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An Approximation with Seji Transform for Some Non-Linear Delay Differential Equations

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ABSTRACT:

In this research paper the SEJI (Sadiq- Emad-Jinan) integral transform technique is applied to obtain a solution for non-linear delay differential equations (DDE).

Definitions of the transformations that have been applied in this paper was obtained in previous work, we intend to find an approximation for some non-linear delay differential equations (DDE). Some examples were introduced to prove that the transformations can be applied on the non-linear class of differential equations. Examples of several SEJI transformations was approached throughout this work.

Keywords: SEJI Integral transform, delay-differential equations (DDEs) with non-constant coefficients, non-linear delay differential equations, the inverse of SEJI integral transform, the SEJI integral transform applied on derivatives, the inverse SEJI integral technique for derivatives, Adomain polynomials.

1. INTRODUCTION

The transforms with integration techniques [1] such as Laplace, Fourier and Sudumu transforms [2]-[3], Elzaki, ZZ transform [4]-[8], Natural, cryptography logarithm function [13], and Aboodh transforms [9-11] can be applied to differential and integral equations, some exact solutions or close approximations for them were calculated using those transforms similar to the analytical way. The "SEJI integral complex transform" [12] was invented to find approximations for some delay differential equations (DDEs). This technique is related in the manner to the Laplace and Jaffari transforms. The SEJI transform ability to solve (ODEs) with variable coefficients is the main goal to approach. In this research paper, the SEJI transform is adopted to find a solution of non-linear delay differential equations with a solution so close to the exact solution and it becomes more accurate as the number of iterations increase, several SEJI transforms were evaluated and used to find out other transforms and their inverses during the work.

2. MAIN CONCEPTS

Definition [1]: Let h(x) be an integrable function defined for all $t \geq 0$, where $p(s) \neq 0$ and q(s) are complex parameter functions $\mathrm{Im}(q(s))<0$ (imaginary part) , i is a complex number; the "SEJI" integral transform, $H_g^c(s)$ of $h(\tau)$ via the formula:

$$T_g^c\{h(\tau);\, q(s)\} = H_g^c(s) = p(s)\int_{\tau=0}^{\infty} e^{-iq(s)x} h(\tau)\,d\tau \qquad \mathrm{Im}(q(s))<0 ,$$

Provided the integral exists for some complex with negative imaginary part parameter function q(s). [12]

Proposition [1]: let $h(\tau)$ be a real function with these features:

1. $h(\tau)$ is a piecewise continuous for every $0 < \tau < \tau_1$ $(\tau_1 > 0)$.
2. $h(\tau)$ is of exponential order, i.e., there exist α, M greater than 0, and $\tau_0 > 0 \ni$

$$e^{\alpha x}\left|h(\tau)\right| \prec \sin^{-1}\theta M \quad \text{for all } \tau > \tau_0 .$$

Then the SEJI integral transform exist $\forall$ $s > \alpha$. [12]

3. SEJI TECHNIQUES FOR SOME IMPORTANT ELEMENTARY FUNCTIONS [12]

In the following formulas of some important functions in the SEJI integral technique

1. $$\mathrm{T}_g^c\{1\} = \frac{-ip(s)}{q(s)} \qquad q(s) \succ 0$$

2. $$\mathrm{T}_g^c\{x^n\} = \frac{(-i)^{n+1} n!\, p(s)}{[q(s)]^{n+1}} \qquad q(s) \succ 0$$

3. $$\mathrm{T}_g^c\{e^{\alpha x}\} = -p(s)\left[\frac{\alpha}{\alpha^2 + (q(s))^2} + i\frac{q(s)}{\alpha^2 + (q(s))^2}\right], \qquad q(s) \succ \alpha$$

4. $$\mathrm{T}_g^c\{\sin(\alpha x)\} = \frac{-\alpha\, p(s)}{(q(s))^2 - \alpha^2}, \qquad q(s) \succ |\alpha|$$

5. $$\mathrm{T}_g^c\{\cos(\alpha x)\} = \frac{-ip(s)q(s)}{(q(s))^2 - \alpha^2}, \qquad q(s) \succ |\alpha|$$

6. $$\mathrm{T}_g^c\{\cosh(\alpha x)\} = \frac{-ip(s)q(s)}{(q(s))^2 + \alpha^2}, \qquad q(s) \succ 0$$

7. $$\mathrm{T}_g^c\{\sinh(\alpha x)\} = \frac{-\alpha\, p(s)}{(q(s))^2 + \alpha^2}, \qquad q(s) \succ 0$$

4. INVERSE OF SEJI TRANSFORM FOR SOME ESSENTIAL FUNCTIONS [12]

1. $$\mathrm{T}_g^{c-1}\left\{\frac{-ip(s)}{q(s)}\right\} = 1 \qquad q(s) \succ 0$$

2. $$\mathrm{T}_g^{c-1}\left\{\frac{(-i)^{n+1} n!\, p(s)}{[q(s)]^{n+1}}\right\} = \frac{x^n}{n!} \qquad n \succ 0$$

$$\mathrm{T}_g^{c-1}\left\{\frac{(-i)^{n+1} n!\, p(s)}{[q(s)]^{n+1}}\right\} = \frac{x^n}{\Gamma(n+1)} \qquad n \succ -1$$

3. $$\mathrm{T}_g^{c-1}\left\{-p(s)\left[\frac{\alpha}{\alpha^2 + (q(s))^2} + \frac{q(s)}{\alpha^2 + (q(s))^2}\right]\right\} = e^{\alpha x}, \qquad q(s) \succ \alpha$$

4. $$\mathrm{T}_g^{c-1}\left\{\frac{-\alpha\, p(s)}{(q(s))^2 - \alpha^2}\right\} = \sin(\alpha x), \qquad q(s) \succ |\alpha|$$

5. $$\mathrm{T}_g^{c-1}\left\{\frac{-ip(s)q(s)}{(q(s))^2 - \alpha^2}\right\} = \cos(\alpha x), \qquad q(s) \succ |\alpha|$$

6. $$\mathrm{T}_g^{c-1}\left\{\frac{-ip(s)q(s)}{\left(q(s)\right)^2+\alpha^2}\right\}=\cosh(\alpha x), \qquad q(s)\succ 0$$

7. $$\mathrm{T}_g^{c}\left\{\frac{-\alpha\, p(s)}{\left(q(s)\right)^2+\alpha^2}\right\}=\sinh(\alpha x), \qquad q(s)\succ 0$$

5. THE GENERAL COMPLEX INTEGRAL TECHNIQUE FOR DERIVATIVES [12]

Function h(τ) defined as the SEJI integral transform of $\mathrm{T}_g\{h(x)\}=H_g(s)$ then:

1. $$\mathrm{T}_g^{c}\left\{h'(x)\right\}=iq(s)H_g^c(s)-p(s)h(0)$$

2. $$\mathrm{T}_g^{c}\left\{h''(x)\right\}=\left(iq(s)\right)^2 H_g^c(s)-p(s)h'(0)-iq(s)p(s)h(0)$$

3. $$\mathrm{T}_g^{c}\left\{h'''(x)\right\}=\left(iq(s)\right)^3 H_g^c(s)-p(s)\left[h''(0)+iq(s)h'(0)+\left(iq(s)\right)^2 h(0)\right]$$

$$\mathrm{T}_g^{c}\left\{h^{(n)}(x)\right\}=\left(iq(s)\right)^n H_g^c(s)-p(s)\left[\sum_{J=1}^{n}\left(iq(s)\right)^{J-1}h^{(n-J)}(0)\right]$$

6. SOLVING NON-LINEAR (DDE) USING SEJI TRANSFORM METHOD COMBINED WITH ADOMIAN POLYNOMIALS

Consider the general non-linear ordinary differential equation (ODE)

$$\frac{d^n y(t)}{dt^n}+P(y)+\square\,(t-\tau)=g(t) \qquad n=1,2,3,...... \tag{0.1}$$

with initial conditions $y^k(0)=y_0^k$, P is a linear operator, □ non-linear operator and g(t) is continuous function.

Apply Seji transformation to both sides and represent Y and □ in a series expansion with Adomian polynomials for the non-linear part □ we get the following equations;

$$T_g^c\left[\frac{d^n y(t)}{dt^n}\right]+T_g^c\left[P(y)\right]+T_g^c\left[\square\,(t-\tau)\right]=T_g^c\left[g(t)\right] \tag{0.2}$$

$$T_g^c\left[\frac{d^n y(t)}{dt^n}\right]=\left(iq(s)\right)^n Y_g^c(s)-p(s)\left[\sum_{k=1}^{n}\left(iq(s)\right)^{k-1}y^{(n-k)}(0)\right] \tag{0.3}$$

Provided that;

$$T_g^c\left[P(y)\right]=T_g^c\left[P\left(\sum_{n=0}^{\infty}y_n\right)\right],$$

$$T_g^c\left[\square\,(t-\tau)\right]=T_g^c\left[P\left(\sum_{n=0}^{\infty}B_n\right)\right] \qquad B_i:\ \textit{Adomian polynomials}$$

$$\left(iq(s)\right)^n Y_g^c(s)-p(s)\left[\sum_{k=1}^{n}\left(iq(s)\right)^{k-1}y^{(n-k)}(0)\right]=-T_g^c\left[p\left(\sum_{n=0}^{\infty}y_n\right)\right]-T_g^c\left[\sum_{n=0}^{\infty}B_n\right]+T_g^c\left[g(t)\right]$$

We get;

$$T_g^c\left(\sum_{n=0}^{\infty} y_n\right) = p(s)\frac{\left[\sum_{k=1}^{n}\left(iq(s)\right)^{k-1} y^{(n-k)}(0)\right]}{\left(iq(s)\right)^n} - \frac{T_g^c\left[p\left(\sum_{n=0}^{\infty} y_n\right)\right]}{\left(iq(s)\right)^n} - \frac{T_g^c\left[\sum_{n=0}^{\infty} B_n\right]}{\left(iq(s)\right)^n} + \frac{T_g^c\left[g(t)\right]}{\left(iq(s)\right)^n} \quad (0.4)$$

Starting an iteration method to find $y_0, y_1, y_2, ...$; we obtain.

$$T_g^c\left(y_0\right) = p(s)\frac{\left[\sum_{k=1}^{n}\left(iq(s)\right)^{k-1} y^{(n-k)}(0)\right]}{\left(iq(s)\right)^n} + \frac{T_g^c\left[g(t)\right]}{\left(iq(s)\right)^n}$$

$$T_g^c\left(y_1\right) = -\frac{T_g^c\left[p\left(y_0\right)\right]}{\left(iq(s)\right)^n} - \frac{T_g^c\left[B_0\right]}{\left(iq(s)\right)^n}$$

$$T_g^c\left(y_2\right) = -\frac{T_g^c\left[p\left(y_1\right)\right]}{\left(iq(s)\right)^n} - \frac{T_g^c\left[B_1\right]}{\left(iq(s)\right)^n}$$

$$T_g^c\left(y_n\right) = -\frac{T_g^c\left[p\left(y_{n-1}\right)\right]}{\left(iq(s)\right)^n} - \frac{T_g^c\left[B_{n-1}\right]}{\left(iq(s)\right)^n} \qquad n \geq 1$$

7. ILLUSTRATIVE EXAMPLES

Example 1: non-linear (DDE)

$$\frac{dy(t)}{dt} = 1 + 2y^2\left(\frac{t}{2}\right) \qquad 0 \leq t \leq 1 \qquad y(0) = 0 \quad (0.5)$$

Applying SEJI transform to both sides

$$T_g^c\left\{\frac{dy}{dt}\right\} = T_g^c\left\{1\right\} + 2T_g^c\left\{y^2\left(\frac{t}{2}\right)\right\}$$

$$iq(s)T_g^c\left\{y(t)\right\} - p(s)y(0) = \frac{-ip(s)}{q(s)} + 2T_g^c\left\{y^2\left(\frac{t}{2}\right)\right\}$$

$$T_g^c\left\{y(t)\right\} = \frac{-ip(s)}{iq^2(s)} + 2\frac{T_g^c\left\{y^2\left(\frac{t}{2}\right)\right\}}{iq(s)} \quad (0.6)$$

Now, we use the inverse of SEJI transform technique ;

$$y(t) = T_g^{c-1}\left\{\frac{-p(s)}{q^2(s)}\right\} + 2T_g^{c-1}\left\{\frac{T_g^c\left\{y^2\left(\frac{t}{2}\right)\right\}}{iq(s)}\right\} \quad (0.7)$$

Starting the iteration calculations combined with Adomian polynomials for the non-linear part we have;

$$y_0(t) = T_g^{c-1}\left\{\frac{-p(s)}{q^2(s)}\right\} = t \qquad y_0(\frac{t}{2}) = \frac{t}{2}$$

But

$$y^2\left(\frac{t}{2}\right) = \sum_{n=0}^{\infty} B_n \qquad B_j : Adomian\ Polynomials$$

$$y_{n+1}(t) = 2T_g^{c-1}\left\{\frac{T_g^c\{B_n\}}{iq(s)}\right\} \qquad (0.8)$$

The Adomian polynomials formulas we will are;

$$B_0 = f(y_0)$$

$$B_1 = y_1 f'(y_0)$$

$$B_2 = y_2 f'(y_0) + \frac{1}{2} y_1^2 f''(y_0)$$

$$B_3 = y_3 f'(y_0) + y_1 y_2 f''(y_0) + \frac{1}{3!} y_1^3 f'''(y_0)$$

Apply them on $f(y) = y^2$ we have,

$$B_0 = f(y_0) = y_0^2\left(\frac{t}{2}\right) = \frac{t^2}{4}$$

$$B_1 = y_1\left(\frac{t}{2}\right) f'(y_0) = y_1\left(\frac{t}{2}\right) y_0\left(\frac{t}{2}\right)$$

$$B_2 = y_2\left(\frac{t}{2}\right) f'(y_0) + \frac{1}{2} y_1^2\left(\frac{t}{2}\right) f''(y_0)$$

$$B_3 = y_3\left(\frac{t}{2}\right) f'(y_0) + y_1\left(\frac{t}{2}\right) y_2\left(\frac{t}{2}\right) f''(y_0) + \frac{y_1^3}{3!} f'''\left(y_0\right)$$

So using those equations for our non-linear function $f(y) = y^2$

$$B_0 = f(y_0) = y_0^2\left(\frac{t}{2}\right) = \frac{t^2}{4}$$

$$B_1 = y_1\left(\frac{t}{2}\right)\left(y_0^2\left(\frac{t}{2}\right)\right)' = 2y_1\left(\frac{t}{2}\right) y_0\left(\frac{t}{2}\right)$$

$$B_2 = y_2\left(\frac{t}{2}\right)\left(y_0^2\left(\frac{t}{2}\right)\right)' + \frac{1}{2} y_1^2\left(\frac{t}{2}\right)\left(y_0^2\right)''$$

$$B_3 = y_3\left(\frac{t}{2}\right)\left(y_0^2\right)' + y_1\left(\frac{t}{2}\right) y_2\left(\frac{t}{2}\right)\left(y_0^2\right)'' + \frac{y_1^3}{3!}\left(y_0^2\right)'''$$

Now starting to evaluate the values of $y_0, y_1, y_2, \ldots\ldots$ using SEJI and the inverse of SEJI we get,

$$y_0(t) = t \qquad y_0\left(\frac{t}{2}\right) = \frac{t}{2}$$

$$y_{n+1} = 2T_g^{c-1}\left\{\frac{T_g^c\left[B_n\right]}{iq(s)}\right\} \qquad n \geq 1 \tag{0.9}$$

$$B_0 = y_0^2\left(\frac{t}{2}\right) = \frac{t^2}{4}$$

$$y_1 = 2T_g^{c-1}\left\{\frac{T_g^c\left[\frac{t^2}{4}\right]}{iq(s)}\right\} \qquad n \geq 1$$

$$y_1 = 2T_g^{c-1}\left\{\frac{(-i)^3\, 2!P(s)}{4\left(q(s)\right)^3 iq(s)}\right\}$$

$$y_1 = T_g^{c-1}\left\{\frac{P(s)}{\left(q(s)\right)^4}\right\} = \frac{t^3}{3!} \qquad n \geq 1$$

$$y_1(t) = \frac{t^3}{3!} \qquad y_1\left(\frac{t}{2}\right) = \frac{t^3}{48} \tag{0.10}$$

$$B_1 = 2y_1y_0 = 2\frac{t^3}{48}\frac{t}{2} = \frac{t^4}{48} \tag{0.11}$$

Now we calculate, y_2

$$y_2 = 2T_g^{c-1}\left\{\frac{T_g^c\left\{B_1\right\}}{iq(s)}\right\} = 2T_g^{c-1}\left\{\frac{T_g^c\left\{\frac{t^4}{48}\right\}}{iq(s)}\right\} = 2T_g^{c-1}\left\{\frac{(-i)^5\, 4!P(s)}{48\left(q(s)\right)^5 iq(s)}\right\}$$

$$y_2 = 2T_g^{c-1}\left\{\frac{T_g^c\left\{B_1\right\}}{iq(s)}\right\} = 2T_g^{c-1}\left\{\frac{-p(s)}{q^6(s)}\right\} = \frac{t^5}{5!} \qquad y_2\left\{\frac{t}{2}\right\} = \frac{t^5}{2^5 \times 5!} \tag{0.12}$$

$$B_2 = 2y_2y_0 + y_1^2 = 2\frac{t^5}{2^5 \times 5!}\frac{t}{2} + \left(\frac{t^3}{48}\right)^2 = \frac{t^6}{1440} \tag{0.13}$$

Now we calculate, y_3

$$y_3 = 2T_g^{c-1}\left\{\frac{T_g^c\{B_2\}}{iq(s)}\right\} = 2T_g^{c-1}\left\{\frac{T_g^c\left\{\frac{t^6}{1440}\right\}}{iq(s)}\right\} = \frac{1}{720}T_g^{c-1}\left\{\frac{(-i)^7\,6!P(s)}{(q(s))^5\,iq(s)}\right\} = \frac{t^7}{7!} \quad (0.14)$$

$$y_3\left(\frac{t}{2}\right) = \frac{t^7}{2^7\times 7!} \quad (0.15)$$

$y = y_0 + y_1 + +y_2 + y_3 + ...$ and hence the exact solution is:

$$y = \frac{t}{2} + \frac{t^3}{2^3.3!} + \frac{t^5}{2^5.5!} + \frac{t^7}{2^7.7!} + ... = \sinh\left(\frac{t}{2}\right)$$

Example 2: linear (DDE) with exponential coefficient

Similarly to the previous example we apply SEJI transformation and the inverse SEJI transformation to the DDE;

$$\frac{dy(t)}{dt} = e^t y\left(\frac{t}{2}\right) + y(t) \qquad 0 \le t \le 1 \qquad y(0) = 1 \quad (0.16)$$

$$T_g^c\left\{\frac{dy}{dt}\right\} = T_g^c\left\{e^t y\left(\frac{t}{2}\right) + y(t)\right\} \quad (0.17)$$

$$iq(s)T_g^c\{y(t)\} - p(s)y(0) = T_g^c\left\{e^t y\left(\frac{t}{2}\right)\right\} + T_g^c\{y(t)\} \quad (0.18)$$

$$T_g^c\{y(t)\} = \frac{p(s)}{iq(s)} + \frac{T_g^c\left\{e^t y\left(\frac{t}{2}\right)\right\}}{iq(s)} + \frac{T_g^c\{y(t)\}}{iq(s)} \quad (0.19)$$

$$T_g^c\{y_0(t)\} = \frac{p(s)}{iq(s)} = \frac{-ip(s)}{q(s)} \qquad y_0(t) = 1 \qquad q(s) \succ 0 \quad (0.20)$$

Applying the SEJI and the inverse SEJI transformation we have;

$$T_g^c\{y_{n+1}(t)\} = \frac{T_g^c\left\{e^t y_n\left(\frac{t}{2}\right)\right\}}{iq(s)} + \frac{T_g^c\{y_n(t)\}}{iq(s)} \quad (0.21)$$

$$T_g^c\{y_1(t)\} = \frac{T_g^c\left\{e^t y_0\left(\frac{t}{2}\right)\right\}}{iq(s)} + \frac{T_g^c\{y_0(t)\}}{iq(s)} \quad (0.22)$$

$$T_g^c\{y_1(t)\} = \frac{T_g^c\{e^t\}}{iq(s)} + \frac{T_g^c\{1\}}{iq(s)} \quad (0.23)$$

$$T_g^c\{y_1(t)\} = -p(s)\left[\frac{\frac{1}{1+q^2(s)} + i\frac{q(s)}{1+q^2(s)}}{iq(s)} + \frac{-i\frac{p(s)}{q(s)}}{iq(s)}\right] \quad (0.24)$$

$$T_g^c\left\{y_1(t)\right\}=-\frac{p(s)}{iq(s)}\left[\frac{1}{1+q^2(s)}+i\frac{q(s)}{1+q^2(s)}\right]-\frac{p(s)}{q^2(s)} \quad (0.25)$$

We use here the series expansion to evaluate the terms so we obtain the following;

$$\frac{-P(s)}{iq(s)\left(1+q^2(s)\right)}=\frac{-p(s)}{iq^3(s)}\left[\frac{1}{\frac{1}{q^2(s)}+1}\right]=\frac{-p(s)}{iq^3(s)}\sum_{n=0}^{\infty}\left(-\frac{1}{q^2(s)}\right)^n=-\frac{1}{i}\sum_{n=0}^{\infty}\frac{P(s)}{(-1)^n q^{2n+3}(s)}$$

$$=-\frac{1}{i}\sum_{n=0}^{\infty}\frac{(-i)^{2n+3}P(s)}{(-1)^n(-i)^{2n+3}q^{2n+3}(s)} \quad (0.26)$$

$$T_g^{c-1}\left[\frac{-P(s)}{iq(s)\left(1+q^2(s)\right)}\right]=\sum_{n=1}^{\infty}\frac{t^{2n}}{(2n)!}=\cosh(t)-1 \quad (0.27)$$

$$\frac{-p(s)}{iq(s)}\left[\frac{iq(s)}{1+q^2(s)}\right]=\frac{-p(s)}{1+q^2(s)}=\frac{-p(s)}{q^2(s)}\left[\frac{1}{\frac{1}{q^2(s)}+1}\right]$$

$$=\frac{-p(s)}{q^2(s)}\sum_{n=0}^{\infty}\left(-\frac{1}{q^2(s)}\right)^n=\sum_{n=0}^{\infty}\frac{-p(s)}{(-1)^n q^{2n+2}(s)}$$

$$=\sum_{n=0}^{\infty}\frac{-p(s)(-i)^{2n+2}}{(-1)^n(-i)^{2n+2}q^{2n+2}(s)} \quad (0.28)$$

$$T_g^{c-1}\left[\frac{-p(s)}{iq(s)}\left[\frac{iq(s)}{1+q^2(s)}\right]\right]=\sum_{n=0}^{\infty}\frac{t^{2n+1}}{(2n+1)!}=\sinh(t) \quad (0.29)$$

$$T_g^{c-1}\left[\frac{-p(s)}{q^2(s)}\right]=t \quad (0.30)$$

$$T_g^{c-1}\left\{T_g^c\left\{y_1(t)\right\}\right\}=y_1=\cosh(t)-1+\sinh(t)+t=e^t+t-1 \quad (0.31)$$

$$y_1(t)=e^t-1+t \qquad y_1\left(\frac{t}{2}\right)=e^{\frac{t}{2}}-1+\frac{t}{2} \quad (0.32)\text{y}$$

Now we start calculating y_2;

$$T_g^c\left\{y_2(t)\right\}=\frac{T_g^c\left\{e^t y_1(\frac{t}{2})\right\}}{iq(s)}+\frac{T_g^c\left\{y_1(t)\right\}}{iq(s)} \quad (0.33)$$

$$T_g^c\left\{y_2\right\}=\frac{1}{iq(s)}T_g^c\left\{e^t\left(e^{\frac{t}{2}}-1+\frac{t}{2}\right)\right\}+\frac{1}{iq(s)}T_g^c\left\{e^t+t-1\right\} \quad (0.34)$$

$$T_g^c\left\{y_2(t)\right\}=-\frac{p(s)}{iq(s)}\left[\frac{\frac{3}{2}}{\left(\frac{3}{2}\right)^2+q^2(s)}+i\frac{q(s)}{\left(\frac{3}{2}\right)^2+q^2(s)}\right]-\frac{(-p(s))}{iq(s)}\left[\frac{1}{1+q^2(s)}+i\frac{q(s)}{1+q^2(s)}\right]$$

$$+\frac{1}{2iq(s)}T_g^c\left\{te^t\right\}-\frac{p(s)}{iq(s)}\left[\frac{1}{1+q^2(s)}+i\frac{q(s)}{1+q^2(s)}\right]+\frac{1}{iq(s)}\left[\frac{-p(s)}{q^2(s)}\right]-\frac{1}{iq(s)}\left[\frac{-ip(s)}{q(s)}\right]$$

(0.35)

We use here the series expansion to evaluate the terms so we obtain the following;

$$\frac{-p(s)}{iq(s)}\frac{\frac{3}{2}}{\left(\frac{3}{2}\right)^2+q^2(s)}=\frac{-p(s)}{iq(s)}\frac{\frac{3}{2}}{\left(\frac{3}{2}\right)^2\left[1+\frac{q^2(s)}{\left(\frac{3}{2}\right)^2}\right]}$$

$$=\frac{-p(s)}{iq(s)}\frac{\frac{3}{2}}{q^2(s)\left[1+\frac{1}{\frac{q^2(s)}{\left(\frac{3}{2}\right)^2}}\right]}=\frac{-p(s)}{iq(s)}\frac{\frac{3}{2}}{q^2(s)}\sum_{n=0}^{\infty}\left(\frac{-1}{\frac{q^2(s)}{\left(\frac{3}{2}\right)^2}}\right)^n \quad (0.36)$$

$$=\frac{-p(s)}{iq(s)}\frac{\frac{3}{2}}{q^2(s)}\sum_{n=0}^{\infty}\left(\frac{-1}{\frac{q^2(s)}{\left(\frac{3}{2}\right)^2}}\right)^n=\frac{-p(s)}{iq(s)}\frac{\frac{3}{2}}{q^2(s)}\sum_{n=0}^{\infty}\frac{(-1)^n\left(\frac{3}{2}\right)^{2n}}{q^{2n}(s)}=\sum_{n=0}^{\infty}\frac{p(s)(-1)^{n+1}\left(\frac{3}{2}\right)^{2n+1}}{iq^{2n+3}(s)}$$

$$=\sum_{n=0}^{\infty}\frac{p(s)(-1)^{n+1}(-i)^{2n+3}\left(\frac{3}{2}\right)^{2n+1}}{iq^{2n+3}(s)(-i)^{2n+3}}$$

(0.37)

$$T_g^{c-1}\left[\sum_{n=0}^{\infty}\frac{p(s)(-1)^{n+1}(-i)^{2n+3}\left(\frac{3}{2}\right)^{2n+1}}{iq^{2n+3}(s)(-i)^{2n+3}}\right]=\sum_{n=0}^{\infty}\frac{t^{2n+2}\left(\frac{3}{2}\right)^{2n+1}}{(2n+2)!}$$

$$=\sum_{n=0}^{\infty}\frac{t^{2n+2}\left(\frac{3}{2}\right)^{2n+1}}{(2n+2)!}=\sum_{n=1}^{\infty}\frac{t^{2n}\left(\frac{3}{2}\right)^{2n-1}}{2n!}=\frac{2}{3}\sum_{n=1}^{\infty}\frac{t^{2n}\left(\frac{3}{2}\right)^{2n}}{2n!}=\frac{2}{3}\left[\cosh\left(\frac{3t}{2}\right)-1\right] \quad (0.38)$$

$$\left(\frac{-p(s)}{iq(s)}\right)\left(\frac{iq(s)}{\left(\frac{3}{2}\right)^{2}+q^{2}(s)}\right)=\frac{-p(s)}{q^{2}(s)}\sum_{n=0}^{\infty}\left(\frac{1}{\frac{\frac{1}{q^{2}(s)}}{\left(\frac{3}{2}\right)^{2}}+1}\right)^{n}$$

$$=\frac{-p(s)}{q^{2}(s)}\sum_{n=0}^{\infty}\left(-\frac{\left(\frac{3}{2}\right)^{2}}{q^{2}(s)}\right)^{n}=\sum_{n=0}^{\infty}\frac{-p(s)(-1)^{n}\left(\frac{3}{2}\right)^{2n}}{q^{2n+2}(s)} \quad (0.39)$$

$$T_g^{c-1}\left[\sum_{n=0}^{\infty}\frac{-p(s)(-1)^{n}\left(\frac{3}{2}\right)^{2n}}{q^{2n+2}(s)}\right]=T_g^{c-1}\left[\sum_{n=0}^{\infty}\frac{-p(s)(-1)^{n}(-i)^{2n+2}\left(\frac{3}{2}\right)^{2n}}{q^{2n+2}(s)(-i)^{2n+2}}\right]$$

$$=\frac{2}{3}\sum_{n=0}^{\infty}\frac{t^{2n+1}\left(\frac{3}{2}\right)^{2n+1}}{(2n+1)!}=\frac{2}{3}\sinh\left(\frac{3t}{2}\right) \quad (0.40)$$

$$T_g^{c-1}\left[\frac{-p(s)}{iq(s)}\left[\frac{\frac{3}{2}}{\left(\frac{3}{2}\right)^{2}+q^{2}(s)}+i\frac{q(s)}{\left(\frac{3}{2}\right)^{2}+q^{2}(s)}\right]\right]=\frac{2}{3}\cosh\left(\frac{3t}{2}\right)+\frac{2}{3}\sinh\left(\frac{3t}{2}\right)=\frac{2}{3}\left[e^{\frac{3}{2}t}-1\right] \quad (0.41)$$

We need to calculate $T_g^{c}\left\{te^{t}\right\}$

$$T_g^c\left\{te^t\right\}=p(s)\int_0^\infty e^{-iq(s)t}te^t dt=p(s)\int_0^\infty te^t e^{-iq(s)t}dt=p(s)\int_0^\infty te^{-(iq(s)-1)t}tdt$$

$$=p(s)\left[\left.\frac{te^{-(iq(s)-1)t}}{-(iq(s)-1)}\right|_0^\infty-\int_0^\infty\frac{e^{-(iq(s)-1)t}}{-(iq(s)-1)}dt\right]$$

$$=p(s)\left[\left.\frac{e^{-(iq(s)-1)t}}{-(iq(s)-1)^2}\right|_0^\infty\right]=\frac{p(s)}{(iq(s)-1)^2}$$

Note;

$$\frac{d}{dq}\left(\frac{1}{iq(s)-1}\right)=\frac{d}{dq}\left(\frac{1}{iq(s)\left(1-\frac{1}{iq(s)}\right)}\right)=\frac{d}{dq}\left(\frac{1}{iq(s)}\sum_{n=0}^{\infty}\left(\frac{1}{iq(s)}\right)^n\right)$$

This imply

$$\frac{-i}{(iq(s)-1)^2}=\frac{-i}{(iq(s))^3}\sum_{n=1}^{\infty}n\left(\frac{1}{iq(s)}\right)^{n-1}+\frac{-i}{(iq(s))^2}\sum_{n=0}^{\infty}\left(\frac{1}{iq(s)}\right)^n$$

$$\frac{1}{(iq(s)-1)^2}=\sum_{n=1}^{\infty}n\frac{1}{(i)^{n+2}q^{n+2}(s)}+\sum_{n=0}^{\infty}\frac{1}{(i)^{n+2}q^{n+2}(s)}$$

$$T_g^{c-1}\left[\frac{p(s)}{iq(s)(iq(s)-1)^2}\right]=T_g^{c-1}\left[\sum_{n=1}^{\infty}n\frac{P(s)}{(i)^{n+3}q^{n+3}(s)}+\sum_{n=0}^{\infty}\frac{p(s)}{(i)^{n+3}q^{n+3}(s)}\right]$$

$$T_g^{c-1}\left[\frac{1}{iq(s)(iq(s)-1)^2}\right]=T_g^{c-1}\left[\sum_{n=1}^{\infty}n\frac{p(s)(-i)^{n+3}}{(i)^{n+3}(-i)^{n+3}q^{n+3}(s)}+\sum_{n=0}^{\infty}\frac{p(s)(-i)^{n+3}}{(i)^{n+3}(-i)^{n+3}q^{n+3}(s)}\right]$$

$$T_g^{c-1}\left[\frac{1}{iq(s)(iq(s)-1)^2}\right]=\sum_{n=1}^{\infty}n\frac{t^{n+2}}{(n+2)!}+\sum_{n=0}^{\infty}\frac{t^{n+2}}{(n+2)!}=\sum_{n=3}^{\infty}(n-2)\frac{t^n}{n!}+\sum_{n=2}^{\infty}\frac{t^n}{n!}=te^t-e^t+1$$

$$T_g^{c-1}\left[\frac{T_g^c\left\{te^t\right\}}{iq(s)}\right]=te^t-e^t+1 \qquad (0.42)$$

$$T_g^{c-1}T_g^c\left[\frac{e^t}{iq(s)}\right]=\sum_{n=1}^{\infty}\frac{t^n}{n!}=e^t-1 \qquad (0.43)$$

$$T_g^{c-1}\left[\frac{T_g^c\{t\}}{iq(s)}\right]=T_g^{c-1}\left[\frac{-p(s)}{iq^3(s)}\right]=T_g^{c-1}\left[\frac{ip(s)}{q^3(s)}\right]=\frac{t^2}{2!} \qquad (0.44)$$

$$T_g^{c-1}\left[\frac{T_g^c\{1\}}{iq(s)}\right]=T_g^{c-1}\left[\frac{1}{iq(s)}\left[\frac{-ip(s)}{q(s)}\right]\right]=t \qquad (0.45)$$

Hence adding up (3.27-3.30) we obtain;

$$y_2(t)=\frac{2}{3}\left[e^{\frac{3t}{2}}-1\right]+\frac{1}{2}\left(te^t-e^t+1\right)+\frac{t^2}{2}-t \quad (0.46)$$

$$y_2(t)=\frac{2}{3}\left[e^{\frac{3t}{2}}-1\right]+\frac{1}{2}\left(te^t-e^t+1\right)+\frac{t^2}{2}-t$$

$$y_2(\frac{t}{2})=\frac{2}{3}e^{\frac{3t}{4}}-\frac{2}{3}+\frac{1}{2}\left(\frac{t}{2}e^{\frac{t}{2}}-e^{\frac{t}{2}}+1\right)+\frac{t^2}{8}-\frac{t}{2}$$

Now we calculate y_3 ;

$$T_g^c\left\{y_3(t)\right\}=\frac{T_g^c\left\{e^t y_2(\frac{t}{2})\right\}}{iq(s)}+\frac{T_g^c\left\{y_2(t)\right\}}{iq(s)}$$

$$T_g^c\left\{y_3(t)\right\}=\frac{T_g^c\left[e^t\left[\frac{2}{3}e^{\frac{3t}{4}}-\frac{2}{3}+\frac{1}{2}\left(\frac{t}{2}e^{\frac{t}{2}}-e^{\frac{t}{2}}+1\right)+\frac{t^2}{8}-\frac{t}{2}\right]\right]}{iq(s)}$$

$$+\frac{T_g^c\left[\frac{2}{3}e^{\frac{3t}{2}}-\frac{2}{3}+\frac{1}{2}\left(te^t-e^t+1\right)+\frac{t^2}{2}-t\right]}{iq(s)}$$

$$T_g^c\left\{y_3(t)\right\}=\frac{T_g^c\left[\frac{2}{3}e^{\frac{7t}{4}}-\frac{2}{3}e^t+\frac{1}{2}\left(\frac{t}{2}e^{\frac{3t}{2}}-e^{\frac{3t}{2}}+e^t\right)+\frac{t^2}{8}e^t-\frac{t}{2}e^t\right]}{iq(s)}$$

$$+\frac{T_g^c\left[\frac{2}{3}e^{\frac{3t}{2}}-\frac{2}{3}+\frac{1}{2}\left(te^t-e^t+1\right)+\frac{t^2}{2}-t\right]}{iq(s)}$$

$$T_g^c\left\{y_3(t)\right\}=\frac{T_g^c\left[\frac{2}{3}e^{\frac{7t}{4}}+\frac{1}{6}e^{\frac{3t}{2}}+\frac{t}{4}e^{\frac{3t}{2}}+\frac{t^2}{8}e^t-\frac{2}{3}e^t+\frac{t^2}{2}-t-\frac{1}{6}\right]}{iq(s)}$$

$$T_g^c\left\{y_3(t)\right\}=\frac{\frac{2}{3}T_g^c\left[e^{\frac{7t}{4}}\right]+\frac{1}{6}T_g^c\left[e^{\frac{3t}{2}}\right]+\frac{1}{4}T_g^c\left[te^{\frac{3t}{2}}\right]+\frac{1}{8}T_g^c\left[t^2e^t\right]-\frac{2}{3}T_g^c\left[e^t\right]+\frac{1}{2}T_g^c\left[t^2\right]-T_g^c\left[t\right]+\frac{1}{2}T_g^c\left[1\right]}{iq(s)}$$

$$T_g^c\left\{y_3(t)\right\}=\frac{\frac{-2p(s)}{3}\left[\frac{\frac{7}{4}}{\left(\frac{7}{4}\right)^2+q^2(s)}+i\frac{q(s)}{\left(\frac{7}{4}\right)^2+q^2(s)}\right]+\frac{-p(s)}{6}\left[\frac{\frac{3}{2}}{\left(\frac{3}{2}\right)^2+q^2(s)}+i\frac{q(s)}{\left(\frac{3}{2}\right)^2+q^2(s)}\right]}{iq(s)}+$$
$$+\frac{+\frac{1}{4}T_g^c\left[te^{\frac{3t}{2}}\right]+\frac{1}{8}T_g^c\left[t^2e^t\right]-\frac{2}{3}\left[\frac{1}{1+q^2(s)}+i\frac{q(s)}{1+q^2(s)}\right]+\frac{1}{2}\left[\frac{6ip(s)}{q^3(s)}\right]-\left[\frac{p(s)}{q^2(s)}\right]+\frac{1}{2}T_g^c\left[\frac{-ip(s)}{q(s)}\right]}{iq(s)}$$

(0.47)

There are several transforms we need to evaluate,

$$T_g^c\left\{te^{\frac{3t}{2}}\right\}=p(s)\int_0^\infty e^{-iq(s)t}te^{\frac{3t}{2}}dt=p(s)\int_0^\infty te^{\frac{3t}{2}}e^{-iq(s)t}dt=p(s)\int_0^\infty te^{-\left(iq(s)-\frac{3}{2}\right)t}tdt$$

$$T_g^c\left\{te^{\frac{3t}{2}}\right\}=p(s)\left[\left.\frac{te^{-\left(iq(s)-\frac{3}{2}\right)t}}{-\left(iq(s)-\frac{3}{2}\right)}\right|_0^\infty-\int_0^\infty\frac{e^{-\left(iq(s)-\frac{3}{2}\right)t}}{-\left(iq(s)-\frac{3}{2}\right)}dt\right]=p(s)\left[\left.\frac{e^{-\left(iq(s)-\frac{3}{2}\right)t}}{-\left(iq(s)-\frac{3}{2}\right)^2}\right|_0^\infty\right]=\frac{p(s)}{\left(iq(s)-\frac{3}{2}\right)^2}$$

(0.48)

Note;

$$\frac{d}{dq}\left(\frac{1}{iq(s)-\frac{3}{2}}\right)=\frac{d}{dq}\left(\frac{1}{iq(s)\left(1-\frac{\frac{3}{2}}{iq(s)}\right)}\right)=\frac{d}{dq}\left(\frac{1}{iq(s)}\sum_{n=0}^{\infty}\left(\frac{\frac{3}{2}}{iq(s)}\right)^n\right)$$

This imply

$$\frac{-i}{\left(iq(s)-\frac{3}{2}\right)^2}=\frac{-\frac{3}{2}i}{\left(iq(s)\right)^3}\sum_{n=1}^{\infty}n\left(\frac{\frac{3}{2}}{iq(s)}\right)^{n-1}+\frac{-i}{\left(iq(s)\right)^2}\sum_{n=0}^{\infty}\left(\frac{\frac{3}{2}}{iq(s)}\right)^n$$

$$\frac{1}{\left(iq(s)-\frac{3}{2}\right)^2}=\sum_{n=1}^{\infty}n\frac{\left(\frac{3}{2}\right)^n}{(i)^{n+2}q^{n+2}(s)}+\sum_{n=0}^{\infty}\frac{\left(\frac{3}{2}\right)^n}{(i)^{n+2}q^{n+2}(s)}$$

$$T_g^{c-1}\left[\frac{p(s)}{iq(s)\left(iq(s)-\frac{3}{2}\right)^2}\right]=T_g^{c-1}\left[\sum_{n=1}^{\infty}n\frac{P(s)\left(\frac{3}{2}\right)^n}{(i)^{n+3}q^{n+3}(s)}+\sum_{n=0}^{\infty}\frac{p(s)\left(\frac{3}{2}\right)^n}{(i)^{n+3}q^{n+3}(s)}\right]$$

$$T_g^{c-1}\left[\frac{1}{iq(s)\left(iq(s)-\frac{3}{2}\right)^2}\right]=T_g^{c-1}\left[\sum_{n=1}^{\infty}n\frac{p(s)\left(\frac{3}{2}\right)^n(-i)^{n+3}}{(i)^{n+3}(-i)^{n+3}q^{n+3}(s)}+\sum_{n=0}^{\infty}\frac{p(s)\left(\frac{3}{2}\right)^n(-i)^{n+3}}{(i)^{n+3}(-i)^{n+3}q^{n+3}(s)}\right]$$

$$T_g^{c-1}\left[\frac{1}{iq(s)\left(iq(s)-\frac{3}{2}\right)^2}\right]=\sum_{n=3}^{\infty}\frac{\left(\frac{3}{2}\right)^{n-2}t^n}{(n-1)!}-2\sum_{n=3}^{\infty}\frac{\left(\frac{3}{2}\right)^{n-2}t^n}{n!}+\sum_{n=2}^{\infty}\frac{t^n\left(\frac{3}{2}\right)^{n-2}}{n!}$$

$$T_g^{c-1}\left[\frac{1}{iq(s)\left(iq(s)-\frac{3}{2}\right)^2}\right]=\frac{2}{3}t\sum_{n=3}^{\infty}\frac{\left(\frac{3}{2}\right)^{n-1}t^{n-1}}{(n-1)!}-2\left(\frac{2}{3}\right)^2\sum_{n=3}^{\infty}\frac{\left(\frac{3}{2}\right)^n t^n}{n!}+\left(\frac{2}{3}\right)^2\sum_{n=2}^{\infty}\frac{t^n\left(\frac{3}{2}\right)^n}{n!}$$

$$T_g^{c-1}\left[\frac{1}{iq(s)\left(iq(s)-\frac{3}{2}\right)^2}\right]=\frac{2}{3}t\sum_{n=2}^{\infty}\frac{\left(\frac{3}{2}\right)^n t^n}{n!}-2\left(\frac{2}{3}\right)^2\sum_{n=3}^{\infty}\frac{\left(\frac{3}{2}\right)^n t^n}{n!}+\left(\frac{2}{3}\right)^2\sum_{n=2}^{\infty}\frac{t^n\left(\frac{3}{2}\right)^n}{n!}$$

$$T_g^{c-1}\left[\frac{1}{iq(s)\left(iq(s)-\frac{3}{2}\right)^2}\right]=\frac{2}{3}t\left[e^{\frac{3t}{2}}-\frac{3}{2}t-1\right]$$

$$-2\left(\frac{2}{3}\right)^2\left[e^{\frac{3t}{2}}-\frac{\left(\frac{3}{2}t\right)^2}{2}-\frac{3}{2}t-1\right]+\left(\frac{2}{3}\right)^2\left[e^{\frac{3t}{2}}-\frac{3}{2}t-1\right]$$

$$T_g^{c-1}\left[\frac{T_g^c\left\{te^{\frac{3t}{2}}\right\}}{iq(s)}\right]=\frac{2}{3}te^{\frac{3t}{2}}-\frac{4}{9}e^{\frac{3t}{2}}-\frac{1}{4}t+\frac{4}{9}$$

(0.49)

We need to find out $T_g^c\left\{t^2e^t\right\}$

$$T_g^c\left\{t^2e^t\right\}=p(s)\int_0^\infty t^2e^te^{-iq(s)t}dt=p(s)\int_0^\infty t^2e^{-(iq(s)-1)t}dt$$

$$T_g^c\left\{t^2e^t\right\}=p(s)\left[t^2\frac{e^{-(iq(s)-1)t}}{-(iq(s)-1)}\Bigg|_0^\infty+2\int_0^\infty t\frac{e^{-(iq(s)-1)t}}{(iq(s)-1)}dt\right]$$

$$T_g^c\left\{t^2e^t\right\}=p(s)\left[2t\frac{e^{-(iq(s)-1)t}}{-(iq(s)-1)^2}\Bigg|_0^\infty-2\frac{e^{-(iq(s)-1)t}}{(iq(s)-1)^3}\Bigg|_0^\infty\right]=\frac{2}{(iq(s)-1)^3}$$

From Previous steps;

$$\frac{d}{dq}\left(\frac{1}{iq(s)-1}\right)=\frac{d}{dq}\left(\frac{1}{iq(s)\left(1-\frac{1}{iq(s)}\right)}\right)=\frac{d}{dq}\left(\frac{1}{iq(s)}\sum_{n=0}^\infty\left(\frac{1}{iq(s)}\right)^n\right)$$

This imply

$$\frac{1}{(iq(s)-1)^2}=\frac{1}{(iq(s))^3}\sum_{n=1}^\infty n\left(\frac{1}{iq(s)}\right)^{n-1}+\frac{1}{(iq(s))^2}\sum_{n=0}^\infty\left(\frac{1}{iq(s)}\right)^n$$

$$\frac{d}{dq}\left(\frac{1}{(iq(s)-1)^2}\right)=\frac{d}{dq}\left(\frac{1}{(iq(s))^3}\sum_{n=1}^\infty n\left(\frac{1}{iq(s)}\right)^{n-1}+\frac{1}{(iq(s))^2}\sum_{n=0}^\infty\left(\frac{1}{iq(s)}\right)^n\right)$$

$$\frac{2}{(iq(s)-1)^3}=\frac{3}{(iq(s))^4}\sum_{n=1}^\infty n\left(\frac{1}{(i)q(s)}\right)^{n-1}+\frac{1}{(iq(s))^5}\sum_{n=2}^\infty n(n-1)\left(\frac{1}{(i)q(s)}\right)^{n-2}$$

$$+\frac{2}{(iq(s))^3}\sum_{n=0}^\infty\left(\frac{1}{(i)q(s)}\right)^n+\frac{1}{(iq(s))^4}\sum_{n=1}^\infty n\left(\frac{1}{(i)q(s)}\right)^{n-1}$$

$$\frac{2}{(iq(s)-1)^3}=3\sum_{n=1}^\infty n\left(\frac{1}{(i)q(s)}\right)^{n+3}+\sum_{n=2}^\infty n(n-1)\left(\frac{1}{(i)q(s)}\right)^{n+3}$$

$$+2\sum_{n=0}^\infty\left(\frac{1}{(i)q(s)}\right)^{n+3}+\sum_{n=0}^\infty n\left(\frac{1}{(i)q(s)}\right)^{n+3}$$

$$\frac{2p(s)}{iq(s)(iq(s)-1)^3}=3p(s)\sum_{n=1}^\infty n\left(\frac{1}{(i)q(s)}\right)^{n+4}+p(s)\sum_{n=2}^\infty n(n-1)\left(\frac{1}{(i)q(s)}\right)^{n+4}$$

$$+2p(s)\sum_{n=0}^\infty\left(\frac{1}{(i)q(s)}\right)^{n+4}+p(s)\sum_{n=0}^\infty n\left(\frac{1}{(i)q(s)}\right)^{n+4}$$

$$\frac{2p(s)}{iq(s)\left(iq(s)-1\right)^3}=4p(s)\sum_{n=1}^{\infty}n\left(\frac{1}{(i)q(s)}\right)^{n+4}+p(s)\sum_{n=2}^{\infty}n(n-1)\left(\frac{1}{(i)q(s)}\right)^{n+4}$$
$$+2p(s)\sum_{n=0}^{\infty}\left(\frac{1}{(i)q(s)}\right)^{n+4}$$

$$T_g^{c-1}T_g^c\left[\frac{2p(s)}{iq(s)\left(iq(s)-1\right)^3}\right]=4\sum_{n=1}^{\infty}n\frac{t^{n+3}}{(n+3)!}+\sum_{n=2}^{\infty}n(n-1)\frac{t^{n+3}}{(n+3)!}$$
$$+2\sum_{n=0}^{\infty}\frac{t^{n+3}}{(n+3)!}$$

$$T_g^{c-1}T_g^c\left[\frac{2p(s)}{iq(s)\left(iq(s)-1\right)^3}\right]=4\sum_{n=4}^{\infty}(n-3)\frac{t^n}{n!}+\sum_{n=5}^{\infty}(n-3)(n-4)\frac{t^n}{n!}$$
$$+2\sum_{n=3}^{\infty}\frac{t^n}{n!}$$

$$T_g^{c-1}T_g^c\left[\frac{2p(s)}{iq(s)\left(iq(s)-1\right)^3}\right]=4\sum_{n=4}^{\infty}n\frac{t^n}{n!}-12\sum_{n=4}^{\infty}\frac{t^n}{n!}+\sum_{n=5}^{\infty}n^2\frac{t^n}{n!}-7\sum_{n=5}^{\infty}n\frac{t^n}{n!}+12\sum_{n=5}^{\infty}\frac{t^n}{n!}$$
$$+2\sum_{n=3}^{\infty}\frac{t^n}{n!}$$

$$T_g^{c-1}T_g^c\left[\frac{2p(s)}{iq(s)\left(iq(s)-1\right)^3}\right]=4\sum_{n=4}^{\infty}\frac{t^n}{(n-1)!}-12\sum_{n=4}^{\infty}\frac{t^n}{n!}+\sum_{n=5}^{\infty}n\frac{t^n}{(n-1)!}-7\sum_{n=5}^{\infty}\frac{t^n}{(n-1)!}+12\sum_{n=5}^{\infty}\frac{t^n}{n!}$$
$$+2\sum_{n=3}^{\infty}\frac{t^n}{n!}$$

(0.50)

$$T_g^{c-1}T_g^c\left[\frac{2p(s)}{iq(s)\left(iq(s)-1\right)^3}\right]=4\sum_{n=3}^{\infty}\frac{t^{n+1}}{n!}-12\sum_{n=4}^{\infty}\frac{t^n}{n!}+\sum_{n=4}^{\infty}(n+1)\frac{t^{n+1}}{n!}-7\sum_{n=4}^{\infty}\frac{t^{n+1}}{n!}+12\sum_{n=5}^{\infty}\frac{t^n}{n!}$$
$$+2\sum_{n=3}^{\infty}\frac{t^n}{n!}$$

$$T_g^{c-1}T_g^c\left[\frac{2p(s)}{iq(s)\left(iq(s)-1\right)^3}\right]=4t\sum_{n=3}^{\infty}\frac{t^n}{n!}-12\sum_{n=4}^{\infty}\frac{t^n}{n!}+\sum_{n=4}^{\infty}n\frac{t^{n+1}}{n!}+\sum_{n=4}^{\infty}\frac{t^{n+1}}{n!}$$
$$-7\sum_{n=4}^{\infty}\frac{t^{n+1}}{n!}+12\sum_{n=5}^{\infty}\frac{t^n}{n!}+2\sum_{n=3}^{\infty}\frac{t^n}{n!}$$

$$T_g^{c-1}T_g^c\left[\frac{2p(s)}{iq(s)\left(iq(s)-1\right)^3}\right]=4t\sum_{n=3}^{\infty}\frac{t^n}{n!}-12\sum_{n=4}^{\infty}\frac{t^n}{n!}+\sum_{n=4}^{\infty}\frac{t^{n+1}}{(n-1)!}+t\sum_{n=4}^{\infty}\frac{t^n}{n!}$$
$$-7\sum_{n=4}^{\infty}\frac{t^{n+1}}{n!}+12\sum_{n=5}^{\infty}\frac{t^n}{n!}+2\sum_{n=3}^{\infty}\frac{t^n}{n!}$$

$$T_g^{c-1}T_g^c\left[\frac{2p(s)}{iq(s)\left(iq(s)-1\right)^3}\right]=4t\sum_{n=3}^{\infty}\frac{t^n}{n!}-12\sum_{n=4}^{\infty}\frac{t^n}{n!}+\sum_{n=4}^{\infty}\frac{t^{n+2}}{n!}+t\sum_{n=4}^{\infty}\frac{t^n}{n!}$$

$$-7t\sum_{n=4}^{\infty}\frac{t^n}{n!}+12\sum_{n=5}^{\infty}\frac{t^n}{n!}+2\sum_{n=3}^{\infty}\frac{t^n}{n!}$$

(0.51)

$$T_g^{c-1}T_g^c\left[\frac{2p(s)}{iq(s)\left(iq(s)-1\right)^3}\right]=4t\left[e^t-1-t-\frac{t^2}{2}\right]-12\left[e^t-1-t-\frac{t^2}{2}-\frac{t^3}{6}\right]$$

$$+t^2\left[e^t-1-t-\frac{t^2}{2}-\frac{t^3}{6}\right]-6t\left[e^t-1-t-\frac{t^2}{2}-\frac{t^3}{6}\right]$$

$$+12\left[e^t-1-t-\frac{t^2}{2}-\frac{t^3}{6}-\frac{t^4}{24}\right]+2\left[e^t-1-t-\frac{t^2}{2}\right]$$

(0.52)

$$T_g^{c-1}\left[\frac{T_g^c\left\{t^2e^t\right\}}{iq(s)}\right]=t^2e^t-2te^t+2e^t-\frac{t^5}{6}-2$$

(0.53)

$$T_g^c\left\{y_3(t)\right\}=\frac{\frac{2}{3}T_g^c\left[e^{\frac{7t}{4}}\right]+\frac{1}{6}T_g^c\left[e^{\frac{3t}{2}}\right]+\frac{1}{4}T_g^c\left[te^{\frac{3t}{2}}\right]+\frac{1}{8}T_g^c\left[t^2e^t\right]-\frac{2}{3}T_g^c\left[e^t\right]+\frac{1}{2}T_g^c\left[t^2\right]-T_g^c\left[t\right]+\frac{1}{2}T_g^c\left[1\right]}{iq(s)}$$

(0.54)

And so,

$$y_3(t)=\frac{8}{21}e^{\frac{7t}{4}}+\frac{1}{6}te^{\frac{3t}{2}}+\frac{1}{8}t^2e^t-\frac{1}{4}te^t-\frac{5}{12}e^t+\frac{t^3}{6}-\frac{t^2}{2}+\frac{15t}{16}+\frac{1}{28}\qquad(0.55)$$

$$y=y_0+y_1++y_2+y_3+....$$

$$y_1=e^t+t-1\qquad(0.56)$$

$$y_2(t)=\frac{2}{3}\left[e^{\frac{3t}{2}}-1\right]+\frac{1}{2}\left(te^t-e^t+1\right)+\frac{t^2}{2}-t$$

(0.57)

$$y_3(t)=\frac{8}{21}e^{\frac{7t}{4}}+\frac{1}{6}te^{\frac{3t}{2}}+\frac{2}{3}e^{\frac{3t}{2}}+\frac{1}{8}t^2e^t-\frac{1}{2}te^t+\frac{1}{12}e^t+\frac{t^3}{6}+\frac{15t}{16}-\frac{95}{84}$$

$$y(t)=\frac{8}{21}e^{\frac{7t}{4}}+\frac{1}{6}te^{\frac{3t}{2}}+\frac{1}{8}t^2e^t-\frac{1}{4}te^t-\frac{5}{12}e^t+\frac{t^3}{6}-\frac{t^2}{2}+\frac{15t}{16}+\frac{1}{28}+....$$

8. CONCLUSION

After applying SEJI transform to different forms of delay differential equations (DEE), we can conclude that the transform is valid for those kinds of equations, in some cases it gives the exact solutions, and other cases if evaluate more iterations it gives results closer to the exact solution, non-linearity do not guarantee an accurate exact solution. In the future we will apply this transformation on fractional non-linear differential equations to see if we can obtain some valid results.

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CONFLICT OF INTEREST:

Authors do not have any conflict of interest.

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