International Journal of Research Publication and Reviews

Journal homepage: www.ijrpr.com ISSN 2582-7421

Importance of Metric Space

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ABSTRACT:

Since the last century, the postulational method and an abstract point of view have played a vital role in the development of modern mathematics. The experience gained from the earlier concrete studies of analysis point to the importance of passage to the limit. The basis of this operation is the notion of distance between any two points of the line or the complex plane. The algebraic properties of underlying sets often play no role in the development of analysis; this situation naturally leads to the study of metric spaces. Some key types of metric spaces are Euclidean Metric Spaces, Normed Metric Spaces and Discrete Metric Space etc. The abstraction not only simplifies and elucidates mathematical ideas that recur in different guises, but also helps economize the intellectual effort involved in learning them. However, such an abstract approach is likely to overlook the special features of particular mathematical developments, especially those not taken into account while forming the larger picture. Hence, the study of particular mathematical developments is hard to overemphasize.

Keywords: Metric, Discrete, Normed

Introduction and Concept:

The metric space is the transitional stage between real line R and the topological space. An essential pre-requisite for metric is clear understanding of basic topology on R i.e. the notion of open sets, closed sets, limit points, convergence and continuity in R metric Space, the idea is to carry these and other related concepts from real line to a general set.
If I wish to travel from Araria to Patna, then I may be interested in one or more of the following numbers..
(1) The distance, in kilometers, from Araria to Patna by road.
(2) The time, in minutes, of the shortest journey from Araria to Patna by rail.
(3) The cost of the cheapest journey from Araria to Patna by rail.

Each of these numbers is of interest to someone and none of them is easily obtained from another. However, they do have certain properties in common which we try to isolate in the following definition.

Metric spaces can be thought of as very basic spaces, with only a few axioms, where the ideas of convergence and continuity exist. The fundamental ingredient that is needed to make these concepts rigorous is that of a distance, also called a metric, which is a measure of how close elements are to each other.

Definition:

Let X be a non empty set and d be a real valued function on $X \times X$ i.e. $d: X \times X \rightarrow R$ a function with the following properties:-
(i) $d(x, y) \geq 0$ for all $x, y \in X$.
(ii) Null condition
$d(x, y) = 0$ if and only if $x = y$.
(iii) Symmetric condition
$(x, y) = d(y, x)$ for all $x, y \in X$.
(iv) Triangular inequality
$d(x, y) + d(y, z) \geq d(x, z)$ for all $x, y, z \in X$. (This is called the triangle inequality after the result in Euclidean geometry that the sum of the lengths of two sides of a triangle is at least as great as the length of the third side.)
Then we say that d is a metric on X and that (X, d) is a metric space.

Definition:

Let V be a vector space over F (with F = R or F = C) and N: V → R a map such that, writing N(u) = ||u||, the following results hold.
(i) $\|u\| \geqslant 0$ for all $u \in V$
(ii) If $\|u\| = 0$, then $u = 0$
(iii) If $\lambda \in F$ and $u \in V$, then $\| \lambda u \| = | \lambda | \, \|u\|$
(iv) [Triangle law.] If $u, v \in V$, then $\|u\| + \|v\| \geqslant \|u + v\|$
Then we call || || a norm and say that (V, || ||) is a Normed Metric Space.

Discrete metric space:

We take any set X and on it the so-called discrete metric for X, defined by
$d(x, x) = 0$,
$d(x, y) = 1 \quad (x \neq y)$
This space (x, d) is called a discrete metric space. This is useful to explain in finer point of theory.

Examples-
Real line R- This is the set of all real numbers, taken with the
usual metric defined by

$$d(x, y) = | x - y| \text{ where } x, y \in R$$

Then d is called usual metric. This real line metric space is model for generalization to metric space.

Example
Let R^2 be set of ordered pairs of real numbers and we define the following functions
Let $x = (x_1, x_2) \in R^2$ and $y = (y_1, y_2) \in R^2$
Define

A- $d_1 (x, y) = | x_1 - y_1| + | x_2 - y_2|$

B- $d_2 (x, y) = \sqrt{|x_1 - y_1|^2 + |x_2 - y_2|^2}$

C- $d_\infty (x, y) = \max. \{| x_1 - y_1| , | x_2 - y_2|\}$

Then each of d_1, d_2, d_∞ is a metric on plane R^2. The metric space R^2 (the natural distance between two points in a plane) called the Euclidean plane.

Proof:
A: $d_1 (x,y)$ is a metric.
Now we verify condition of metric space one by one
Let $x,y, z \in R^2$

1- Symmetric condition
$d_1 (x, y) = | x_1 - y_1| + | x_2 - y_2|$
$= | y_1 - x_1| + | y_2 - x_2|$
$= d_1 (y, x)$

2- Null condition
$d_1 (x, y) = 0$
$\Leftrightarrow | x_1 - y_1| + | x_2 - y_2| = 0$
$\Leftrightarrow | x_1 - y_1| = 0$ and $| x_2 - y_2| = 0$
$\Leftrightarrow x_1 = y_1$ and $x_2 = y_2$
$\Leftrightarrow (x_1, x_2) = (y_1, y_2)$
$\Leftrightarrow x=y$

3- Triangular inequality
$d_1 (x, y) = | x_1 - y_1| + | x_2 - y_2|$
since $| x_1 - y_1| \leq | x_1 - z_1| + | z_1 - y_1|$
and $| x_2 - y_2| \leq | x_2 - z_2| + | z_2 - y_2|$
by the Triangular inequality or R with usual metric
Therefore $d_1 (x, y) \leq | x_1 - z_1| + | z_1 - y_1| + | x_2 - z_2| + | z_2 - y_2|$
$= d_1 (x, z) + d_1 (z, y)$
Thus the all condition of metric space holds. Hence $d_1 (x, y)$ is a metric space.

B- $d_2 (x, y)$ is a metric.
1- Symmetric condition
$d_2 (x, y) = \sqrt{|x_1 - y_1|^2 + |x_2 - y_2|^2}$
$= \sqrt{|y_1 - x_1|^2 + |y_2 - x_2|^2}$
By the symmetric condition on R with usual metric

$d_2(x, y) = d_2(y, x)$
2- Null condition
$d_2(x, y) = 0$
$\Leftrightarrow \sqrt{|x_1 - y_1|^2 + |x_2 - y_2|^2} = 0$
$\Leftrightarrow |x_1 - y_1|^2 + |x_2 - y_2|^2 = 0$
$\Leftrightarrow |x_1 - y_1|^2 = 0$ and $|x_2 - y_2|^2 = 0$
$\Leftrightarrow x_1 = y_1$ and $x_2 = y_2$
$\Leftrightarrow (x_1, x_2) = (y_1, y_2)$
$\Leftrightarrow x = y$

3- Triangular inequality
Now we have to show that
$d_2(x, y) \le d_2(x, z) + d_2(z, y)$
We take $x_1 - z_1 = a_1$, $x_2 - z_2 = a_2$, $z_1 - y_1 = b_1$, $z_2 - y_2 = b_2$
Then above condition can be written as
$d_2(x, y) = \sqrt{|x_1 - y_1|^2 + |x_2 - y_2|^2}$
$= \sqrt{|a_1 + b_1|^2 + |a_2 + b_2|^2}$
$d_2(x, z) = \sqrt{|x_1 - z_1|^2 + |x_2 - z_2|^2} = \sqrt{{a_1}^2 + {a_2}^2}$
$d_2(z, y) = \sqrt{|z_1 - y_1|^2 + |z_2 - y_2|^2} = \sqrt{{b_1}^2 + {b_2}^2}$

Thus in assume form we have to show that $\sqrt{|a_1 + b_1|^2 + |a_2 + b_2|^2} \le \sqrt{{a_1}^2 + {a_2}^2} + \sqrt{{b_1}^2 + {b_2}^2}$

On squaring both side, we get
$|a_1 + b_1|^2 + |a_2 + b_2|^2 \le {a_1}^2 + {a_2}^2 + {b_1}^2 + {b_2}^2 + 2\sqrt{{a_1}^2 + {a_2}^2} \times \sqrt{{b_1}^2 + {b_2}^2}$
$\Rightarrow {a_1}^2 + {b_1}^2 + 2a_1b_1 + {a_2}^2 + {b_2}^2 + 2a_2b_2 \le {a_1}^2 + {a_2}^2 + {b_1}^2 + {b_2}^2 + \sqrt{{a_1}^2 + {a_2}^2} \times \sqrt{{b_1}^2 + {b_2}^2}$
$a_1b_1 + a_2b_2 \le \sqrt{{a_1}^2 + {a_2}^2} \times \sqrt{{b_1}^2 + {b_2}^2}$

again squaring on both side

$(a_1b_1 + a_2b_2)^2 \le ({a_1}^2 + {a_2}^2) \times ({b_1}^2 + {b_2}^2)$
$\Rightarrow {a_1}^2{b_1}^2 + {a_2}^2{b_2}^2 + 2\,a_1b_1\,a_2b_2 \le {a_1}^2{b_1}^2 + {a_2}^2{b_2}^2 + {a_1}^2{b_2}^2 + {a_2}^2{b_1}^2$
$\Rightarrow 0 \le {a_1}^2{b_2}^2 + {a_2}^2{b_1}^2 - 2\,a_1b_1\,a_2b_2$
$= (a_1b_2 - a_2b_1)^2$
This is true because whole square is always positive.
Thus the all condition of metric space holds. Hence $d_2(x, y)$ is a metric space.

C- $d_\infty(x, y)$ is metric.
We verify all condition one by one.
Let $x, y, z \in R^2$
1- Symmetric condition
$d_\infty(x, y) = \max.\{|x_1 - y_1|, |x_2 - y_2|\}$
$= \max.\{|y_1 - x_1|, |y_2 - x_2|\}$
$= d_\infty(y, x)$
2- Null condition
$d_\infty(x, y) = \max.\{|x_1 - y_1|, |x_2 - y_2|\} = 0$
$\Leftrightarrow |x_1 - y_1| = 0$ and $|x_2 - y_2| = 0$
$\Leftrightarrow (x_1, x_2) = (y_1, y_2)$
$\Leftrightarrow x = y$

3-Triangular inequality
Since $|x_1 - y_1| \le |x_1 - z_1| + |z_1 - y_1|$
$\le d_\infty(x, z) + d_\infty(z, y)$
And $|x_2 - y_2| \le |x_2 - z_2| + |z_2 - y_2|$
$\le d_\infty(x, z) + d_\infty(z, y)$ By triangular in R
$\max.\{|x_1 - y_1|, |x_2 - y_2|\} \le d_\infty(x, z) + d_\infty(z, y)$
$d_\infty(x, y) \le d_\infty(x, z) + d_\infty(z, y)$
Thus the all condition of metric space holds. Hence $d_\infty(x, y)$ is a metric space.

Euclidean Metric Space:

Euclidean plane is obtained if we take the set of ordered pairs of real numbers, written $x = (\xi_1, \xi_2)$, and $y = (\eta_1, \eta_2)$, etc., and the Euclidean metric defined by

$$d(x, y) = \sqrt{(\xi_1 - \eta_1)^2 + (\xi_2 - \eta_2)^2}$$

Another metric space is obtained if we choose the same set as before but another metric d_1 defined by

$$d(x, y) = |(\xi_1 - \eta_1) + (\xi_2 - \eta_2)|$$

Sequence metric space:

Now we define the following set
$S = \{x = (x_k) : (x_k)$ is a sequence of real number or complex number$\}$

$l\infty$ = {x = (x_k): (x_k) is a bounded sequence of real number or complex number}
c = {x = (x_k): (x_k) is a convergent sequence of real number or complex number}
c_0 = {x = (x_k): (x_k) is a null sequence of real number or complex number, i.e. $x_k \to 0$ as $k \to \infty$}
Here it is clear that
$c_0 \subset c \subset l\infty \subset S$
On c_0, c and l_∞, we define the metric
$d_\infty (x, y) = \sup_{j \geq n} |x_j - y_j|$
Where d_∞ is a metric on c_0, c and l^∞ and can be proved in the same way as example A, B and C. c_0 and c are relatively metric sub space of l^∞.

Conclusion:

In many branches of mathematics, it is convenient to have available a notion of distance between elements of an abstract set. For example, the proofs of some of the theorems in real analysis or analytic function theory depend only on a few properties of the distance between points and not on the fact that the points are in R or C. When these properties of distance are abstracted, they lead to the concept of a metric space. The notion of distance between points of an abstract set leads naturally to the discussion of convergence of sequences and Cauchy sequences in the set.

These are fundamental in functional analysis because they play a role similar to that of the real line R in calculus. In fact, they generalize R and have been created in order to provide a basis for a unified treatment of important problems from various branches of analysis.

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