiency of estimators constructed for the estimation of unknown population parameters. The literature describes several ratio, regression and exponential methods utilizing the auxiliary variable at the estimation stage. Many prominent authors developed several modified and improved ratio, regression and exponential type estimators by using the population information of the auxiliary variable x. However, the information about the population mean of the auxiliary variable is not always available. In the aforesaid environment, the most popular sampling scheme is the two-phase sampling scheme which was first established by Neyman (1938) to accumulate information on sampling. It is customarily acquired when the accumulation of information on study variable is very costly but relatively cheaper to accumulate information on auxiliary variables that are correlated with the study variables. Due to these reasons, the two-phase sampling becomes a powerful and cost-effective scheme for obtaining the authentic estimate in the one-phase sample for the unknown parameters of the auxiliary variable. Authors like Kumar and Bahl (2006), Singh and Vishwakarma (2007), Ozgul and Cingi (2014), Kalita *et al.* (2016), Noor-Al-Amin *et al.* (2016), Bazad and Bazad (2019), Bhushan and Gupta (2019), Adamu *et al* (2019) and Bhushan *et al.* (2023) have extensively worked under two-phase sampling. However, the applicability of aforementioned estimators depends on the complete availability of sample information measured on both the study and auxiliary variables. Reduction in the sizes of sample information due to non-response decreases the efficiency of these estimators. One of the approaches to tackle the problem of non-response is imputation techniques.

Survey such as in medical and social sciences etc. conducted by human are often characterized by non-response. Hansen and Hurwitz (1946) first discussed the issue of non-response and imputation methods to deal with non-response issues were suggested by several scholars like Signh and Horn (2000), Signh and Deo (2003), Ahmed *et al.* (2006), Wang and Wang (2006), Toutenburg *et al.* (2008), Signh (2009), Diana and Perri (2010), Al-Omar et al. (2013), Mishra and Signh (2017), Signh and Gogoi (2018), Prasad (2018), Audu *et al.* (2020b), Audu *et al* .(2021b), Audu *et al.* (2021c), Audu *et al.* (2020c), are some of the most recent imputation methods. However, all the estimators of the schemes proposed by aforementioned authors are functions of population mean of auxiliary variable $\bar{X}$ and if $\bar{X}$ is unknown, the schemes cannot be applied to real life situations.

This study, therefore utilizes the concept of two-phase sampling in which a large sample of size $n_1\left(n_1 \le N\right)$ is taking to estimate $\bar{X}$ thereby addressing the problem of complete information about the auxiliary variable and introduce robust outliers measures to enhance the efficiency of the modified estimators in case the auxiliary variable information are characterized with outliers or extreme values.

2.0 LITERATURE REVIEW

2.1 Some Existing Imputation Estimators of Population Mean under Two-Phase Sampling

Following Lee at al. (1994), values found missing in the study variable are to be replaced by values obtained using the expression $\hat{\beta}=\sum_{i=1}^{r} y_i / \sum_{i=1}^{r} x_i = \bar{y}_r / \bar{x}_r$. The study variable thereafter, takes the form given as

$$y_{.i}=\begin{cases} y_i & \text{if} \quad i\in R \\ \hat{\beta}\mathrm{x_i} & \text{if} \quad i\in R^c \end{cases} \tag{2.1}$$

Under the method of imputation, ratio method estimator of population mean denoted by $\hat{t}_1$ can be derived as

$$T_1^*=T_0\frac{\bar{x}_n}{\bar{x}_r} \tag{2.2}$$

where

$$\bar{x}_r=\frac{1}{r}\sum_{i\in R}x_i,\ \bar{x}_n=\frac{1}{n}\sum_{i\in S}x_i$$

The MSE of T_1^* up $\mathrm{O}\left(n^{-1}\right)$ is given as:

$$MSE(T_1^*)_I=MSE\left(T_0\right)+\psi_{r,n}\left(R^2S_X^2-2RS_{YX}\right) \tag{2.3}$$

$$MSE\left(T_2^*\right)_{II}=\bar{Y}^2\left(\frac{1}{r}-\frac{1}{N}\right)C_Y^2+\left(\frac{1}{r}+\frac{1}{n}-\frac{2}{N}\right)C_X^2-2\left(\frac{1}{r}-\frac{1}{N}\right)\rho_{YX}C_YC_X \tag{2.4}$$

where $S_{YX}=\rho_{YX}S_YS_X$, $\rho_{YX}=\frac{S_{YX}}{S_YS_X}$, $S_X^2=\frac{1}{N-1}\sum_{i=1}^{N}\left(x_i-\bar{X}\right)^2$, $\bar{X}=\frac{1}{N}\sum_{i=1}^{N}x_i$, $\psi_{r,n}=\frac{1}{r}-\frac{1}{n}$, $R=\frac{\bar{Y}}{\bar{X}}$

Singh and Horn (2000) utilized information from imputed values for responding and non-responding units as well, thereafter giving study variable the form given by (2.94)

$$y_{.i}=\begin{cases} \lambda\frac{n}{r}y_i+(1-\lambda)\hat{\beta}x_i & \text{if} \quad i\in R \\ (1-\lambda)\hat{\beta}x_i & \text{if} \quad i\in R^c \end{cases} \tag{2.5}$$

Under this method of imputation, estimator of population mean denoted by T_2^* can be derived as

$$T_2^*=T_0\left(\lambda+\left(1-\lambda\right)\frac{\bar{x}_n}{\bar{x}_r}\right) \tag{2.6}$$

T_2^* attained optimum when $\lambda=1-RS_{YX}/S_X^2$ and the $MSE\left(T_2^*\right)_{\min}$ is given by

$$MSE\left(T_2^*\right)_{\min}=MSE\left(T_0\right)-\psi_{r,n}\beta_{reg}\rho_{YX}S_YS_X \tag{2.7}$$

$$MSE\left(T_2^*\right)_{II\min} = \bar{Y}^2 C_y^2 \left(\psi_{r,N} - \frac{\psi_{r,N}{}^2 \rho^2}{\left(\psi_{n,N} + \psi_{r,N}\right)} \right) \tag{2.8}$$

Kadilar and Cingi (2008) modified the work of Kadilar and Cingi (2004) in the case of missing observations and suggested the following estimators of population mean

$$T_3^* = \left(T_0 + \hat{\beta}_{reg}\left(\bar{x}_n - \bar{x}_r\right)\right)\frac{\bar{x}_n}{\bar{x}_r} \tag{2.9}$$

$$MSE\left(T_3^*\right)_I = MSE\left(T_0\right) + \psi_{r,n}\left(S_X^2\left(R + \beta_{reg}\right)^2 - 2\left(R + \beta_{reg}\right)S_{YX}\right) \tag{2.10}$$

$$MSE\left(T_3^*\right)_{II} = \bar{Y}^2\left(\frac{1}{r} - \frac{1}{N}\right)C_Y^2 + \left(\bar{Y} + \beta_{reg}\right)^2\left(\frac{1}{r} + \frac{1}{n} - \frac{2}{N}\right)C_X^2 - 2\bar{Y}\left(\bar{Y} + \beta_{reg}\right)\left(\frac{1}{r} - \frac{1}{N}\right)\rho_{YX}C_Y C_X \tag{2.11}$$

Singh, (2009), proposed a new method of imputation method as

$$y_{.i} = \begin{cases} y_i & if \quad i \in R \\ \bar{y}_i\left[\dfrac{(n-r)\bar{x}_n + \vartheta r\left(\bar{x}_n - \bar{x}_r\right)}{\vartheta \bar{x}_r + (1-\vartheta)\bar{x}_n}\right]\dfrac{x_i}{\sum_i \in R^C} & if \quad i \in R^C \end{cases} \tag{2.12}$$

The point estimator of population mean $\bar{Y}$ under the proposed method of imputation becomes

$$T_4^* = \frac{\bar{y}_r \bar{x}_n}{\vartheta \bar{x}_r + (1-\vartheta)\bar{x}_n} \tag{2.13}$$

where ϑ is an unknown parameter to be estimated.

The bias, MSE and min MSE of T_4^* are

$$Bias\left(T_4^*\right) = \bar{Y}\begin{bmatrix} \left(\frac{1}{n} - \frac{1}{N}\right)\rho C_y C_x + \vartheta^2\left(\frac{1}{r} - \frac{1}{N}\right)C_x^2 + (1-\vartheta)^2\left(\frac{1}{n} - \frac{1}{N}\right)C_x^2 \\ -\vartheta\left\{\left(\frac{1}{r} - \frac{1}{N}\right)\rho C_y C_x + \left(\frac{1}{n} - \frac{1}{N}\right)C_x^2\right\} + 2\vartheta(1-\vartheta) \\ \left(\frac{1}{n} - \frac{1}{N}\right)C_x^2 - (1-\vartheta)\left\{\left(\frac{1}{n} - \frac{1}{N}\right)\rho C_y C_x + \left(\frac{1}{n} - \frac{1}{N}\right)C_x^2\right\} \end{bmatrix} \tag{2.14}$$

$$MSE\left(T_4^*\right)_I = MSE\left(T_1^*\right) - \left(\frac{1}{r} - \frac{1}{N}\right)S_x^2\left(\frac{S_{xy}}{S_X^2} - \frac{\bar{Y}}{\bar{X}}\right)^2 \tag{2.15}$$

$$MSE\left(T_4^*\right)_{II} = \bar{Y}^2 C_y^2\left(\psi_{r,N} - \frac{\psi_{r,N}{}^2\rho^2}{\left(\psi_{n,N} + \psi_{r,N}\right)}\right) \tag{2.16}$$

Audu *et al.* (2021c) the following imputation scheme is proposed.

$$y_{.i} = \begin{cases} f_1 \dfrac{n}{r} y_i & \mathrm{i} \in R \\ \dfrac{n}{n-r}\bar{y}_r \left(f_2 \left(\dfrac{\bar{x}_n}{\bar{x}_r} \right)^{\tau_1} + f_3 \exp\left(\dfrac{\tau_2 \left(\bar{x}_n - \bar{x}_r \right)}{\bar{x}_n + \bar{x}_r} \right) \right) & \mathrm{i} \in R^{c} \end{cases} \tag{2.17}$$

where $\tau_1 \tau_2 \in (-1,1)$, $f_i, i = 1,2,3$ are filtration parameters, $\sum_{i=1}^{3} f_i = 1$.

$$T_5^* = \bar{y}_r \left(f_1 + f_2 \left(\frac{\bar{x}_n}{\bar{x}_r} \right)^{\tau_1} + f_3 \exp\left(\tau_2 \left(\frac{\bar{x}_n - \bar{x}_r}{\bar{x}_n + \bar{x}_r} \right) \right) \right) \tag{2.18}$$

$$MSE\left(T_5^*\right)_{I\min} = S_y^2 \left(\psi_{r,N} - \left(\psi_{r,N} - \psi_{n,N} \right) \rho^2 \right) \tag{2.19}$$

where

$$\left. \begin{aligned} f_1 &= 1 - \left[\left(\gamma_3 - \gamma_2 \right) \rho_{YX} S_y S_x^{-1} R^{-1} \right] / \left[\left(\tau_1 \gamma_3 - 2^{-1} \tau_2 \gamma_2 \right) \right] \\ f_2 &= \gamma_3 \rho_{YX} S_y S_x^{-1} R^{-1} / \left(\tau_1 \gamma_3 - 2^{-1} \tau_2 \gamma_2 \right) \\ f_3 &= -\gamma_2 \rho_{YX} S_y S_x^{-1} R^{-1} / \left(\tau_1 \gamma_3 - 2^{-1} \tau_2 \gamma_2 \right) \end{aligned} \right\}$$

$$\left. \begin{aligned} \gamma_1 &= Bias\left(\bar{y}_r \right) = 0 \\ \gamma_2 &= Bias\left(\bar{y}_r \left(\frac{\bar{x}_n}{\bar{x}_r} \right)^{\tau_1} \right) = \frac{R}{\bar{\bar{Y}}} \left(\psi_{r,N} \left(\frac{\tau_1 + \tau_1^2}{2} R S_x^2 - \tau_1 \rho_{YX} S_y S_x \right) - \psi_{n,N} \left(\tau_1 \rho_{YX} S_y S_x - \tau_1^2 R S_x^2 \right) \right) \\ \gamma_3 &= Bias\left(\bar{y}_r \exp\left(\tau_2 \frac{\bar{x}_n - \bar{x}_r}{\bar{x}_n + \bar{x}_r} \right) \right) = \frac{R}{\bar{\bar{Y}}} \left(\left(\frac{\tau_2}{4} + \frac{\tau_2^2}{8} \right) \psi_{r,N} \left(R C_x^2 - \rho_{YX} S_y S_x \right) - \frac{\tau_2}{2} \psi_{n,N} \rho_{YX} S_y S_x \right) \end{aligned} \right)$$

$$MSE\left(T_5^*\right)_{II\min} = S_y^2 \left(\psi_{r,N} - \frac{\psi_{r,N}^2 \rho_{YX}^2}{\left(\psi_{n,N} + \psi_{r,N} \right)} \right) \tag{2.20}$$

where

$$\left. \begin{aligned} f_1 &= 1 - \left[\left(\vartheta_3 - \vartheta_2 \right) \psi_{r,N} \left(\psi_{n,N} + \psi_{r,N} \right)^{-1} \rho C_y C_x^{-1} \right] / \left[\left(\tau_1 \vartheta_3 - 2^{-1} \tau_2 \gamma_2 \right) \right] \\ f_2 &= \vartheta_3 \lambda \left(\psi_{n,N} + \psi_{r,N} \right)^{-1} \rho C_y C_x^{-1} / \left(\tau_1 \vartheta_3 - 2^{-1} \tau_2 \vartheta_2 \right) \\ f_3 &= -\vartheta_2 \psi_{r,N} \left(\psi_{n,N} + \psi_{r,N} \right)^{-1} \rho C_y C_x^{-1} / \left(\tau_1 \vartheta_3 - 2^{-1} \tau_2 \vartheta_2 \right) \end{aligned} \right\} \tag{2.21}$$

$$\left.\begin{array}{l}\vartheta_1 = Bias\left(\bar{y}_r\right) = 0 \\ \vartheta_2 = Bias\left(\bar{y}_r\left(\dfrac{\bar{x}_n}{\bar{x}_r}\right)^{\tau_1}\right) = \bar{Y}^{-1}R\left(\psi_{r,N}\left(\dfrac{\tau_1+\tau_1^2}{2}RS_x^2 - \tau_1\rho_{YX}S_yS_x\right) - \psi_{n,N}\left(\dfrac{\tau_1^2-\tau_1}{2}RS_x^2\right)\right) \\ \vartheta_3 = Bias\left(\bar{y}_r\exp\left(\tau_2\dfrac{\bar{x}_n-\bar{x}_r}{\bar{x}_n+\bar{x}_r}\right)\right) = \dfrac{R}{\bar{Y}}\left\{\psi_{r,N}\left(\left(\dfrac{\tau_2}{4}+\dfrac{\tau_2^2}{8}\right)RS_x^2 - \dfrac{\tau_2}{2}\rho_{YX}S_yS_x\right) + \psi_{n,N}\left(\dfrac{\tau_2}{4}-\dfrac{\tau_2^2}{8}\right)RS_x^2\right\}\end{array}\right) \quad (2.22)$$

Musa *et al.* (2023) suggested regression-based class of imputation schemes using auxiliary information under two-phase sampling schemes as extension of Audu and Singh (2021) imputation scheme which assumed known population mean of auxiliary variable given in (2.26) as

$$y_{.i} = \begin{cases} y_i & \mathrm{i} \in R \\ \left(\bar{y}_r + \hat{\beta}_{rg}\left(\bar{x}_n - \bar{x}_r\right)\right)\dfrac{\bar{X}}{\bar{x}_r}\exp\left(\dfrac{\tau_1\left(\bar{x}_n-\bar{x}_r\right)}{\tau_1\left(\bar{x}_n+\bar{x}_r\right)+2\tau_2}\right) & \mathrm{i} \in R^{\mathrm{c}} \end{cases} \quad (2.23)$$

where τ_1 and τ_2 are known functions of auxiliary variables like coefficients of skewness $\beta_1(x)$, kurtosis $\beta_2(x)$, variation C_X, standard deviation S_X, $f_i,\ i = 1,2,3$.

The estimator of the proposed scheme by Musa et al. (2023) as well the MSEs under cases I and II are given in (2.24), (2.25) and (2.26) respectively

$$T_6^* = \frac{r}{n}\bar{y}_r + \left(1-\frac{r}{n}\right)\left(v\left(\bar{y}_r + \hat{\beta}_{rg}\left(\bar{x}_n-\bar{x}_r\right)\right) + (1-v)\bar{y}_r\frac{\bar{x}_n}{\bar{x}_r}\right)\exp\left(\frac{\tau_1\left(\bar{x}_n-\bar{x}_r\right)}{\tau_1\left(\bar{x}_n+\bar{x}_r\right)+2\tau_2}\right) \quad (2.24)$$

$$MSE\left(\hat{t}\right)_{I\min} = \bar{Y}^2C_y^2\left(\psi_{r,N} - \left(\psi_{r,N}-\psi_{n,N}\right)\rho^2\right) \quad (2.25)$$

$$MSE\left(T_6^*\right)_{II\min} = \bar{Y}^2C_y^2\left(\psi_{r,N} - \frac{\psi_{r,N}{}^2\rho^2}{\left(\psi_{n,N}+\psi_{r,N}\right)}\right) \quad (2.26)$$

$$v_{opt(I)} = \frac{1}{\left(\beta_{rg} - \dfrac{\bar{Y}}{\bar{X}}\right)}\left(\frac{\bar{Y}\rho C_y}{\bar{X}C_x\left(1-\dfrac{r}{n}\right)} - \frac{\bar{Y}\tau_1}{2\left(\tau_1\bar{X}+\tau_2\right)} - \frac{\bar{Y}}{\bar{X}}\right)$$

where

$$v_{opt(II)} = \frac{1}{\left(\beta_{rg} - \dfrac{\bar{Y}}{\bar{X}}\right)}\left(\frac{\bar{Y}\lambda\rho C_y}{\left(\psi_{n,N}+\psi_{r,N}\right)\bar{X}C_x\left(1-\dfrac{r}{n}\right)} - \frac{\bar{Y}\tau_1}{2\left(\tau_1\bar{X}+\tau_2\right)} - \frac{\bar{Y}}{\bar{X}}\right)$$

3.0 MATERIAL AND METHODS

3.1Robust Outlier-free Measure to be used in the Study

Robust Outlier-free parameters that will be used in the study include:

Gini's mean method

Downton's Method

Method of probability weighted moments

3.1.1 Gini's mean method (Nair,1936)

Let $\Delta Xi, = X_{(i+1)} - X_{(i)}$, where $x_{(i)}$ is the order statistics, so that $\Delta Xi, i = 1, ..., n-1$ is the distance between adjacent observations. Then

$$G = \tfrac{4}{N-1} \sum_{i=1}^{N} \left(\frac{2i - N - 1}{2N} \right) \Delta Xi \qquad (3.1)$$

3.1.2 Downton's Method (Downton, 1966)

Let $X_1, X_2, ..., X_n$, be a random sample from a normal distribution with mean μ and variance σ^2; that is, $X_1, X_2, ..., X_n \sim N(\mu, \sigma^2)$. Let $X_{(1)} \le X_{(2)} \le ... X_{(n)}$ denote the corresponding order statistics. The Downton estimator, D, is given as

$$D = c \sum_{i=1}^{n} \frac{(2i - n - 1) X_{(i)}}{n(n-1)} \qquad (3.2)$$

Where $C = \sqrt{\pi} = 1.772453851$ for a Normal distribution and does not depend on the sample size n. By applying a little algebra to Eq. (3.2), the Downton estimator, D, can be written in another form as follows:

$$D = \frac{2\sqrt{\pi}}{N-1} \sum_{i=1}^{N} \left(i - \frac{N+1}{2N} \right) X_{(i)} \qquad (3.3)$$

3.1.3 Method of probability weighted moments

Let $X_1, X_2, ..., X_n$, be a random sample from a normal distribution with mean μ and variance σ^2; that is, $X_1, X_2, ..., X_n \sim N(\mu, \sigma^2)$. Let $X_{(1)} \le X_{(2)} \le ... X_{(n)}$ denote the corresponding order statistics. The PWMs is by defined Greenwood *et al*. (1979) and is given by:

$$S_{pw} = \frac{\sqrt{\pi}}{N^2} \sum_{i=1}^{N} (2i - N - 1) X_{(i)} \qquad (3.4)$$

G, D and S_{pw} are the Gini's mean difference, Downtown's method and Probability weighted moments, which are the nonconventional robust measures free from the influence of outliers.

3.2 MODIFIED IMPUTATION SCHEMES UNDERTWO-PHASE SAMPLING

Modifying the works of Audu *et al*. (2021c) and Musa *et al*.(2023) when population mean $\bar{X}$ of auxiliary variable X is unknown due to incomplete availability of its information, then the schemes proposed in subsection 2.3 are not applicable in real life situation. To overcome this situation, an estimate of $\bar{X}$ denoted by $\bar{x}_1$ is obtained by preliminary large sample of size $n_1 (n_1 > n)$ and substitute it for $\bar{X}$ in the schemes proposed in subsection 2.3 which lead to the following schemes under two-phase sampling. The schemes are studied under two cases;

Case I: when secondary sample S_2 is a subset of preliminary sample S_1 i.e. $S_2 \subset S_1$.

Case II: when secondary sample S_2 is independent of preliminary sample S_1 i.e. $S_2 \subset \Omega_N$.

modified imputation scheme under two-phase sampling is

$$y_{.i} = \begin{cases} y_i & if \quad i \in R \\ \dfrac{\bar{y}_r + b_{rg}\left(\bar{x}_1 - \bar{x}_r\right)}{\left(\phi_j \bar{x}_r + \phi_k\right)}\left(\phi_j \bar{x}_1 + \phi_k\right)\exp\left(\dfrac{\eta_1\left(\left(\phi_j \bar{x}_1 + \phi_k\right) - \left(\phi_j \bar{x}_r + \phi_k\right)\right)}{\left(\phi_j \bar{x}_1 + \phi_k\right) + \left(\phi_j \bar{x}_r + \phi_k\right)}\right) & if \quad i \in R^c \end{cases} \tag{3.5}$$

3.3 ESTIMATORS OF THE MODIFIED IMPUTATION SCHEMES

The estimators of the proposed modified imputation schemes denoted by $T_1^{(d)}$ (3.6).

$$T_1^{(d)} = \frac{1}{n}\left(\sum_{i \in R} y_i + \sum_{i \in R^c} \frac{\bar{y}_r + b_{rg}\left(\bar{x}_1 - \bar{x}_r\right)}{\left(\phi_j \bar{x}_r + \phi_k\right)}\left(\phi_j \bar{x}_1 + \phi_k\right)\exp\left(\frac{\eta_1\left(\left(\phi_j \bar{x}_1 + \phi_k\right) - \left(\phi_j \bar{x}_r + \phi_k\right)\right)}{\left(\phi_j \bar{x}_1 + \phi_k\right) + \left(\phi_j \bar{x}_r + \phi_k\right)}\right)\right) \tag{3.6}$$

Simplify (3.7) and (3.8), the estimators $T_1^{(d)} and T_2^{(d)}$ and are obtained as

$$T_1^{(d)} = \frac{r}{n}\bar{y}_r + \left(1 - \frac{r}{n}\right)\frac{\bar{y}_r + b_{rg}\left(\bar{x}_1 - \bar{x}_r\right)}{\left(\phi_j \bar{x}_r + \phi_k\right)}\left(\phi_j \bar{x}_1 + \phi_k\right)\exp\left(\frac{\eta_1\left(\left(\phi_j \bar{x}_1 + \phi_k\right) - \left(\phi_j \bar{x}_r + \phi_k\right)\right)}{\left(\phi_j \bar{x}_1 + \phi_k\right) + \left(\phi_j \bar{x}_r + \phi_k\right)}\right) \tag{3.7}$$

3.4 PROPERTIES (BIASES AND MSES) OF ESTIMATORS OF THE MODIFIED IMPUTATION SCHEMES

To obtain the biases and MSEs of estimators of the modified imputation schemes, the following error terms $\Delta_0 = \dfrac{\bar{y}_r - \bar{Y}}{\bar{Y}}$, $\Delta_1 = \dfrac{\bar{x}_r - \bar{X}}{\bar{X}}$ and $\Delta_2 = \dfrac{\bar{x}_1 - \bar{X}}{\bar{X}}$ are defined such that $\left|\Delta_i\right| \approx 0,\ i = 0,1,2$ and $\bar{y}_r = \left(1 + \Delta_0\right)\bar{Y},\ \bar{x}_r = \left(1 + \Delta_1\right)\bar{X},\ \bar{x}_1 = \left(1 + \Delta_2\right)\bar{X}$

The expectation of $\left|\Delta_i\right| \approx 0,\ i = 0,1,2$ under the two cases of two-phase sampling are given below

Case I: when $S_2 \subset S_1$.

$$\left.\begin{aligned} &E\left(\Delta_0\right) = E\left(\Delta_1\right) = E\left(\Delta_2\right) = 0, E\left(\Delta_0^2\right) = \left(\frac{1}{r} - \frac{1}{N}\right)C_Y^2, E\left(\Delta_1^2\right) = \left(\frac{1}{r} - \frac{1}{N}\right)C_X^2 \\ &E\left(\Delta_2^2\right) = \left(\frac{1}{n_1} - \frac{1}{N}\right)C_X^2,\ E\left(\Delta_0\Delta_1\right) = \left(\frac{1}{r} - \frac{1}{N}\right)\rho_{YX}C_Y C_X \\ &E\left(\Delta_0\Delta_2\right) = \left(\frac{1}{n_1} - \frac{1}{N}\right)\rho_{YX}C_Y C_X,\ E\left(\Delta_1\Delta_2\right) = \left(\frac{1}{n_1} - \frac{1}{N}\right)C_X^2 \end{aligned}\right\} \tag{3.8}$$

Case II: when $S_2 \subset \Omega_N$.

$$\left.\begin{aligned} &E\left(\Delta_0\right) = E\left(\Delta_1\right) = E\left(\Delta_2\right) = 0, E\left(\Delta_0^2\right) = \left(\frac{1}{r} - \frac{1}{N}\right)C_Y^2, E\left(\Delta_1^2\right) = \left(\frac{1}{r} - \frac{1}{N}\right)C_X^2 \\ &E\left(\Delta_2^2\right) = \left(\frac{1}{n_1} - \frac{1}{N}\right)C_X^2,\ E\left(\Delta_0\Delta_1\right) = \left(\frac{1}{r} - \frac{1}{N}\right)\rho_{YX}C_Y C_X, E\left(\Delta_0\Delta_2\right) = 0,\ E\left(\Delta_1\Delta_2\right) = 0 \end{aligned}\right\} \tag{3.9}$$

3.5.1 Theoretical Bias and MSE of the estimator $T_1^{(d)}$

Express (3.5) in term of error terms $\left|\Delta_i\right| \approx 0,\ i = 0,1,2$ as

$$T_1^{(d)} = \frac{r}{n}\left(1+\Delta_0\right)\bar{Y} + \left(1-\frac{r}{n}\right)\frac{\left[\left(1+\Delta_0\right)\bar{Y} + b_{reg}\left(\left(1+\Delta_2\right)\bar{X} - \left(1+\Delta_1\right)\bar{X}\right)\right]}{\left(\phi_j\left(1+\Delta_1\right)\bar{X} + \phi_k\right)}\left(\phi_j\left(1+\Delta_2\right)\bar{X} + \phi_k\right)$$
$$\exp\left(\frac{\eta\left(\phi_j\left(1+\Delta_2\right)\bar{X} + \phi_k\right) - \left(\phi_j\left(1+\Delta_1\right)\bar{X} + \phi_k\right)}{\left(\phi_j\left(1+\Delta_2\right)\bar{X} + \phi_k\right) - \left(\phi_j\left(1+\Delta_1\right)\bar{X} + \phi_k\right)}\right) \quad (3.10)$$

Simplify (3.10)

$$T_1^{(d)} = \frac{r}{n}\left(1+\Delta_0\right)\bar{Y} + \left(1-\frac{r}{n}\right)\frac{\left[\bar{Y} + \bar{Y}\Delta_0 + \bar{X}b_{reg}\left(\Delta_2 - \Delta_1\right)\right]}{\left(\phi_j\bar{X} + \phi_k\right)\left[1 + \frac{\phi_j\bar{X}\Delta_1}{\phi_j\bar{X} + \phi_k}\right]}\left(\phi_j\bar{X} + \phi_k\right)\left[1 + \frac{\phi_j\bar{X}\Delta_2}{\phi_j\bar{X} + \phi_k}\right]$$
$$\times\exp\left(\frac{\eta_1\left(\phi_j\bar{X} + \phi_k\right) + \phi_j\bar{X}\Delta_2 - \left(\phi_j\bar{X} + \phi_k\right) - \phi_j\bar{X}\Delta_1}{\left(\phi_j\bar{X} + \phi_k\right) + \phi_j\bar{X}\Delta_2 + \left(\phi_j\bar{X} + \phi_k\right) + \phi_j\bar{X}\Delta_1}\right) \quad (3.11)$$

$$T_1^{(d)} = \frac{r}{n}\left(1+\Delta_0\right)\bar{Y} + \left(1-\frac{r}{n}\right)\frac{\left[\bar{Y} + \bar{Y}\Delta_0 + \bar{X}b_{reg}\left(\Delta_2 - \Delta_1\right)\right]}{\left(\phi_j\bar{X} + \phi_k\right)\left[1 + \frac{\phi_j\bar{X}\Delta_1}{\phi_j\bar{X} + \phi_k}\right]}\left(1+\Theta\Delta_2\right)\left(1+\Theta\Delta_1\right)^{-1}$$
$$\times\exp\left(\frac{\eta_1\phi_j\bar{X}\left(\Delta_2 - \Delta_1\right)}{2\left(\phi_j\bar{X} + \phi_k\right) + \left[1 + \frac{\phi_j\bar{X} + \Delta_2}{2\left(\phi_j\bar{X} + \phi_k\right)} + \frac{\phi_j\bar{X} + \Delta_1}{2\left(\phi_j\bar{X} + \phi_k\right)}\right]}\right) \quad (3.12)$$

Let $\Theta = \dfrac{\phi_j\bar{X}}{\phi_j\bar{X} + \phi_k}$, then (3.12) becomes

$$T_1^{(d)} = \frac{r}{n}\left(1+\Delta_0\right)\bar{Y} + \left(1-\frac{r}{n}\right)\left[\bar{Y} + \bar{Y}\Delta_0 + \bar{X}b_{reg}\left(\Delta_2 - \Delta_1\right)\right]\left(1+\Theta\Delta_2\right)\left(1+\Theta\Delta_1 + \Theta^2\Delta_1^2\right)$$
$$\exp\left(\eta_1\frac{\Theta}{2}\left(\Delta_2 - \Delta_1\right)\left(1 + \frac{\Theta\Delta_2}{2} + \frac{\Theta\Delta_1}{2}\right)^{-1}\right) \quad (3.13)$$

$$T_1^{(d)} = \frac{r}{n}\left(1+\Delta_0\right)\bar{Y} + \left(1-\frac{r}{n}\right)\left[\bar{Y} + \bar{Y}\Delta_0 + \bar{X}b_{reg}\left(\Delta_2 - \Delta_1\right)\right]\left(\begin{array}{l}1 - \Theta\Delta_1 + \Theta^2\Delta_1^2 + \Theta\Delta_2 \\ -\Theta^2\Delta_1\Delta_2\end{array}\right)$$
$$\exp\left(\eta_1\frac{\Theta}{2}\left(\Delta_2 - \Delta_1\right)\left(1 - \frac{\Theta\Delta_1}{2} - \frac{\Theta\Delta_2}{2} + \frac{\Theta^2\Delta_1^2}{4} + \frac{\Theta^2\Delta_2^2}{4} + \frac{\Theta^2\Delta_1\Delta_2}{2}\right)\right) \quad (3.14)$$

Simplify (3.14) to second degree approximation

$$T_1^{(d)}=\frac{r}{n}\left(1+\Delta_0\right)\bar{Y}+\left(1-\frac{r}{n}\right)\left[\bar{Y}+\bar{Y}\Delta_0+\bar{X}b_{reg}\Delta_2-\bar{X}b_{reg}\Delta_1\right]\begin{pmatrix}1-\Theta\Delta_1+\Theta^2\Delta_1^2+\Theta\Delta_2\\ -\Theta^2\Delta_1\Delta_2\end{pmatrix}$$
$$\exp\left(\eta_1\frac{\Theta}{2}\left(\Delta_2-\Delta_1\right)\left(\Delta_2-\frac{\Theta\Delta_1\Delta_2}{2}-\frac{\Theta\Delta_2^2}{2}-\Delta_1+\frac{\Theta\Delta_1^2}{2}++\frac{\Theta\Delta_1\Delta_2}{2}\right)\right) \quad (3.15)$$

Simplify the exponential term to second degree approximation

$$T_1^{(d)}=\frac{r}{n}\left(1+\Delta_0\right)\bar{Y}+\left(1-\frac{r}{n}\right)\begin{bmatrix}\bar{Y}-\left(\Theta\bar{Y}+\bar{X}b_{reg}\right)\Delta_1+\left(\Theta\bar{Y}+\bar{X}b_{reg}\right)\Delta_2\\ +\left(\Theta^2\bar{Y}+\bar{X}b_{reg}\right)\Delta_1^2+\Theta\bar{X}b_{reg}\Delta_2^2\\ -\Theta\bar{Y}\Delta_1\Delta_2-\left(\Theta^2\bar{Y}+2\Theta\bar{X}b_{reg}\right)\Delta_1\Delta_2+\bar{Y}\Delta_0+\Theta\bar{Y}\Delta_0\Delta_2\end{bmatrix}$$
$$\times\left(1+\frac{\eta_1\Theta}{2}\Delta_2-\frac{\eta_1\Theta}{2}\Delta_1+\left(\frac{\eta_1\Theta^2}{4}+\frac{\eta_1^2\Theta^2}{8}\right)\Delta_2^2+\left(\frac{\eta_1\Theta^2}{4}+\frac{\eta_1^2\Theta^2}{8}\right)\Delta_1^2-\frac{\eta_1^2\Theta^2}{4}\Delta_1\Delta_2\right) \quad (3.16)$$

Expand (3.16) to second degree approximation to obtain (3.17) as

$$T_1^{(d)}-\bar{Y}=\bar{Y}\Delta_0+\left(1-\frac{r}{n}\right)\begin{bmatrix}\left(\frac{\Theta\bar{Y}\eta_1}{2}+\Theta\bar{Y}+\bar{X}b_{reg}\right)\Delta_2-\left(\frac{\Theta\bar{Y}\eta_1}{2}+\Theta\bar{Y}+\bar{X}b_{reg}\right)\Delta_1+\\ \left(\frac{3\Theta^2\bar{Y}\eta_1}{4}+\frac{\Theta^2\bar{Y}\eta_1^2}{8}+\Theta^2\bar{Y}+\left(1+\frac{\eta_1}{2}\right)\Theta\bar{X}b_{reg}\right)\Delta_1^2+\\ \left(\frac{3\Theta^2\bar{Y}\eta_1}{4}+\frac{\Theta^2\bar{Y}\eta_1^2}{8}+\left(1+\frac{\eta_1}{2}\right)\Theta\bar{X}b_{reg}\right)\Delta_2^2-\left(\frac{\Theta\bar{Y}\eta_1}{2}+\Theta\bar{Y}\right)\Delta_0\Delta_1\\ +\left(\frac{\Theta\bar{Y}\eta_1}{2}+\Theta\bar{Y}\right)\Delta_0\Delta_2-\left(\frac{5\Theta^2\bar{Y}\eta_1}{4}+\Theta^2\bar{Y}+\left(2+\eta_1\right)\Theta\bar{X}b_{reg}\right)\Delta_1\Delta_2\end{bmatrix} \quad (3.17)$$

Take expectation of (3.17) to obtain the bias of $T_1^{(d)}$

$$Bias\left(T_1^{(d)}\right)=\bar{Y}E\left(\Delta_0\right)+\left(1-\frac{r}{n}\right)E\begin{bmatrix}\left(\frac{\Theta\bar{Y}\eta_1}{2}+\Theta\bar{Y}+\bar{X}b_{reg}\right)\Delta_2-\left(\frac{\Theta\bar{Y}\eta_1}{2}+\Theta\bar{Y}+\bar{X}b_{reg}\right)\Delta_1+\\ \left(\frac{3\Theta^2\bar{Y}\eta_1}{4}+\frac{\Theta^2\bar{Y}\eta_1^2}{8}+\Theta^2\bar{Y}+\left(1+\frac{\eta_1}{2}\right)\Theta\bar{X}b_{reg}\right)\Delta_1^2+\\ \left(\frac{3\Theta^2\bar{Y}\eta_1}{4}+\frac{\Theta^2\bar{Y}\eta_1^2}{8}+\left(1+\frac{\eta_1}{2}\right)\Theta\bar{X}b_{reg}\right)\Delta_2^2\\ -\left(\frac{\Theta\bar{Y}\eta_1}{2}+\Theta\bar{Y}\right)\Delta_0\Delta_1+\left(\frac{\Theta\bar{Y}\eta_1}{2}+\Theta\bar{Y}\right)\Delta_0\Delta_2\\ -\left(\frac{5\Theta^2\bar{Y}\eta_1}{4}+\Theta^2\bar{Y}+\left(2+\eta_1\right)\Theta\bar{X}b_{reg}\right)\Delta_1\Delta_2\end{bmatrix} \quad (3.18)$$

Apply the results of (3.8) in (3.18), the bias of $T_1^{(d)}$ under case I denoted by $Bias\left(T_1^{(d)}\right)_I$ is obtained as

$$Bias\left(T_1^{(d)}\right)_I=\left(1-\frac{r}{n}\right)\left[\begin{array}{l}\left(\frac{3\Theta^2\bar{Y}\eta_1}{4}+\frac{\Theta^2\bar{Y}\eta_1^2}{8}+\Theta^2\bar{Y}+\left(1+\frac{\eta_1}{2}\right)\Theta\bar{X}b_{reg}\right)\left(\frac{1}{r}-\frac{1}{N}\right)C_X^2\\+\left(\frac{3\Theta^2\bar{Y}\eta_1}{4}+\frac{\Theta^2\bar{Y}\eta_1^2}{8}+\left(1+\frac{\eta_1}{2}\right)\Theta\bar{X}b_{reg}\right)\left(\frac{1}{n_1}-\frac{1}{N}\right)C_X^2\\-\left(\frac{\Theta\bar{Y}\eta_1}{2}+\Theta\bar{Y}\right)\left(\frac{1}{r}-\frac{1}{N}\right)\rho_{YX}C_YC_X+\left(\frac{\Theta\bar{Y}\eta_1}{2}+\Theta\bar{Y}\right)\left(\frac{1}{n_1}-\frac{1}{N}\right)C_X^2\\-\left(\frac{5\Theta^2\bar{Y}\eta_1}{4}+\Theta^2\bar{Y}+\left(2+\eta_1\right)\Theta\bar{X}b_{reg}\right)\left(\frac{1}{n_1}-\frac{1}{N}\right)\rho_{YX}C_YC_X\end{array}\right] \quad (3.19)$$

$$Bias\left(T_1^{(d)}\right)_I=\left(1-\frac{r}{n}\right)\left[\begin{array}{l}\left(\left(\frac{3\eta_1}{4}+1\right)\Theta^2\bar{Y}+\frac{\Theta^2\bar{Y}\eta_1^2}{8}+\left(1+\frac{\eta_1}{2}\right)\Theta\bar{X}b_{reg}\right)\left(\frac{1}{r}-\frac{1}{N}\right)C_X^2\\-\left(\left(\frac{\eta_1}{2}+1\right)\Theta^2\bar{Y}-\frac{\Theta^2\bar{Y}\eta_1^2}{8}+\left(1+\frac{\eta_1}{2}\right)\Theta\bar{X}b_{reg}\right)\left(\frac{1}{n_1}-\frac{1}{N}\right)C_X^2\\-\left(\frac{\Theta\bar{Y}\eta_1}{2}+\Theta\bar{Y}\right)\left(\frac{1}{r}-\frac{1}{n_1}\right)\rho_{YX}C_YC_X\end{array}\right] \quad (3.20)$$

Also, apply the results of (3.9) in (3.18), the bias of $T_1^{(d)}$ under case II denoted by $Bias\left(T_1^{(d)}\right)_{II}$ is obtained as

$$Bias\left(T_1^{(d)}\right)_{II}=\left(1-\frac{r}{n}\right)\left[\begin{array}{l}\left(\frac{3\Theta^2\bar{Y}\eta_1}{4}+\frac{\Theta^2\bar{Y}\eta_1^2}{8}+\Theta^2\bar{Y}+\left(1+\frac{\eta_1}{2}\right)\Theta\bar{X}b_{reg}\right)\left(\frac{1}{r}-\frac{1}{N}\right)C_X^2\\+\left(\frac{3\Theta^2\bar{Y}\eta_1}{4}+\frac{\Theta^2\bar{Y}\eta_1^2}{8}+\left(1+\frac{\eta_1}{2}\right)\Theta\bar{X}b_{reg}\right)\left(\frac{1}{n_1}-\frac{1}{N}\right)C_X^2\\-\left(\frac{\Theta\bar{Y}\eta_1}{2}+\Theta\bar{Y}\right)\left(\frac{1}{r}-\frac{1}{N}\right)\rho_{YX}C_YC_X\end{array}\right] \quad (3.21)$$

$$Bias\left(T_1^{(d)}\right)_{II}=\left(1-\frac{r}{n}\right)\left[\begin{array}{l}\left(\left(\frac{3\eta_1}{4}+1\right)\Theta^2\bar{Y}+\frac{\Theta^2\bar{Y}\eta_1^2}{8}+\left(1+\frac{\eta_1}{2}\right)\Theta\bar{X}b_{reg}\right)\left(\frac{1}{r}+\frac{1}{n_1}-\frac{2}{N}\right)C_X^2\\+\Theta^2\bar{Y}\left(\frac{1}{r}-\frac{1}{N}\right)C_X^2-\left(\frac{\Theta\bar{Y}\eta_1}{2}+\Theta\bar{Y}\right)\left(\frac{1}{r}-\frac{1}{N}\right)\rho_{YX}C_YC_X\end{array}\right] \quad (3.22)$$

Square (3.18) and take expectation of the results to obtain the MSE of $T_1^{(d)}$

$$MSE\left(T_1^{(d)}\right)=E\left(\bar{Y}\left(\Delta_0\right)+\left(1-\frac{r}{n}\right)\begin{bmatrix}\left(\frac{\Theta\bar{Y}\eta_1}{2}+\Theta\bar{Y}+\bar{X}b_{reg}\right)\Delta_2-\left(\frac{\Theta\bar{Y}\eta_1}{2}+\Theta\bar{Y}+\bar{X}b_{reg}\right)\Delta_1+\\ \left(\frac{3\Theta^2\bar{Y}\eta_1}{4}+\frac{\Theta^2\bar{Y}\eta_1^2}{8}+\Theta^2\bar{Y}+\left(1+\frac{\eta_1}{2}\right)\Theta\bar{X}b_{reg}\right)\Delta_1^2+\\ \left(\frac{3\Theta^2\bar{Y}\eta_1}{4}+\frac{\Theta^2\bar{Y}\eta_1^2}{8}+\left(1+\frac{\eta_1}{2}\right)\Theta\bar{X}b_{reg}\right)\Delta_2^2\\ -\left(\frac{\Theta\bar{Y}\eta_1}{2}+\Theta\bar{Y}\right)\Delta_0\Delta_1+\left(\frac{\Theta\bar{Y}\eta_1}{2}+\Theta\bar{Y}\right)\Delta_0\Delta_2\\ -\left(\frac{5\Theta^2\bar{Y}\eta_1}{4}+\Theta^2\bar{Y}+\left(2+\eta_1\right)\Theta\bar{X}b_{reg}\right)\Delta_1\Delta_2\end{bmatrix}\right)^2 \quad (3.23)$$

Apply the results of (3.9) in (3.23), the MSE of $T_1^{(d)}$ under case I denoted by $MSE\left(T_1^{(d)}\right)_I$ is obtained as

$$MSE\left(T_1^{(d)}\right)_I=\bar{Y}^2\left(\frac{1}{r}-\frac{1}{N}\right)C_Y^2+\left(1-\frac{r}{n}\right)\begin{bmatrix}\left(\frac{\Theta\bar{Y}\eta_1}{2}+\Theta\bar{Y}+\bar{X}b_{reg}\right)^2\left(\left(\frac{1}{r}-\frac{1}{N}\right)-\left(\frac{1}{n_1}-\frac{1}{N}\right)\right)\\ \left(1-\frac{r}{n}\right)C_X^2+2\bar{Y}\left(\frac{\Theta\bar{Y}\eta_1}{2}+\Theta\bar{Y}+\bar{X}b_{reg}\right)\\ \left(\left(\frac{1}{n_1}-\frac{1}{N}\right)-\left(\frac{1}{r}-\frac{1}{N}\right)\right)\rho_{YX}C_YC_X\end{bmatrix} \quad (3.24)$$

$$MSE\left(T_1^{(d)}\right)_I=\bar{Y}^2\left(\frac{1}{r}-\frac{1}{N}\right)C_Y^2+\left(1-\frac{r}{n}\right)\left(\frac{1}{r}-\frac{1}{n_1}\right)\begin{bmatrix}\left(\frac{\Theta\bar{Y}\eta_1}{2}+\Theta\bar{Y}+\bar{X}b_{reg}\right)^2\left(1-\frac{r}{n}\right)C_X^2\\ -2\bar{Y}\left(\frac{\Theta\bar{Y}\eta_1}{2}+\Theta\bar{Y}+\bar{X}b_{reg}\right)\rho_{YX}C_YC_X\end{bmatrix} \quad (3.25)$$

Differentiate (3.25) partially with respect to η_1 and equate to zero to obtain optimum expression for η_1 denoted by $\eta_{1(opt)I}$

$$\frac{\partial MSE\left(T_1^{(d)}\right)_I}{\partial\eta_1}=\left(1-\frac{r}{n}\right)\left(\frac{1}{r}-\frac{1}{n_1}\right)\begin{pmatrix}\left(1-\frac{r}{n}\right)\left(\frac{\Theta\bar{Y}\eta_1}{2}+\Theta\bar{Y}+\bar{X}b_{reg}\right)\Theta\bar{Y}C_X^2\\ -2\Theta\bar{Y}^2\rho_{YX}C_YC_X\end{pmatrix}=0 \quad (3.26)$$

Solve for η_1, the expression for $\eta_{1(opt)I}$ is obtained as in (3.30)

$$\eta_{1(opt)I}=2\left(\frac{\rho_{YX}C_Y}{\left(1-r/n\right)\Theta C_X}-1-\frac{\bar{X}b_{reg}}{\Theta\bar{Y}}\right) \quad (3.27)$$

Substitute the result in (3.27) in (3.25), the minimum MSE of $T_1^{(d)}$ under case I denoted by $MSE\left(T_1^{(d)}\right)_{I(\min)}$ is obtained as

$$MSE\left(T_1^{(d)}\right)_{I(\min)} = \bar{Y}^2 C_Y^2 \left(\left(\frac{1}{r} - \frac{1}{N}\right) - \left(\frac{1}{r} - \frac{1}{n_1}\right)\rho_{YX}^2 \right) \tag{3.28}$$

Similarly, apply the results of (3.9) to (3.18), the MSE of $T_1^{(d)}$ under case II denoted by $MSE\left(T_1^{(d)}\right)_{II}$ is obtained as

$$MSE\left(T_1^{(d)}\right)_{II} = \bar{Y}^2\left(\frac{1}{r} - \frac{1}{N}\right)C_Y^2 + \left(1 - \frac{r}{n}\right)\left[\begin{array}{l}\left(\frac{\Theta\bar{Y}\eta_1}{2} + \Theta\bar{Y} + \bar{X}b_{reg}\right)^2\left(1 - \frac{r}{n}\right)\left(\frac{1}{r} + \frac{1}{n_1} - \frac{2}{N}\right)C_X^2 \\ -2\bar{Y}\left(\frac{\Theta\bar{Y}\eta_1}{2} + \Theta\bar{Y} + \bar{X}b_{reg}\right)\left(\frac{1}{r} - \frac{1}{N}\right)\rho_{YX}C_Y C_X\end{array}\right] \tag{3.29}$$

Differentiate (3.29) partially with respect to η_1 and equate to zero to obtain optimum expression for η_1 denoted by $\eta_{1(opt)II}$

$$\frac{\partial MSE\left(T_1^{(d)}\right)_{II}}{\partial \eta_1} = \left(1 - \frac{r}{n}\right)\left(\begin{array}{l}\left(1 - \frac{r}{n}\right)\left(\frac{1}{r} + \frac{1}{n_1} - \frac{2}{N}\right)\left(\frac{\Theta\bar{Y}\eta_1}{2} + \Theta\bar{Y} + \bar{X}b_{reg}\right)\Theta\bar{Y}C_X^2 \\ -2\left(\frac{1}{r} - \frac{1}{N}\right)\Theta\bar{Y}^2\rho_{YX}C_Y C_X\end{array}\right) = 0 \tag{3.30}$$

Solve for η_1, the expression for $\eta_{1(opt)II}$ is obtained as in (3.31)

$$\eta_{1(opt)II} = 2\left(\frac{\left(r^{-1} - N^{-1}\right)\rho_{YX}C_Y}{(1 - r/n)\left(r^{-1} + n_1^{-1} - 2N^{-1}\right)\Theta C_X} - 1 - \frac{\bar{X}b_{reg}}{\Theta\bar{Y}}\right) \tag{3.31}$$

Substitute the result in (3.31) in (3.29), the minimum MSE of $T_1^{(d)}$ under case II denoted by $MSE\left(T_1^{(d)}\right)_{II(\min)}$ is obtained as

$$MSE\left(T_1^{(d)}\right)_{II(\min)} = \bar{Y}^2 C_Y^2\left(\frac{1}{r} - \frac{1}{N}\right)\left(1 - \left(\frac{1}{r} - \frac{1}{N}\right)\left(\frac{1}{r} + \frac{1}{n_1} - \frac{2}{N}\right)^{-1}\rho_{YX}^2\right) \tag{3.32}$$

3.5 Theoretical Efficiency Comparison of $T_i^{(d)}, i = 1,2$

In this section, the efficiency of the estimators of the modified imputation schemes are theoretically compared with those estimators of some existing related imputation schemes.

3.5.1 Theoretical Efficiency Comparison of $T_i^{(d)}, i = 1,2$ *under Case I*

i. Lee et al. (1994) estimator T_1^* versus modified estimators $T_i^{(d)},\ i = 1, 2$

$$MSE\left(T_i^{(d)}\right)_{I(\min)} - MSE\left(T_1^*\right)_I < 0 \tag{3.31}$$

$$\bar{Y}^2 C_Y^2\left(\left(\frac{1}{r} - \frac{1}{N}\right) - \left(\frac{1}{r} - \frac{1}{n_1}\right)\rho_{YX}^2\right) - \left(\begin{array}{l}\left(\frac{1}{r} - \frac{1}{N}\right)\bar{Y}^2 C_Y^2 - \left(\frac{1}{r} - \frac{1}{n}\right)\bar{Y}^2 C_X^2 \\ +2\left(\frac{1}{r} - \frac{1}{n}\right)\bar{Y}^2\rho_{YX}C_Y C_X\end{array}\right) < 0 \tag{3.32}$$

$$-\left(\frac{1}{r}-\frac{1}{n_1}\right)C_Y^2\rho_{YX}^2-\left(\frac{1}{r}-\frac{1}{n}\right)C_X^2+2\left(\frac{1}{r}-\frac{1}{n}\right)\rho_{YX}C_YC_X<0 \tag{3.33}$$

$$-\left(\frac{1}{r}-\frac{1}{n_1}\right)C_Y^2\rho_{YX}^2<\left(\frac{1}{r}-\frac{1}{n}\right)\left(C_X^2-2\rho_{YX}C_YC_X\right) \tag{3.34}$$

$$\left(-\left(\frac{1}{r}-\frac{1}{n_1}\right)+\left(\frac{1}{r}-\frac{1}{n}\right)\right)C_Y^2\rho_{YX}^2<\left(\frac{1}{r}-\frac{1}{n}\right)\left(C_X-\rho_{YX}C_Y\right)^2 \tag{3.35}$$

$$\left(\frac{1}{n_1}-\frac{1}{n}\right)<\left(\frac{1}{r}-\frac{1}{n}\right)\left(\frac{C_X}{\rho_{YX}C_Y}-1\right)^2 \tag{3.36}$$

$$\rho_{YX}>\frac{C_Y\left(\sqrt{\left(n_1^{-1}-n^{-1}\right)\left(r^{-1}-n^{-1}\right)^{-1}}+1\right)}{C_X} \tag{3.37}$$

ii. Singh and Horn (2000) estimator T_2^* versus modified estimators $T_i^{(d)},\ i=1,2$

$$MSE\left(T_i^{(d)}\right)_{I(\min)}-MSE\left(T_2^*\right)_I<0 \tag{3.38}$$

$$\bar{Y}^2C_Y^2\left(\left(\frac{1}{r}-\frac{1}{N}\right)-\left(\frac{1}{r}-\frac{1}{n_1}\right)\rho_{YX}^2\right)-\left(\left(\frac{1}{r}-\frac{1}{N}\right)\bar{Y}^2C_Y^2+\left(\frac{1}{r}-\frac{1}{n}\right)B_{reg}\overline{YX}\rho_{YX}C_YC_X\right)<0 \tag{3.39}$$

$$-\bar{Y}C_Y\left(\frac{1}{r}-\frac{1}{n_1}\right)\rho_{YX}+\left(\frac{1}{r}-\frac{1}{n}\right)B_{reg}\bar{X}C_X<0 \tag{3.40}$$

$$\rho_{YX}>\frac{\left(r^{-1}-n^{-1}\right)B_{reg}\bar{X}C_X}{\bar{Y}C_Y\left(r^{-1}-n_1^{-1}\right)} \tag{3.41}$$

iii. Kadilar and Cingi (2008) estimator T_3^* versus modified estimators $T_i^{(d)},\ i=1,2$

$$MSE\left(T_i^{(d)}\right)_{I(\min)}-MSE\left(T_3^*\right)_I<0 \tag{3.42}$$

$$\bar{Y}^2C_Y^2\left(\left(\frac{1}{r}-\frac{1}{N}\right)-\left(\frac{1}{r}-\frac{1}{n_1}\right)\rho_{YX}^2\right)-\left(\left(\frac{1}{r}-\frac{1}{N}\right)\bar{Y}^2C_Y^2+\left(\frac{1}{r}-\frac{1}{n}\right)\begin{pmatrix}\bar{X}^2C_X^2\left(R+B_{reg}\right)^2\\-2\left(R+B_{reg}\right)\overline{YX}\rho_{YX}C_YC_X\end{pmatrix}\right)<0 \tag{3.43}$$

$$-\bar{Y}^2C_Y^2\left(\frac{1}{r}-\frac{1}{n_1}\right)\rho_{YX}^2-\left(\frac{1}{r}-\frac{1}{n}\right)\begin{pmatrix}\bar{X}^2C_X^2\left(R+B_{reg}\right)^2\\-2\left(R+B_{reg}\right)\overline{YX}\rho_{YX}C_YC_X\end{pmatrix}<0 \tag{3.44}$$

$$\bar{Y}^2C_Y^2\rho_{YX}^2\left(-\left(\frac{1}{r}-\frac{1}{n_1}\right)+\left(\frac{1}{r}-\frac{1}{n}\right)\right)<\left(\frac{1}{r}-\frac{1}{n}\right)\left(\bar{X}C_X\left(R+B_{reg}\right)-\bar{Y}\rho_{YX}C_Y\right)^2 \tag{3.45}$$

$$\left(\frac{1}{n_1}-\frac{1}{n}\right)<\left(\frac{1}{r}-\frac{1}{n}\right)\left(\frac{\bar{X}C_X\left(R+B_{reg}\right)}{\bar{Y}\rho_{YX}C_Y}-1\right)^2 \tag{3.46}$$

$$\rho_{YX}>\left(\frac{\bar{Y}C_Y\left(\sqrt{\left(n_1^{-1}-n^{-1}\right)\left(r^{-1}-n^{-1}\right)^{-1}}+1\right)}{\bar{X}C_X\left(R+B_{reg}\right)}\right) \tag{3.47}$$

iv. Singh (2009) estimator T_4^* versus modified estimators $T_i^{(d)},\ i=1,2$

$$MSE\left(T_i^{(d)}\right)_{I(\min)}-MSE\left(T_4^*\right)_I<0 \tag{3.48}$$

$$\bar{Y}^2C_Y^2\left(\left(\frac{1}{r}-\frac{1}{N}\right)-\left(\frac{1}{r}-\frac{1}{n_1}\right)\rho_{YX}^2\right)-\left(\left(\frac{1}{r}-\frac{1}{N}\right)\bar{Y}^2C_Y^2+\left(\frac{1}{r}-\frac{1}{N}\right)S_X^2\left(\frac{\rho_{YX}S_Y}{S_X}-R\right)\right)<0 \tag{3.49}$$

$$-\bar{Y}^2C_Y^2\left(\frac{1}{r}-\frac{1}{n_1}\right)\rho_{YX}^2-\left(\frac{1}{r}-\frac{1}{N}\right)S_X^2\left(\frac{\rho_{YX}S_Y}{S_X}-R\right)^2<0 \tag{3.50}$$

$$-\bar{Y}^2C_Y^2\left(\frac{1}{r}-\frac{1}{n_1}\right)-\left(\frac{1}{r}-\frac{1}{N}\right)\left(S_Y-\frac{S_XR}{\rho_{YX}}\right)^2<0 \tag{3.51}$$

$$\rho_{YX}>\frac{S_XR}{\bar{Y}C_Y\sqrt{\left(r^{-1}-n_1^{-1}\right)\left(r^{-1}-N^{-1}\right)^{-1}}+S_Y} \tag{3.51}$$

v. Audu *et al*. (2021c) estimator T_5^* versus modified estimators $T_i^{(d)},\ i=1,2$

$$MSE\left(T_i^{(d)}\right)_{I(\min)}-MSE\left(T_5^*\right)_I<0 \tag{3.53}$$

$$\bar{Y}^2C_Y^2\left(\left(\frac{1}{r}-\frac{1}{N}\right)-\left(\frac{1}{r}-\frac{1}{n_1}\right)\rho_{YX}^2\right)-\bar{Y}^2C_Y^2\left(\left(\frac{1}{r}-\frac{1}{n}\right)-\left(\left(\frac{1}{r}-\frac{1}{N}\right)-\left(\frac{1}{n}-\frac{1}{N}\right)\right)\rho_{YX}^2\right)<0 \tag{3.54}$$

$$\bar{Y}^2C_Y^2\left(\left(\frac{1}{r}-\frac{1}{N}\right)-\left(\frac{1}{r}-\frac{1}{n_1}\right)\rho_{YX}^2\right)-\bar{Y}^2C_Y^2\left(\frac{1}{r}-\frac{1}{n}\right)\left(1-\rho_{YX}^2\right)<0 \tag{3.55}$$

$$\rho_{YX}>\sqrt{\left(\frac{1}{n}-\frac{1}{N}\right)\left(\frac{1}{n}-\frac{1}{n_1}\right)^{-1}} \tag{3.56}$$

vi. Musa *et al.* (2023) estimator T_6^* versus modified estimators $T_i^{(d)},\ i=1,2$

$$MSE\left(T_i^{(d)}\right)_{I(\min)} - MSE\left(T_6^*\right)_I < 0 \tag{3.57}$$

$$\bar{Y}^2 C_Y^2\left(\left(\frac{1}{r}-\frac{1}{N}\right)-\left(\frac{1}{r}-\frac{1}{n_1}\right)\rho_{YX}^2\right)-\bar{Y}^2 C_Y^2\left(\left(\frac{1}{r}-\frac{1}{N}\right)-\left(\left(\frac{1}{r}-\frac{1}{N}\right)-\left(\frac{1}{n}-\frac{1}{N}\right)\right)\rho_{YX}^2\right)<0 \tag{3.58}$$

$$\left(\frac{1}{n_1}-\frac{1}{N}\right)\rho_{YX}^2-\left(\left(\frac{1}{r}-\frac{1}{N}\right)-\left(\frac{1}{r}-\frac{1}{n}\right)\rho_{YX}^2\right)<0 \tag{3.59}$$

$$\rho_{YX} > \sqrt{\left(\frac{1}{r}-\frac{1}{N}\right)^{-1}\left(\frac{1}{r}+\frac{1}{n_1}-\frac{1}{n}-\frac{1}{N}\right)} \tag{3.60}$$

Equations (3.37), (3.41), (3.47), (3.51), (3.56) and (3.60) are the conditions to be satisfied for the estimators of the modified imputation schemes to be more efficient than the estimators of the existing related imputation schemes by Lee et al. (1994), Singh and Horn (2000), Kadilar and Cingi (2008), Singh (2009), Audu *et al.* (2021c) and Musa *et al.* (2023) respectively under two-phase sampling scheme for case I.

3.6.2 Theoretical Efficiency Comparison of $T_i^{(d)}, i = 1,2$ under Case II

i. Lee et al. (1994) estimator T_1^* versus modified estimators $T_i^{(d)},\ i=1,2$

$$MSE\left(T_i^{(d)}\right)_{II(\min)} - MSE\left(T_1^*\right)_{II} < 0 \tag{3.61}$$

$$\bar{Y}^2\left(\frac{1}{r}-\frac{1}{N}\right)C_Y^2\left(1-\left(\frac{1}{r}-\frac{1}{N}\right)\left(\frac{1}{r}+\frac{1}{n_1}-\frac{2}{N}\right)^{-1}\rho_{YX}^2\right)-\left(\begin{array}{l}\bar{Y}^2\left(\frac{1}{r}-\frac{1}{N}\right)C_Y^2+\left(\frac{1}{r}+\frac{1}{n}-\frac{2}{N}\right)C_X^2\\-2\left(\frac{1}{r}-\frac{1}{N}\right)\rho_{YX}C_YC_X\end{array}\right)<0 \tag{3.62}$$

$$-\bar{Y}^2\left(\frac{1}{r}-\frac{1}{N}\right)C_Y^2\left(\frac{1}{r}-\frac{1}{N}\right)\left(\frac{1}{r}+\frac{1}{n_1}-\frac{2}{N}\right)^{-1}\rho_{YX}^2-\left(\begin{array}{l}\left(\frac{1}{r}+\frac{1}{n_1}-\frac{2}{N}\right)C_X^2\\-2\left(\frac{1}{r}-\frac{1}{N}\right)\rho_{YX}C_YC_X\end{array}\right)<0 \tag{3.63}$$

$$\left(\frac{1}{r}-\frac{1}{N}\right)^2\left(\left(\frac{1}{r}+\frac{1}{n}-\frac{2}{N}\right)^{-1}-\left(\frac{1}{r}+\frac{1}{n_1}-\frac{2}{N}\right)^{-1}\bar{Y}^2\right)<\left(\begin{array}{l}\sqrt{\left(\frac{1}{r}+\frac{1}{n}-\frac{2}{N}\right)}\dfrac{C_X}{\rho_{YX}C_Y}\\-\left(\frac{1}{r}-\frac{1}{N}\right)\dfrac{1}{\sqrt{\left(\frac{1}{r}+\frac{1}{n}-\frac{2}{N}\right)}}\end{array}\right)^2 \tag{3.64}$$

$$\rho_{YX} > \frac{\left(r^{-1}-N^{-1}\right)C_Y\left(\begin{array}{l}\sqrt{\left(r^{-1}+n^{-1}-N^{-2}\right)^{-2}-\left(r^{-1}+n_1^{-1}-N^{-2}\right)^{-1}\left(r^{-1}+n^{-1}-N^{-2}\right)^{-1}\bar{Y}^2}\\+\left(r^{-1}+n^{-1}-N^{-2}\right)^{-1}\end{array}\right)}{C_X} \tag{3.65}$$

ii. Singh and Horn (2000) estimator T_2^* versus modified estimators $T_i^{(d)},\ i=1,2$

$$MSE\left(T_i^{(d)}\right)_{II(\min)} - MSE\left(T_2^*\right)_{II} < 0 \tag{3.66}$$

$$\bar{Y}^2\left(\frac{1}{r}-\frac{1}{N}\right)C_Y^2\left(1-\left(\frac{1}{r}-\frac{1}{N}\right)\left(\frac{1}{r}+\frac{1}{n_1}-\frac{2}{N}\right)^{-1}\rho_{YX}^2\right) - \bar{Y}^2\left(\frac{1}{r}-\frac{1}{N}\right)C_Y^2\left(1-\left(\frac{1}{r}-\frac{1}{N}\right)\left(\frac{1}{r}+\frac{1}{n}-\frac{2}{N}\right)^{-1}\rho_{YX}^2\right) < 0 \tag{3.67}$$

$$-\left(\frac{1}{r}+\frac{1}{n_1}-\frac{2}{N}\right)^{-1} + \left(\frac{1}{r}+\frac{1}{n}-\frac{2}{N}\right)^{-1} < 0 \tag{3.68}$$

$$\frac{1}{n}-\frac{1}{n_1} > 0 \tag{3.69}$$

iii. Kadilar and Cingi (2008) estimator T_3^* versus modified estimators $T_i^{(d)},\ i=1,2$

$$MSE\left(T_i^{(d)}\right)_{II(\min)} - MSE\left(T_3^*\right)_{II} < 0 \tag{3.70}$$

$$\bar{Y}^2\left(\frac{1}{r}-\frac{1}{N}\right)C_Y^2\left(1-\left(\frac{1}{r}-\frac{1}{N}\right)\left(\frac{1}{r}+\frac{1}{n_1}-\frac{2}{N}\right)^{-1}\rho_{YX}^2\right) - \left(\begin{array}{l}\bar{Y}^2\left(\frac{1}{r}-\frac{1}{N}\right)C_Y^2 + \left(\bar{Y}+\beta_{reg}\right)^2\left(\frac{1}{r}+\frac{1}{n}-\frac{2}{N}\right)C_X^2 \\ -2\bar{Y}\left(\bar{Y}+\beta_{reg}\right)\left(\frac{1}{r}-\frac{1}{N}\right)\rho_{YX}C_YC_X\end{array}\right) < 0 \tag{3.71}$$

$$-\bar{Y}^2C_Y^2\left(\frac{1}{r}-\frac{1}{N}\right)^2\left(\frac{1}{r}+\frac{1}{n_1}-\frac{2}{N}\right)^{-1}\rho_{YX}^2 < \left(\begin{array}{l}\left(\bar{Y}+\beta_{reg}\right)\sqrt{\left(\frac{1}{r}+\frac{1}{n}-\frac{2}{N}\right)}C_X \\ -\bar{Y}\left(\frac{1}{r}-\frac{1}{N}\right)\frac{\rho_{YX}C_Y}{\sqrt{\left(\frac{1}{r}+\frac{1}{n}-\frac{2}{N}\right)}}\end{array}\right)^2 - \bar{Y}^2\left(\frac{1}{r}-\frac{1}{N}\right)\frac{\rho_{YX}^2C_Y^2}{\left(\frac{1}{r}+\frac{1}{n}-\frac{2}{N}\right)} \tag{3.72}$$

$$\left(\sqrt{\left(\frac{1}{r}+\frac{1}{n}-\frac{2}{N}\right)^{-1}-\left(\frac{1}{r}+\frac{1}{n_1}-\frac{2}{N}\right)^{-1}} + \sqrt{\left(\frac{1}{r}+\frac{1}{n}-\frac{2}{N}\right)^{-1}}\right)\bar{Y}\left(\frac{1}{r}-\frac{1}{N}\right)\rho_{YX}C_Y < \left(\bar{Y}+\beta_{reg}\right)\sqrt{\left(\frac{1}{r}+\frac{1}{n}-\frac{2}{N}\right)}C_X \tag{3.73}$$

$$\rho_{YX} > \frac{\left(\sqrt{\left(\frac{1}{r}+\frac{1}{n}-\frac{2}{N}\right)^{-1}-\left(\frac{1}{r}+\frac{1}{n_1}-\frac{2}{N}\right)^{-1}} + \sqrt{\left(\frac{1}{r}+\frac{1}{n}-\frac{2}{N}\right)^{-1}}\right)\bar{Y}\left(\frac{1}{r}-\frac{1}{N}\right)C_Y}{\left(\bar{Y}+\beta_{reg}\right)\sqrt{\left(\frac{1}{r}+\frac{1}{n}-\frac{2}{N}\right)}C_X} \tag{3.74}$$

iv. Singh (2009) estimator T_4^* versus modified estimators $T_i^{(d)},\ i=1,2$

$$MSE\left(T_i^{(d)}\right)_{II(\min)} - MSE\left(T_4^*\right)_{II(\min)} < 0 \tag{3.75}$$

$$\bar{Y}^2\left(\frac{1}{r}-\frac{1}{N}\right)C_Y^2\left(1-\left(\frac{1}{r}-\frac{1}{N}\right)\left(\frac{1}{r}+\frac{1}{n_1}-\frac{2}{N}\right)^{-1}\rho_{YX}^2\right) - \bar{Y}^2\left(\frac{1}{r}-\frac{1}{N}\right)C_Y^2\left(1-\left(\frac{1}{r}-\frac{1}{N}\right)\left(\frac{1}{r}+\frac{1}{n}-\frac{2}{N}\right)^{-1}\rho_{YX}^2\right) < 0 \tag{3.76}$$

$$-\left(\frac{1}{r}+\frac{1}{n_1}-\frac{2}{N}\right)^{-1}+\left(\frac{1}{r}+\frac{1}{n}-\frac{2}{N}\right)^{-1} < 0 \tag{3.77}$$

$$\frac{1}{n}-\frac{1}{n_1} > 0 \tag{3.78}$$

v. Audu *et al.* (2021c) estimator T_5^* versus modified estimators $T_i^{(d)},\ i=1,2$

$$MSE\left(T_i^{(d)}\right)_{II(\min)} - MSE\left(T_5^*\right)_{II(\min)} < 0 \tag{3.79}$$

$$\bar{Y}^2\left(\frac{1}{r}-\frac{1}{N}\right)C_Y^2\left(1-\left(\frac{1}{r}-\frac{1}{N}\right)\left(\frac{1}{r}+\frac{1}{n_1}-\frac{2}{N}\right)^{-1}\rho_{YX}^2\right) - \bar{Y}^2\left(\frac{1}{r}-\frac{1}{N}\right)C_Y^2\left(1-\left(\frac{1}{r}-\frac{1}{N}\right)\left(\frac{1}{r}+\frac{1}{n}-\frac{2}{N}\right)^{-1}\rho_{YX}^2\right) < 0 \tag{3.80}$$

$$-\left(\frac{1}{r}+\frac{1}{n_1}-\frac{2}{N}\right)^{-1}+\left(\frac{1}{r}+\frac{1}{n}-\frac{2}{N}\right)^{-1} < 0 \tag{3.81}$$

$$\frac{1}{n}-\frac{1}{n_1} > 0 \tag{3.82}$$

vi. Musa *et al.* (2023) estimator T_6^* versus modified estimators $T_i^{(d)},\ i=1,2$

$$MSE\left(T_i^{(d)}\right)_{II(\min)} - MSE\left(T_6^*\right)_{II(\min)} < 0 \tag{3.83}$$

$$\bar{Y}^2\left(\frac{1}{r}-\frac{1}{N}\right)C_Y^2\left(1-\left(\frac{1}{r}-\frac{1}{N}\right)\left(\frac{1}{r}+\frac{1}{n_1}-\frac{2}{N}\right)^{-1}\rho_{YX}^2\right) - \bar{Y}^2\left(\frac{1}{r}-\frac{1}{N}\right)C_Y^2\left(1-\left(\frac{1}{r}-\frac{1}{N}\right)\left(\frac{1}{r}+\frac{1}{n}-\frac{2}{N}\right)^{-1}\rho_{YX}^2\right) < 0 \tag{3.84}$$

$$-\left(\frac{1}{r}+\frac{1}{n_1}-\frac{2}{N}\right)^{-1}+\left(\frac{1}{r}+\frac{1}{n}-\frac{2}{N}\right)^{-1} < 0 \tag{3.85}$$

$$\frac{1}{n}-\frac{1}{n_1}>0 \quad (3.86)$$

Equations (3.65), (3.69), (3.74), (3.78), (3.82) and (3.86) are the conditions to be satisfied for the estimators of the modified imputation schemes to be more efficient than the estimators of the existing related imputation schemes by Lee *et al.* (1994), Singh and Horn (2000), Kadilar and Cingi (2008), Singh (2009), Audu *et al.* (2021c) and Musa *et al.* (2023) respectively under two-phase sampling scheme for case II.

4.0 RESULTS AND DISCUSSION

4.1 SIMULATION USED

Table 4.1: Populations used for Simulation Study for Positive Relationship

Populations	Auxiliary variable (x)	Study variable (y)
1	$X \sim exp(0.3)$	$Y = X + E$ where $E \sim absolute(Normal(0,1))$
2	$X \sim uniform(10,100)$	

Table4. 2: MSEs and PREs of Estimators $T_{i(j)}^{(d)}$, T_1^*, T_2^*, T_3^*, T_4^*, $T_{5(j)}^*$, and $T_{6(j)}^*$ when $N=1000,\ n_1=300,\ n=200,\ r=150$ using Population I

ESTIMATORS	CASE I			CASE II		
	MSE	PRE	RANK	MSE	PRE	RANK
Sample mean $\bar{y}_r$	0.366678	100		0.770024	100	
Ratio T_1^*	0.2402253	139.0393	13th	1.283373	60.00002	18th
Singh and Horn (2000) T_2^*	0.2797053	125.1053	16th	0.476681	161.5385	6th
Kadilar and Cingi (2008) T_3^*	0.5248172	78.81559	17th	1.246705	61.76472	17th
Singh (2009) T_4^*	32.79433	0.4637965	18th	0.733356	105	16th
Audu *et al*. (2021) T_5^*						
$T_{5(1)}^*$	0.2270429	161.5017	6th	0.2270429	161.5017	7th
$T_{5(2)}^*$	0.2297179	159.621	8th	0.2297179	159.621	9th
$T_{5(3)}^*$	0.2334238	157.0868	11th	0.2334238	157.0868	12th
$T_{5(4)}^*$	0.2230457	164.3959	5th	0.2230457	164.3959	5th
Musa *et al*. (2023) T_6^*						

$T^*_{6(1)}$	0.2391215	153.3438	12th	0.2391215	153.3438	13th
$T^*_{6(2)}$	0.2298595	159.5227	9th	0.2298595	159.5227	10th
$T^*_{6(3)}$	0.2313785	158.4754	7th	0.2313785	158.4754	8th
$T^*_{6(4)}$	0.2292677	159.9345	10th	0.2292677	159.9345	11th
Proposed Estimators $T_1^{*(d)}$						
$T^{*(d)}_{1(1)}$	0.1859372	197.2053	2nd	0.1859372	197.2053	2nd
$T^{*(d)}_{1(2)}$	0.1858529	197.2947	1st	0.1858529	197.2947	1st
$T^{*(d)}_{1(3)}$	0.211284	173.5475	4th	0.211284	173.5475	4th
$T^{*(d)}_{1(4)}$	0.19716	185.9799	3rd	0.19716	185.9799	3rd
$T^{*(d)}_{1(5)}$	0.2690445	136.289	15th	0.2690445	136.289	15th
$T^{*(d)}_{1(6)}$	0.2502166	146.5443	14th	0.2502166	146.5443	14th

COMMENT: Table 4.2 case I shows simulation results of the MSEs and percentage relative efficiency (PREs) of classes of proposed estimators $T^{(d)}_{i(j)},\ i=1,2\ \ j=1,2,3,...,6$ and that of some existing related estimators like Lee *et al.* (1994) T_1^*, Singh and Horn (2000) T_2^*, Kadilar and Cingi (2008) T_3^*, Singh (2009) T_4^*, Audu *et al.* (2021c) $T^*_{5(j)},\ j=1,2,3,4$ and Musa *et al.* (2023) $T^*_{6(j)},\ j=1,2,3,4$ estimators using population I obtained from exponential distribution under simple random sampling scheme in the presence of non-response on both the study and auxiliary variables for $N=1000,\ n_1=300,\ \ n=200,\ \ r=150$. The result revealed that the proposed estimators favourably compete and demonstrated high level of efficiency in comparison with the existing estimators. Using the ranks of the estimators, the proposed estimators $T^{(d)}_{1(2)}, T^{(d)}_{1(1)}, T^{(d)}_{1(4)}, T^{(d)}_{1(3)}$ emerged out of 18 competing estimators first, second, third, and fourth position respectively. Additionally, other proposed estimators performed better than all the existing estimators proposed by Lee *et al.* (1994) T_1^*, Singh and Horn (2000) T_2^*, Kadilar and Cingi (2008) T_3^*, Singh (2009) T_4^*, Audu *et al.* (2021c) $T^*_{5(j)},\ j=1,2,3,4$ and Musa *et al.* (2023) $T^*_{6(j)},\ j=1,2,3,4$. For both cases

Table 4.3: MSEs and PREs of Estimators $T^{(d)}_{i(j)}, T_1^*$, T_2^*, T_3^*, T_4^*, $T^*_{5(j)}$, and $T^*_{6(j)}$ when $N=1000,\ n_1=300,\ \ n=200,\ \ r=150$ using Population II

ESTIMATORS	CASE I			CASE II		
	MSE	PRE	RANK	MSE	PRE	RANK
Sample mean $\bar{y}_r$	3.737614	100		3.737614	100	
Ratio T_1^*	2.518663	148.3967	8th	2.518663	148.3967	8th

Singh and Horn (2000) T_2^*	2.518491	148.4069	7th	2.518491	148.4069	7th
Kadilar and Cingi (2008) T_3^*	3.657948	102.1779	17th	3.657948	102.1779	17th
Singh (2009) T_4^*	5556.424	0.06726654	18th	5556.424	0.06726654	18th
Audu *et al.* (2021) T_5^*						
$T_{5(1)}^*$	2.519341	148.3568	9th	2.519341	148.3568	9th
$T_{5(2)}^*$	2.520187	148.307	11th	2.520187	148.307	11th
$T_{5(3)}^*$	2.520445	148.2918	12th	2.520445	148.2918	12th
$T_{5(4)}^*$	2.519644	148.339	10th	2.519644	148.339	10th
Musa *et al.* (2023) T_6^*						
$T_{6(1)}^*$	2.539053	147.2051	14th	2.539053	147.2051	14th
$T_{6(2)}^*$	2.53989	147.1565	15th	2.53989	147.1565	15th
$T_{6(3)}^*$	2.549409	146.6071	16th	2.549409	146.6071	16th
$T_{6(4)}^*$	2.538546	147.2344	13th	2.538546	147.2344	13th
Proposed Estimators $T_1^{*(d)}$						
$T_{1(1)}^{*(d)}$	1.683007	222.0795	2nd	1.683007	222.0795	2nd
$T_{1(2)}^{*(d)}$	1.682992	222.0815	1st	1.682992	222.0815	1st
$T_{1(3)}^{*(d)}$	1.687228	221.524	4th	1.687228	221.524	4th
$T_{1(4)}^{*(d)}$	1.684961	221.8221	3rd	1.684961	221.8221	3rd
$T_{1(5)}^{*(d)}$	1.855337	201.4521	6th	1.855337	201.4521	6th
$T_{1(6)}^{*(d)}$	1.851549	201.8642	5th	1.851549	201.8642	5th

COMMENT: Table 4.3 case I shows simulation results of the MSEs and percentage relative efficiency (PREs) of classes of proposed estimators $T_{i(j)}^{(d)},\ i=1,2\quad j=1,2,3,...,6$ and that of some existing related estimators like Lee *et al.* (1994) T_1^*, Singh and Horn (2000) T_2^*, Kadilar and

Cingi (2008) T_3^*, Singh (2009) T_4^*, Audu *et al.* (2021c) $T_{5(j)}^*,\ j=1,2,3,4$ and Musa *et al.* (2023) $T_{6(j)}^*,\ j=1,2,3,4$ estimators using population II obtained from uniform distribution under simple random sampling scheme in the presence of non-response on both the study and auxiliary variables for $N=1000,\ n_1=300,\ n=200,\ r=150$. The result revealed that the proposed estimators favourably compete and demonstrated high level of efficiency in comparison with the existing estimators. Using the ranks of the estimators, the proposed estimators $T_{1(2)}^{(d)}, T_{1(1)}^{(d)}, T_{1(4)}^{(d)}, T_{1(3)}^{(d)} T_{1(6)}^{(d)} and T_{1(5)}^{(d)}$ emerged out of 18 competing estimators first, second, third, and fourth position, which performed better than all the existing estimators proposed by Lee *et al.* (1994) T_1^*, Singh and Horn (2000) T_2^*, Kadilar and Cingi (2008) T_3^*, Singh (2009) T_4^*, Audu *et al.* (2021c) $T_{5(j)}^*,\ j=1,2,3,4$ and Musa *et al.* (2023) $T_{6(j)}^*,\ j=1,2,3,4$. For both cases

Table 4.4: Populations used for Simulation Study for Negative Relationship

Populations	Auxiliary variable (x)	Study variable (y)
1	$X \sim exp(0.3)$	$Y=\frac{1}{X}+E$ $where\ \ E \sim absolute\left(Normal(0,1)\right)$
2	$X \sim uniform(10,100)$	

Table 4.5: MSEs and PREs of Estimators $T_{i(j)}^{(d)}, T_1^*$, T_2^*, T_3^*, T_4^*, $T_{5(j)}^*$, **and** $T_{6(j)}^*$ **when** $N=1000,\ n_1=300,\ n=200,\ r=150$ **using Population I**

ESTIMATORS	CASE I			CASE II		
	MSE	PRE	RANK	MSE	PRE	RANK
Sample mean $\bar{y}_r$	0.008576306	100		0.012007	100	
Ratio T_1^*	0.01230551	69.69484	17th	0.017228	69.69484	17th
Singh and Horn (2000) T_2^*	0.008499475	100.9039	15th	0.011899	100.9039	15th
Kadilar and Cingi (2008) T_3^*	0.01039553	82.49996	16th	0.01455374	82.49994	16th
Singh (2009) T_4^*	33.50364	0.02559814	18th	46.905096	0.025598	18th
Audu et al. (2021) T_5^*						
$T_{5(1)}^*$	0.008354977	102.6491	9th	0.0116970	102.6505	9th
$T_{5(2)}^*$	0.008418799	101.8709	14th	0.011786	101.8724	14th
$T_{5(3)}^*$	0.008353781	102.6638	8th	0.0116953	102.6652	8th
$T_{5(4)}^*$	0.008399005	102.111	12th	0.011759	102.1124	12th

Musa *et al.* (2023) T_6^*						
$T_{6(1)}^*$	0.008361747	101.5489	10th	0.011706	102.5674	10th
$T_{6(2)}^*$	0.008363969	102.5387	11th	0.01171	102.5402	11th
$T_{6(3)}^*$	0.008405533	102.0317	13th	0.011768	102.0331	13th
$T_{6(4)}^*$	0.008339501	102.8396	4th	0.011675	102.841	4th
Proposed Estimators $T_1^{*(d)}$						
$T_{1(1)}^{*(d)}$	0.00833068	102.9484	3rd	0.011663	102.9499	3rd
$T_{1(2)}^{*(d)}$	0.008330573	102.9498	2nd	0.011663	102.9512	2nd
$T_{1(3)}^{*(d)}$	0.008314143	103.1532	1st	0.01164	103.1547	1st
$T_{1(4)}^{*(d)}$	0.008344734	102.7751	5th	0.011683	102.7765	5th
$T_{1(5)}^{*(d)}$	0.008353767	102.6639	6th	0.0116953	102.6654	6th
$T_{1(6)}^{*(d)}$	0.00835376	102.664	7th	0.0116953	102.6655	7th

COMMENT: Table4. 5 shows simulation results of the MSEs and percentage relative efficiency (PREs) of classes of proposed estimators $T_{i(j)}^{(d)},\ i=1,2\ \ j=1,2,3,...,6$ and that of some existing related estimators like Lee *et al.* (1994) T_1^*, Singh and Horn (2000) T_2^*, Kadilar and Cingi (2008) T_3^*, Singh (2009) T_4^*, Audu *et al.* (2021c) $T_{5(j)}^*,\ j=1,2,3,4$ and Musa *et al.* (2023) $T_{6(j)}^*,\ j=1,2,3,4$ estimators using population I obtained from exponential distribution under simple random sampling scheme in the presence of non-response on both the study and auxiliary variables for $N=1000,\ n_1=300,\ \ n=200,\ \ r=150$. The result revealed that the proposed estimators favourably compete and demonstrated high level of efficiency in comparison with the existing estimators. Using the ranks of the estimators, the proposed estimators $T_{1(3)}^{(d)}, T_{1(2)}^{(d)}, T_{1(1)}^{(d)}, T_{1(4)}^{(d)}, T_{1(6)}^{(d)}a$ emerged out of 18 competing estimators first, second, third, fifth and sixth position. Which outperformed better than all the existing estimators proposed by Lee *et al.* (1994) T_1^*, Singh and Horn (2000) T_2^*, Kadilar and Cingi (2008) T_3^*, Singh (2009) T_4^*, Audu *et al.* (2021c) $T_{5(j)}^*,\ j=1,2,3,4$ and Musa *et al.* (2023) $T_{6(j)}^*,\ j=1,2,3,4$, for both cases.

Table 4.6: MSEs and PREs of Estimators $T_{i(j)}^{(d)}, T_1^*$, T_2^*, T_3^*, T_4^*, $T_{5(j)}^*$, **and** $T_{6(j)}^*$ **when** $N=1000,\ n_1=300,\ \ n=200,\ \ r=150$ **using Population II**

ESTIMATORS	CASE I			CASE II		
	MSE	PRE	RANK	MSE	PRE	RANK
Sample mean $\bar{y}_r$	0.7810891	100		1.249743	100	

Ratio T_1^*	0.9271915	84.24248	17th	1.483506	84.24248	17th
Singh and Horn (2000) T_2^*	0.7551311	103.4376	8th	1.20821	103.4376	8th
Kadilar and Cingi (2008) T_3^*	0.8317936	93.9042	16th	1.33087	93.9042	16th
Singh (2009) T_4^*	366.0589	0.213378	18th	585.6942	0.213378	18th
Audu *et al.* (2021) T_5^*						
$T_{5(1)}^*$	0.7753945	100.7344	13th	1.240631	100.734	13th
$T_{5(2)}^*$	0.7839908	99.62988	15th	1.254385	99.62988	15th
$T_{5(3)}^*$	0.7552587	103.4201	9th	1.208414	103.4201	9th
$T_{5(4)}^*$	0.7546341	103.5057	7th	1.207415	103.5057	7th
Musa *et al.* (2023) T_6^*						
$T_{6(1)}^*$	0.7731544	101.0263	12th	1.237047	101.0263	12th
$T_{6(2)}^*$	0.7713021	101.2689	10th	1.234083	101.2689	10th
$T_{6(3)}^*$	0.7713161	101.2671	11th	1.234106	101.2671	11th
$T_{6(4)}^*$	0.7812675	99.97717	14th	1.250028	99.97717	14th
Proposed Estimators $T_1^{*(d)}$						
$T_{1(1)}^{*(d)}$	0.7474839	104.4958	2nd	1.195974	104.4958	2nd
$T_{1(2)}^{*(d)}$	0.7472312	104.5311	1st	1.19557	104.5311	1st
$T_{1(3)}^{*(d)}$	0.7496923	104.188	4th	1.199508	104.188	4th
$T_{1(4)}^{*(d)}$	0.7490048	104.2836	3rd	1.198408	104.2836	3rd
$T_{1(5)}^{*(d)}$	0.7532302	103.6986	6th	1.205168	103.6986	6th
$T_{1(6)}^{*(d)}$	0.7503759	104.093	5th	1.200601	104.093	5th

COMMENT: Table 4.6 shows simulation results of the MSEs and percentage relative efficiency (PREs) of classes of proposed estimators $T_{i(j)}^{(d)},\ i=1,2\quad j=1,2,3,...,6$ and that of some existing related estimators like Lee *et al.* (1994) T_1^*, Singh and Horn (2000) T_2^*, Kadilar and Cingi (2008) T_3^*, Singh (2009) T_4^*, Audu *et al.* (2021c) $T_{5(j)}^*,\ j=1,2,3,4$ and Musa *et al.* (2023) $T_{6(j)}^*,\ j=1,2,3,4$ estimators using population II obtained from uniform distribution under simple random sampling scheme in the presence of non-response on both the study and auxiliary variables for $N=1000,\ n_1=300,\ n=200,\ r=150$. The result revealed that the proposed estimators favourably compete and demonstrated high level of efficiency in comparison with the existing estimators. Using the ranks of the estimators, the proposed estimators $T_{1(2)}^{(d)}, T_{1(1)}^{(d)}, T_{1(4)}^{(d)}, T_{1(3)}^{(d)} T_{1(6)}^{(d)} and T_{1(5)}^{(d)}$ emerged out of 18 competing estimators first, second, third, fourth, fifth and sixth position, which performed better than all the existing estimators proposed by Lee *et al.* (1994) T_1^*, Singh and Horn (2000) T_2^*, Kadilar and Cingi (2008) T_3^*, Singh (2009) T_4^*, Audu *et al.* (2021c) $T_{5(j)}^*,\ j=1,2,3,4$ and Musa *et al.* (2023) $T_{6(j)}^*,\ j=1,2,3,4$. For both cases

4.2 REAL LIFE DATA USED

(Kadilar and Cingi, 2008)

Y: Level of apple production**.**

X: Number of apple trees.

$$\mathrm{N}=19;\mathrm{n}=10;\mathrm{r}=8;\bar{\mathrm{Y}}=575;\bar{\mathrm{X}}=13537.68;\mathrm{S}_y=858.36;\mathrm{S}_x=12945.38;\rho_{yx}=0.88;$$

Table 4.7: Table: MSEs and PREs of Estimators $T_{i(j)}^{(d)},\ T_1^*,\ T_2^*,\ T_3^*,\ T_4^*,\ T_{5(j)}^*,$ **and** $T_{6(j)}^*$

ESTIMATORS	CASE I			CASE II		
	MSE	PRE	RANK	MSE	PRE	RANK
	n_1=13					
Sample mean $\bar{y}_r$	53319.74	100		53319.74	100	
Ratio T_1^*	40111.55	132.9287	17th	53319.67	100.0001	18th
Singh and Horn (2000) T_2^*	390451.04	135.1522	15th	28463.76	187.7325	11th
Kadilar and Cingi (2008) T_3^*	46613.82	114.3861	18th	29407.49	181.3135	17th
Singh (2009) T_4^*	37054.99	143.8936	16th	28483.96	187.1922	15th
Audu *et al.* (2021) Estimators T_5^*						
$T_{5(1)}^*$	39055.64	136.5225	7th	28363.76	187.9855	7th
$T_{5(2)}^*$	39084.73	136.4209	8th	28391.54	187.8015	9th
$T_{5(3)}^*$	39142.41	136.2199	9th	28425.71	187.5758	13th
$T_{5(4)}^*$	39196.72	136.0311	12th	28488.32	187.1635	16th

Musa *et al.* (2023) Estimators T_6^*						
$T_{6(1)}^*$	39155.64	136.1738	10th	28396.76	187.7670	10th
$T_{6(2)}^*$	39189.42	136.0565	11th	28402.03	187.7321	12th
$T_{6(3)}^*$	39206.73	135.9964	13th	28475.52	187.2476	14th
$T_{6(4)}^*$	39252.14	135.8391	14th	28517.33	186.9731	8th
Proposed Estimators $T_1^{*(d)}$						
$T_{1(1)}^{*(d)}$	25901.04	205.8595	2nd	22498.34	236.9941	2nd
$T_{1(2)}^{*(d)}$	25888.79	205.9569	1st	22405.68	237.9742	1st
$T_{1(3)}^{*(d)}$	25959.30	205.3974	3rd	22502.79	236.9472	3rd
$T_{1(4)}^{*(d)}$	25989.62	205.1578	4th	22564.64	236.2978	4th
$T_{1(5)}^{*(d)}$	26100.43	204.2868	5th	22592.13	236.0102	5th
$T_{1(6)}^{*(d)}$	26142.57	203.9575	6th	22631.05	235.6044	6th
n_1=15						
Sample mean $\bar{y}_r$	53319.74	100		53319.74	100	
Ratio T_1^*	40111.55	132.9287	17th	53319.67	100.0001	18th
Singh and Horn (2000) T_2^*	39451.04	135.1522	14th	28463.76	187.7325	10th
Kadilar and Cingi (2008) T_3^*	46613.82	114.3861	18th	29407.49	181.3135	17th
Singh (2009) T_4^*	37054.99	143.8936	7th	28363.76	187.9855	7th
Audu *et al.* (2021) Estimators T_5^*						
$T_{5(1)}^*$	39055.64	136.5225	8th	28363.76	187.9855	7th
$T_{5(2)}^*$	39096.23	136.3808	9th	28789.43	185.2050	14th
$T_{5(3)}^*$	39147.46	136.2023	10th	28823.75	184.9854	15th

$T^*_{5(4)}$	39188.71	136.0589	12th	28879.04	184.6313	16th
Musa *et al.* (2023) Estimators T^*_6						
$T^*_{6(1)}$	39155.64	136.1738	11th	28396.85	187.7670	9th
$T^*_{6(2)}$	39215.32	135.9666	13th	28415.63	187.6423	11th
$T^*_{6(3)}$	39651.05	134.4725	15th	28466.21	187.3089	12th
$T^*_{6(4)}$	39967.18	133.4088	16th	28533.78	186.8653	13th
Proposed Estimators $T_1^{*(d)}$						
$T^{*(d)}_{1(1)}$	20094.53	265.3445	2nd	18779.29	283.9284	2nd
$T^{*(d)}_{1(2)}$	20036.85	266.1084	1st	18736.07	284.5834	1st
$T^{*(d)}_{1(3)}$	20163.49	264.4371	3rd	18834.67	283.0936	3rd
$T^{*(d)}_{1(4)}$	20189.16	264.1008	4th	18886.22	282.3209	4th
$T^{*(d)}_{1(5)}$	20205.72	263.8844	5th	18939.84	281.5216	5th
$T^{*(d)}_{1(6)}$	20287.09	262.8260	6th	18971.63	281.0499	6th

COMMENT: Table 4.7 shows results of the MSEs and percentage relative efficiency (PREs) of classes of proposed estimators $T^{(d)}_{i(j)},\ i=1,2\ \ j=1,2,3,...,6$ and that of some existing related estimators like Lee *et al.* (1994) T_1^*, Singh and Horn (2000) T_2^*, Kadilar and Cingi (2008) T_3^*, Singh (2009) T_4^*, Audu *et al.* (2021c) $T^*_{5(j)},\ j=1,2,3,4$ and Musa *et al.* (2023) $T^*_{6(j)},\ j=1,2,3,4$ estimators using the population at different sample sizes.

The result under case I revealed that the proposed estimators favourably compete and demonstrated high level of efficiency in comparison with the existing estimators. Using the ranks of the estimators, the the proposed estimators $T^{(d)}_{1(2)}, T^{(d)}_{1(1)}, T^{(d)}_{1(3)}, T^{(d)}_{1(4)} T^{(d)}_{1(5)} and T^{(d)}_{1(6)}$ emerged out of 18 competing estimators first, second, third, fourth, fifth and sixth position, which performed better than all the existing estimators proposed by Lee *et al.* (1994) T_1^*, Singh and Horn (2000) T_2^*, Kadilar and Cingi (2008) T_3^*, Singh (2009) T_4^*, Audu *et al.* (2021c) $T^*_{5(j)},\ j=1,2,3,4$ and Musa *et al.* (2023) $T^*_{6(j)},\ j=1,2,3,4$. For both cases at different sample sizes of n=13 and n=15

4.3 Conclusion

From the results of the empirical studies, it was revealed that the estimators of proposed schemes compete favourably against the existing ones, which demonstrated high level of efficiency over the existing estimators considered in the study for both presence and non-presence of outliers Moreover, the proposed estimators, members of $T^{(d)}_{2(j)},\ j=1,2$ outperformed in numbers than others that are considered in the study for population I, II, IV and V while the proposed estimators, member of $T^{(d)}_{1(j)},\ j=1,2$ also performed better than others that are considered in the study population III. Results also revealed that the proposed estimators' scheme performed better for both positive and negative relationshin.

For the real life data both proposed Estimators performed equally, they can only be used as substitute for each other

References

Adamu, I.K., Adewara, A.A. & Audu, A. (2019). Modified product Estimator Under Two-Phase Sampling. *Global Journal of Pure and Applied Sciences,* 25, 201-208.

Ahmed, M.S., Al-Titi, O., Al-Rawi & Abu-Dayyeh, W. (2006). Estimation of a population mean using different Imputation Methods. *Transaction*, 7: 1247 -1264.

Al-Omari, A.I., Bouza, C.N. & Herrera, C. (2013). Imputation Methods of Missing Data for Estimating the Population Mean using Simple Random sampling with Known correlation Coefficient. *Quality and Quantity,* 47, 353-365.

Audu, A., Danbaba, A., Ahmad, S. K., Musa, N., Shehu, A., Ndatsu, A. M. & Joseph, A.O. (2021b). On the Efficient of Almost Unbiased Mean Imputation When the Population Mean of Auxiliary Variable is Unknown. *Asian Journal of Probability and Statistics,* 15(4): 235-250.

Audu, A., Singh, R. & Khare, S. (2021c):New Regression-Type Compromised Imputation Class of Estimators with known Parameters of Auxiliary Variable. *Communication in Statistics-Simulation and Computation*, 1-13.

Audu, A., Ishaq, O. O., Isah, U., Muhammed, S., Akintola, K. A., Rashida, A. & Abubakar,

A. (2020b). On the Class of Exponential-Type Imputation Estimators of Population Mean with Known Population Mean of Auxiliary Variable. *NIPES Journal of Science and Technology Research* 2(4), 1 – 11.

Bazad, R. & Bazad, P. (2019). Two phase sampling for estimation of population mean with auxiliary information using exponential type of estimator. *Journal for Statistcal Management System,* 22(5), 969-976.

Bhushan, S. & Gupta, R. (2019) Some new log type class of double sampling estimators.

Int. J. Agric. Stat. Sci., 14(1): 31-39.

Bhushan, S. & Kumar, A. (2023). New efficient class of estimators of population mean using two-phase sampling. *Int. J. Math. Oper. Res.* 24(2): 155-172

Diana, G. & Perri, P. F., (2010). Improved estimators of the population mean for missing data, *Communications in Statistics- Theory and Methods*, 39, 3245-3251.

Downton, F. (1966). Linear estimates with polynomial coefficients. Biometrika 53:129–141.

Hansen, M. N. & Hurwitz, W. N. (1949). The problem of non-response in sample surveys. *Journal of the American Statistical Association*, 41(236):517-529.

Kadilar, C. & Cingi, H. (2008). Ratio Estimators in simple random sampling, *Applied Mathematics and Computation*, 151, 893-902.

Kalita, D., Singh, B. K. & Choudhury, S. (2016). Exponential ratio-cum-exponential dual to ratio estimator in double sampling. *Proc. Math. Stat.*, 124: 211-220.

Kumar, M & Bahl, S. (2006). A class of dual to ratio estimator in double sampling. *Statistical Paper*, 47: 319-326.

Lee, H. Rancourt, E. & Sarndal, C. E. (1994): Experiments with variance Estimation from survey data with imputed values. *Journal of official Statistics,* 10, 231-243

Mishra, M., Singh, B. P. and Singh, R. (2017). Estimation of Population Mean Using two Auxiliary Variables in Stratified Random Sampling. *Journal of Reliability and Statistical Studies,* 10(1), 59-68.

Musa, N., Audu, A., Joseph, O. A., Muhammad, S., Shehu, A., Abubakar, I. & Yunusa, M. A (2023). On the Efficiency of Modified Regression-Type Mean Imputation Scheme under Two-Phase Sampling. *Caliphate Journal of Science and Technology*, 2, 88-97.

Nair, U.A. (1936). The Standard Error of Gini’s Mean Differnce Biometrika, 28(3-4) 428-436.

Neyman J. Contribution to the theory of sampling human populations. *Journal of American Statistical Association*, 1938; 33: 101-116.

Noor ul Amin, M. & Hanif, M. (2016). Some exponential estimators in survey sampling”, *Pakistan Journal of Statistics,* 28 (3); 367–374.

Ozgul, N. & Cingi, H. (2014). A new class of exponential regression cum ratio estimator in two phase sampling. *Hacettepe Journal of Mathematics and Statistics,* 43(1): 131- 140.

Prasad, S. (2018). A study on new methods of ratio exponential type imputation in sample surveys. *Hacettepe Journal of Mathematics and Statistics*, 47(5) 1281-1301

Singh, S. (2003): *Advanced Sampling Theory with Applications*, How Michael “selected” Amy. Kluwer Academic Publishers. The Netherlands, (2003).

Singh, B. (2011) A new ratio estimator for finite population mean in double sampling. *International Journal of Agricultural Statistical Science.* 7(2): 651-656.

Singh, H.P. & Vishwakarma, G.K. (2007): Modified exponential ratio and product estimators for finite population mean in double sampling. *Austrian Journal of Statistics*, 36, 3, 217- 225.

Singh, G. N., Maurya, S. Khetan, M. & Kadilar, C. (2016). Some imputation methods for missing data in sample surveys. *Hacettepe Journal of Mathematics and Statistics*, 45 (6), 1865-1880.

Singh, A. K., Singh, P. & Singh, V. K. (2014). Exponential-Type Compromised Imputation in Survey Sampling, *J. Stat. Appl*. 3 (2), 211-217.

Singh, S. & Deo, B. (2003). Imputation by power transformation. *Statistical Papers*, 44, 555-579.

Singh, S. & Horn, S. (2000). Compromised imputation in survey sampling. *Metrika*, 51, 267-276.

Singh, S., (2009). A new method of imputation in survey sampling. *Statistics*, 43, 499-511.

Singh, B.K. & U. Gogoi, (2018).Estimation of Population Mean using Ratio Cum Product Imputation Techniques in Sample Survey. *Res. Rev,:J.stat.*,7:38-49

Srivenkataramana T. (1980). A dual to ratio estimator in sample surveys. *Biometrik,*. 67: 199-204.

Toutenburg, H., Srivastava, V. K. & Shalabh, A. (2008). Imputation versus imputation of missing values through ratio method in sample surveys. *Statistical Papers*, 49, 237- 247.

Wang , L. & Wang, Q. (2006). Empirical Likelihood for Parametric model under Imputation for Missing Data. *Journal of statistics and Management systems*, 9(1), 1-13.