International Journal of Research Publication and Reviews

Journal homepage: www.ijrpr.com ISSN 2582-7421

ENERGY VARIANT AND INDEX IN THE CONTEXT OF CONNECTED GRAPH

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ABSTRACT :

We propose and study the Reachability energy and Reachability Estrada index of a graph, both of which are based on the eigenvalues of the Reachability matrix. Additionally, we derive upper and lower bounds for these new graph invariants and explore the relationship between them.

Keywords: Energy, Estrada Index, Connected graph, Spectrum of a graph.

1. INTRODUCTION :

The concept of the energy of a simple graph was first introduced by Ivan Gutman in 1978 [14,15]. The energy of a graph, often referred to as the ordinary energy of graph G, is defined as the sum of the absolute values of the eigenvalues of its adjacency matrix [19]. Specifically, for a graph G, its energy E(G) is given by:

$$E(G) = \sum_{i=1}^{n} |\lambda_i|$$

Various other energy measures have been studied based on different matrices [17], such as the Distance matrix [3,13,18], Laplacian matrix [2], Randic matrix [16] and Harary matrix [12].

In 2007, De la Peña et al. [4] introduced the Estrada index of a graph, which is defined as:

$$EE(G) = \sum_{i=1}^{p} e^{\lambda_i},$$

where $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_p$ are the eigenvalues of the adjacency matrix $A(G)$ of G[5,6,7,8,9,10,11]. Denoting by $M_k(G)$ to the k-th moment of the graph G, we get $M_k(G) = \sum_{i=1}^{p} (\lambda_i)^k$and recalling the power series expansion of e^x, we have $EE = \sum_{k=0}^{\infty} \frac{M_k}{k!}$.

It is well known that $M_k(G)$ is equal to the number of closed walks of length k of the graph G [10]. In fact, Estrada index of graphs has an important role in chemistry and physics and there exists a vast literature that studies this special index. In addition to the Estrada's papers depicted above, we may also refer [4,5] to the reader for detail information such as lower and upper bounds for EE in terms of the number of vertices and edges and some inequalities between EE and the energy of G.

In this study, we discuss energy and index for the Reachability matrix of a graph. Also, we derive upper and lower bounds for these new graph invariants and explore the relationship between them.

Definition 1.1:

For a connected graph G with p vertices. The **Reachability matrix** $\mathbb{R} = (r_{ij})$ of a graph G, denoted by $\mathbb{R}(G)$ is the $p \times p$ matrix with

$$r_{ij} = \begin{cases} 1, & t_j \text{ is reachable from } t_i \\ 0, & \text{otherwise} \end{cases}.$$

Definition 1.2:

The **Characteristic polynomial** of the Reachability matrix R(G) is defined as $\phi(G,\beta) = \det(\mathbb{R}(G) - \beta I)$, where I is the idendity matrix. The roots of the equation $\phi(G,\beta) = 0$ is called the eigen values of the reachability matrix. The eigenvalues of the reachability matrix of the graph G are called as

$\mathbb{R}$-eigenvalues of G and denoted by $\beta_1, \beta_2, \dots, \beta_p$. The collection of $\mathbb{R}$- eigenvalues is called **Spectrum** of a graph G[16]. If $\beta_1 \geq \beta_2 \geq \beta_3 \geq \cdots \geq \beta_p$ are the distinct eigenvalues of G with multiplicities $m_1, m_2, \dots, m_p$ respectively then

$$Spec\ (G) = \begin{pmatrix} \beta_1 & \beta_2 & \dots & \beta_p \\ m_1 & m_2 & \dots & m_p \end{pmatrix}.$$

The **largest eigenvalue** of a connected graph G satisfies $\lambda_1 \leq \frac{2m}{n} \leq 1, \lambda_1$is always positive.

Let tr denotes the **trace** of a matrix of G, $tr\ (\mathbb{R}(G)^k) = \sum_{i=1}^{p} (\beta_i)^k$.

Let Det denotes the **determinant** of reachability matrix of G, $Det\ \mathbb{R}(G) = \prod_{i=1}^{p} \beta_i$.

Definition 1.3:

The sum of the absolute values of the eigenvalues of G is known as **Reachability energy** of a graph G, denoted by $\mathbb{R}E(G)$, is defined by

$$\mathbb{R}E(G) = \sum_{i=1}^{p} |\beta_i|$$

Definition 1.4:

If G is a connected graph with p vertices, then the **Reachability Estrada index** of G, denoted by $\mathbb{R}EE(G)$, is defined by

$$\mathbb{R}EE(G) = \sum_{i=1}^{p} e^{\beta_i}$$

where $\beta_1 \geq \beta_2 \geq \beta_3 \geq \cdots \geq \beta_p$ are the $\mathbb{R}$-eigenvalues of G.

Let $V_k = tr\ (\mathbb{R}(G)^k) = \sum_{i=1}^{p} (\beta_i)^k$,

where $V_0 = \sum_{i=1}^{p} (\beta_i)^0 = p; V_1 = \sum_{i=1}^{p} (\beta_i)^1 = 0$ & $V_2 = \sum_{i=1}^{p} (\beta_i)^2 = 2R$

Then $\mathbb{R}EE(G) = \sum_{k=0}^{\infty} \frac{V_k}{k!}$.

2. PRELIMINARIES :

In this section, we discuss some lemmas and using that we will calculate the bounds for largest eigenvalue using reachability matrix.

Lemma 2.1:

Let G be a connected graph of order p and let $\beta_1, \beta_2, \dots, \beta_p$ be eigenvalues of reachability matrix. Then

$$tr\big(\mathbb{R}(G)\big) = \sum_{i=1}^{p} \beta_i = 0 \ \&\ tr(\mathbb{R}(G)^2) = \sum_{i=1}^{p} {\beta_i}^2 = 2R, where\ R = \sum_{1 \leq i < j \leq p} (r_{ij})^2$$

Lemma 2.2:

Let G be a connected graph with diameter less than or equal to 2 and let $\beta_1, \beta_2, \dots, \beta_p$ be eigenvalues of reachability matrix. Then $\sum_{i=1}^{p} {\beta_i}^2 = p(p-1)$.

Lemma 2.3:

Let G represents a graph with p vertices then

$$|Det\ \mathbb{R}(G)| \leq (2\ R)^{\frac{p}{2}}.$$

Proof:

We know that, $|Det\ \mathbb{R}(G)| = \prod_{i=1}^{p} |\beta_i|$

$$= |\beta_1||\beta_2| \dots |\beta_p| \leq |\beta_1||\beta_1| \dots |\beta_1| \leq |\beta_1|^p \leq \left(\sqrt{2 \sum_{1 \leq i < j \leq p} (r_{ij})^2}\right)^p \leq (2\ R)^{\frac{p}{2}}$$

$$|Det\ \mathbb{R}(G)| \leq (2\ R)^{\frac{p}{2}}.$$

Lemma 2.4:

If G is any graph with p vertices then $\beta_1 \leq \sqrt{\frac{2\,(p-1)R}{p}}$.

Proof:

Using *Cauchy Schwartz inequality* and lemma 2.1, by setting $a_i = 1$ and $b_i = \beta_i$ for $i = 2,3, \dots, p$ then we get,

$$\left(\sum_{i=2}^{p} |\beta_i|\right)^2 \leq (p-1)\left(\sum_{i=1}^{p} {\beta_i}^2\right)$$

$$(-\beta_1)^2 \le (p-1)\left(2\sum_{1\le i<j\le p}(r_{ij})^2 - {\beta_1}^2\right)$$

$$\Rightarrow\ \beta_1 \le \sqrt{\frac{2\,(p-1)R}{p}}.$$

Lemma 2.5:

Consider a connected graph G with p vertices. The largest eigenvalue β_1 of G satisfies the inequality $|\beta_1| \ge\ |Det\ \mathbb{R}(G)|^{\frac{1}{p}}$.

Proof:

Using *Arithmetic- Geometric Mean Inequality* for the values $|\beta_1|, |\beta_2|, \dots, |\beta_p|$, we get,

$$\frac{|\beta_1+\beta_2+\dots+\beta_p|}{p} \ge |\beta_1\beta_2\dots\beta_p|^{\frac{1}{p}}$$

$$\Rightarrow \frac{|\beta_1|+|\beta_1|+\cdots+|\beta_1|}{p}\ \ge\ |Det\ \mathbb{R}(G)|^{\frac{1}{p}}$$

$$|\beta_1| \ge\ |Det\ \mathbb{R}(G)|^{\frac{1}{p}}.$$

Lemma 2.6:

Let G be a connected graph with p vertices, the largest eigenvalue β_1 of G satisfies the inequality $|\beta_1| \ge \frac{|Det\ \mathbb{R}(G)|^{\frac{1}{p}}}{\sqrt{p}}$.

Proof:

Using *Arithmetic- Geometric Mean Inequality* for the values $|\beta_1|, |\beta_2|, \dots, |\beta_p|$, we get,

$$\frac{|\beta_1|+|\beta_2|+\cdots+|\beta_p|}{p} \ge |\beta_1\beta_2\dots\beta_p|^{\frac{1}{p}}$$

$$\frac{|\beta_1|+|\beta_2|+\cdots+|\beta_p|}{\sqrt{p}} \ge \frac{|\beta_1|+|\beta_2|+\cdots+|\beta_p|}{p} \ge |\beta_1\beta_2\dots\beta_p|^{\frac{1}{p}}$$

$$\Rightarrow |\beta_1| \ge \frac{|Det\ \mathbb{R}(G)|^{\frac{1}{p}}}{\sqrt{p}}.$$

3. CONSTRAINTS ON THE REACHABILITY ENERGY

In this section, we obtain the reachability energy of the graph and establish its bounds of it.

3.1 REACHABILITY ENERGY

Theorem 3.1.1:

Let G be any connected graph of order p then reachability energy of the graph is $2(p-1)$.

Proof:

Consider a connected graph with p vertices. Using definition 1.1 and 1.2, we get,

The spectrum of $\mathbb{R}(G)$ is

$$\begin{pmatrix} -1 & p-1 \\ p-1 & 1 \end{pmatrix}$$

Using definition 1.3,

$$\mathbb{R}E(G) = \sum_{i=1}^{p}|\beta_i| = 2(p-1).$$

$$\mathbb{R}E(G) = 2(p-1).$$

This completes the proof.

3.2. BOUNDS FOR REACHABILITY ENERGY

In this subsection, we obtain upper bound and lower bound for the reachability energy of the connected graph.

Theorem 3.2.1:

If G be a connected graph then $\sqrt{2R} \le \mathbb{R}E(G) \le \sqrt{2pR}$.

Proof:

By *Cauchy-Schwartz inequality*,

$$\left(\sum_{i=1}^{p} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{p} {a_i}^2\right)\left(\sum_{i=1}^{p} {b_i}^2\right)$$

Consider, $a_i = 1$ and $b_i = |\beta_i|$, then $\mathbb{R}E(G) \leq \sqrt{2pR}$.

From inequality, $\left(\mathbb{R}E(G)\right)^2 = \left(\sum_{i=1}^{p}|\beta_i|\right)^2$

$$\mathbb{R}E(G) \geq \sqrt{2R}.$$

$$\therefore \sqrt{2R} \leq \mathbb{R}E(G) \leq \sqrt{2pR}$$

Hence the result.

Theorem 3.2.2:

Let G be a connected graph and let $|Det\ \mathbb{R}(G)|$ be the absolute value of the determinant of the reachability matrix $\mathbb{R}(G)$ of a graph then

$$\sqrt{2R + p(p-1)|Det\ \mathbb{R}(G)|^{\frac{2}{p}}} \leq \mathbb{R}E(G) \leq \sqrt{2pR}.$$

Proof:

According to Theorem 3.1.1, an upper bound for $\mathbb{R}(G)$ is established. We will now demonstrate the lower bound for $\mathbb{R}(G)$), thereby concluding the proof.

By the definition of reachability energy,

$$\left(\mathbb{R}E(G)\right)^2 = \left(\sum_{i=1}^{p}|\beta_i|\right)^2 = \sum_{i=1}^{p}|\beta_i|^2 + 2\sum_{1\leq i<j\leq p}|\beta_i|\,|\beta_j| = 2\sum_{1\leq i<j\leq p}\left(r_{ij}\right)^2 + \sum_{i\neq j}|\beta_i|\,|\beta_j|$$

From *Arithmetic – Geometric mean inequality*, we have,

$$\frac{1}{p(p-1)}\sum_{i\neq j}|\beta_i|\,|\beta_j| \geq |Det\ \mathbb{R}(G)|^{\frac{2}{p}}.$$

which gives

$$\mathbb{R}E(G) \geq \sqrt{2\sum_{1\leq i<j\leq p}\left(r_{ij}\right)^2 + p(p-1)|Det\ \mathbb{R}(G)|^{\frac{2}{p}}}\ .$$

$$\therefore \sqrt{2R + p(p-1)|Det\ \mathbb{R}(G)|^{\frac{2}{p}}} \leq \mathbb{R}E(G) \leq \sqrt{2pR}.$$

Hence the result.

Theorem 3.2.3:

Let G be a connected graph with p vertices and $\mathbb{R}(G)$ be a non-singular reachability matrix then

$$p|Det\ \mathbb{R}(G)|^{\frac{1}{p}} \leq \mathbb{R}E(G) \leq \frac{2pR}{|Det\ \mathbb{R}(G)|^{\frac{1}{p}}}.$$

Proof:

Using *Arithmetic- Geometric Mean Inequality* for the values $|\beta_1|, |\beta_2|, \ldots, |\beta_p|$, we get,

$$\frac{|\beta_1| + |\beta_2| + \cdots + |\beta_p|}{p} \geq |\beta_1\beta_2 \ldots \beta_p|^{\frac{1}{p}}$$

$$\sum_{i=1}^{p}|\beta_i| \geq p|Det\ \mathbb{R}(G)|^{\frac{1}{p}}$$

$$\mathbb{R}E(G) \geq p|Det\ \mathbb{R}(G)|^{\frac{1}{p}}$$

which gives a lower bound for $\mathbb{R}E(G)$.

We have $|\beta_1| \geq |Det\ \mathbb{R}(G)|^{\frac{1}{p}}$

$$|\beta_1|\sum_{i=1}^{p}|\beta_i| \geq |Det\ \mathbb{R}(G)|^{\frac{1}{p}}\sum_{i=1}^{p}|\beta_i|$$

$$\Rightarrow p|\beta_1|^2 \geq |Det\ \mathbb{R}(G)|^{\frac{1}{p}}\left(\mathbb{R}E(G)\right)$$

$$\mathbb{R}E(G) \leq \frac{p|\beta_1|^2}{|Det\ \mathbb{R}(G)|^{\frac{1}{p}}}$$

Since $|\beta_1|^2 \leq 2R$

$$\mathbb{R}E(G) \leq \frac{2pR}{|Det\ \mathbb{R}(G)|^{\frac{1}{p}}}$$

which gives an upper bound for $\mathbb{R}E(G)$.

$$\therefore p|Det\ \mathbb{R}(G)|^{\frac{1}{p}} \leq \mathbb{R}E(G) \leq \frac{2pR}{|Det\ \mathbb{R}(G)|^{\frac{1}{p}}}.$$

Hence the theorem.

Theorem 3.2.4:

If G represents a graph with p vertices and $\mathbb{R}(G)$ be a non-singular reachability matrix then

$$p^{\frac{p-1}{p}}|Det\ \mathbb{R}(G)|^{\frac{1}{p}} \leq \mathbb{R}E(G) \leq \frac{(4R)^{p^2}}{|Det\ \mathbb{R}(G)|^{p-1}}.$$

Proof:

Consider the following matrix form

$$\begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1p} \\ x_{21} & x_{22} & \cdots & x_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ x_{p1} & x_{p2} & \cdots & x_{pp} \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{|\beta_1|} & \cdots & \frac{1}{|\beta_1|} \\ \frac{1}{|\beta_2|} & 1 & \cdots & \frac{1}{|\beta_2|} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{|\beta_p|} & \frac{1}{|\beta_p|} & \cdots & 1 \end{pmatrix}$$

Using Holder's Inequality: For positive real numbers x_{ij} $(i = 1,2,\dots,p)(j = 1,2,\dots p)$ then

$$\prod_{i=1}^{p}\left(\sum_{j=1}^{p} x_{ij}\right)^{\frac{1}{p}} \geq \sum_{j=1}^{p}\left(\prod_{i=1}^{p} x_{ij}^{\frac{1}{p}}\right)$$

Consider, $\prod_{i=1}^{p}(\sum_{j=1}^{p} x_{ij})^{\frac{1}{p}}$

$$\prod_{i=1}^{p}\left(\sum_{j=1}^{p} x_{ij}\right)^{\frac{1}{p}} = \left(1+\frac{1}{|\beta_1|}+\cdots+\frac{1}{|\beta_1|}\right)^{\frac{1}{p}}\left(1+\frac{1}{|\beta_2|}+\cdots+\frac{1}{|\beta_2|}\right)^{\frac{1}{p}}\dots\left(1+\frac{1}{|\beta_p|}+\cdots+\frac{1}{|\beta_p|}\right)^{\frac{1}{p}}$$

$$\prod_{i=1}^{p}\left(\sum_{j=1}^{p} x_{ij}\right)^{\frac{1}{p}} \leq \left(1+\frac{p-1}{|\beta_1|}\right)\left(1+\frac{p-1}{|\beta_2|}\right)\dots\left(1+\frac{p-1}{|\beta_p|}\right)$$

Since $|\beta_1| \geq |\beta_i|\ \forall i$,

$$\Rightarrow \prod_{i=1}^{p}\left(\sum_{j=1}^{p} x_{ij}\right)^{\frac{1}{p}} \leq \left(\frac{|\beta_1|+p-1}{|\beta_1|}\right)\left(\frac{|\beta_1|+p-1}{|\beta_2|}\right)\dots\left(\frac{|\beta_1|+p-1}{|\beta_p|}\right)$$

$$\prod_{i=1}^{p}\left(\sum_{j=1}^{p} x_{ij}\right)^{\frac{1}{p}} \leq \frac{(|\beta_1|+p-1)^p}{|\beta_1\beta_2\dots\beta_p|}$$

Since $p-1 < p < 2R\ and\ |\beta_1| \leq 2R$,we get

$$\prod_{i=1}^{p}\left(\sum_{j=1}^{p} x_{ij}\right)^{\frac{1}{p}} \leq \frac{(4R)^p}{|Det\ \mathbb{R}(G)|}.$$

Consider, $\sum_{j=1}^{p}\left(\prod_{i=1}^{p} x_{ij}^{\frac{1}{p}}\right)$

$$\sum_{j=1}^{p}\left(\prod_{i=1}^{p} x_{ij}^{\frac{1}{p}}\right) = \frac{1}{|\beta_2|^{\frac{1}{p}}|\beta_3|^{\frac{1}{p}}\dots|\beta_p|^{\frac{1}{p}}} + \frac{1}{|\beta_1|^{\frac{1}{p}}|\beta_3|^{\frac{1}{p}}\dots|\beta_p|^{\frac{1}{p}}} + \cdots + \frac{1}{|\beta_1|^{\frac{1}{p}}|\beta_2|^{\frac{1}{p}}\dots|\beta_{p-1}|^{\frac{1}{p}}}$$

$$\Rightarrow \sum_{j=1}^{p}\left(\prod_{i=1}^{p} x_{ij}^{\frac{1}{p}}\right) = \frac{|\beta_1|^{\frac{1}{p}}+|\beta_2|^{\frac{1}{p}}+\cdots+|\beta_p|^{\frac{1}{p}}}{|\beta_1\beta_2\dots\beta_p|^{\frac{1}{p}}}$$

$$\Rightarrow \sum_{j=1}^{p}\left(\prod_{i=1}^{p} x_{ij}^{\frac{1}{p}}\right) = \frac{\sum_{i=1}^{p}|\beta_1|^{\frac{1}{p}}}{|Det\ \mathbb{R}(G)|^{\frac{1}{p}}}$$

Therefore,

$$\frac{(4R)^p}{|Det\ \mathbb{R}(G)|} \geq \frac{\sum_{i=1}^{p}|\beta_1|^{\frac{1}{p}}}{|Det\ \mathbb{R}(G)|^{\frac{1}{p}}}$$

$$\Rightarrow \sum_{i=1}^{p}|\beta_1|^{\frac{1}{p}} \leq \frac{(4R)^p}{|Det\ \mathbb{R}(G)|^{1-\frac{1}{p}}}$$

Since $\sum_{i=1}^{p}|\beta_1|^{\frac{1}{p}} \geq \left(\sum_{i=1}^{p}|\beta_i|\right)^{\frac{1}{p}}$

$$\Rightarrow \left(\sum_{i=1}^{p}|\beta_i|\right)^{\frac{1}{p}} \leq \frac{(4R)^p}{|Det\ \mathbb{R}(G)|^{1-\frac{1}{p}}}$$

$$\Rightarrow \mathbb{R}E(G) \leq \frac{(4R)^{p^2}}{|Det\ \mathbb{R}(G)|^{p-1}}.$$

which gives an upper bound for $\mathbb{R}E(G)$

Consider the following matrix form and apply Holder's Inequality.

$$\begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1p} \\ x_{21} & x_{22} & \cdots & x_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ x_{p1} & x_{p2} & \cdots & x_{pp} \end{pmatrix} = \begin{pmatrix} |\beta_1| & |\beta_1| & \cdots & |\beta_1| \\ |\beta_2| & |\beta_2| & \cdots & |\beta_2| \\ \vdots & \vdots & \ddots & \vdots \\ |\beta_p| & |\beta_p| & \cdots & |\beta_p| \end{pmatrix}$$

We get,

$$(p|\beta_1|)^{\frac{1}{p}} + (p|\beta_2|)^{\frac{1}{p}} + \cdots + (p|\beta_p|)^{\frac{1}{p}} \geq p(|\beta_1||\beta_2| \dots |\beta_p|)^{\frac{1}{p}}$$

$$\Rightarrow \left(\sum_{i=1}^{p}|\beta_i|\right)^{\frac{1}{p}} \geq p^{1-\frac{1}{p}}|Det\ \mathbb{R}(G)|^{\frac{1}{p}}$$

$$\sum_{i=1}^{p}|\beta_i| \geq \left(\sum_{i=1}^{p}|\beta_i|\right)^{\frac{1}{p}} \geq p^{1-\frac{1}{p}}|Det\ \mathbb{R}(G)|^{\frac{1}{p}}$$

$$\sum_{i=1}^{p}|\beta_i| \geq p^{\frac{p-1}{p}}|Det\ \mathbb{R}(G)|^{\frac{1}{p}}$$

$$\mathbb{R}E(G) \geq p^{\frac{p-1}{p}}|Det\ \mathbb{R}(G)|^{\frac{1}{p}}.$$

which gives the lower bound for $\mathbb{R}E(G)$.

$$\therefore p^{\frac{p-1}{p}}|Det\ \mathbb{R}(G)|^{\frac{1}{p}} \leq \mathbb{R}E(G) \leq \frac{(4R)^{p^2}}{|Det\ \mathbb{R}(G)|^{p-1}}.$$

Hence the theorem.

Theorem 3.2.5:

If G represents a graph with $p \geq 2$ vertices and $\mathbb{R}(G)$ be a non-singular reachability matrix then

$$\frac{2R}{p} + \left[\frac{|Det\ \mathbb{R}(G)|}{\frac{2R}{p}}\right]^{\frac{1}{p-1}} \leq \mathbb{R}E(G) \leq \sqrt{2R} + \frac{(p-1)2R}{|Det\ \mathbb{R}(G)|^{\frac{1}{p}}}.$$

Proof:

Using *Arithmetic- Geometric Mean Inequality* for $(p-1)\ real\ numbers\ |\beta_2|, |\beta_3|, \dots, |\beta_p|$, we get,

$$\frac{|\beta_2| + |\beta_3| + \cdots + |\beta_p|}{p-1} \geq |\beta_2\beta_3 \dots \beta_p|^{\frac{1}{p-1}}$$

$$|\beta_2| + |\beta_3| + \cdots + |\beta_p| \geq \frac{|\beta_2| + |\beta_3| + \cdots + |\beta_p|}{p-1} \geq \frac{|\beta_1\beta_2\beta_3 \dots \beta_p|^{\frac{1}{p-1}}}{|\beta_1|^{\frac{1}{p-1}}}$$

$$|\beta_2| + |\beta_3| + \cdots + |\beta_p| \geq \frac{|Det\ \mathbb{R}(G)|^{\frac{1}{p-1}}}{|\beta_1|^{\frac{1}{p-1}}}$$

$$\Rightarrow \sum_{i=2}^{p}|\beta_i| - |\beta_1| \geq \frac{|Det\ \mathbb{R}(G)|^{\frac{1}{p-1}}}{|\beta_1|^{\frac{1}{p-1}}}$$

Since $|\beta_1| \geq \frac{2R}{p}$

$$\Rightarrow \mathbb{R}E(G) \geq \frac{2R}{p} + \left[\frac{|Det\ \mathbb{R}(G)|}{\frac{2R}{p}}\right]^{\frac{1}{p-1}}.$$

which gives the lower bound for $\mathbb{R}E(G)$.

We know, $|\beta_1| \geq |Det\ \mathbb{R}(G)|^{\frac{1}{p}}$

$$|\beta_1|\sum_{i=2}^{p}|\beta_i| \geq |Det\ \mathbb{R}(G)|^{\frac{1}{p}}\sum_{i=2}^{p}|\beta_i|$$

Since $|\beta_1| \geq |\beta_i|\ \forall i,$

$$\Rightarrow (p-1)|\beta_1|^2 \geq |Det\ \mathbb{R}(G)|^{\frac{1}{p}}[\mathbb{R}E(G) - |\beta_1|]$$

$$\mathbb{R}E(G) \leq \frac{(p-1)|\beta_1|^2}{|Det\ \mathbb{R}(G)|^{\frac{1}{p}}} + |\beta_1|$$

Since $|\beta_1|^2 \leq 2R$

Thus,

$$\mathbb{R}E(G) \leq \frac{(p-1)2R}{|Det\ \mathbb{R}(G)|^{\frac{1}{p}}} + \sqrt{2R}.$$

Which gives an upper bound for $\mathbb{R}E(G)$.

$$\therefore \frac{2R}{p} + \left[\frac{|Det\ \mathbb{R}(G)|}{\frac{2R}{p}}\right]^{\frac{1}{p-1}} \leq \mathbb{R}E(G) \leq \sqrt{2R} + \frac{(p-1)2R}{|Det\ \mathbb{R}(G)|^{\frac{1}{p}}}.$$

Hence the theorem.

4. CONSTRAINTS ON THE REACHABILITY ESTRADA INDEX :

In this section, we derive bounds for Reachability Estrada index. Additionally, we will determine an upper bound for the Reachability Estrada index in terms of the reachability energy of graphs.

Theorem 4.1:

Let G be a connected graph with diameter less than or equal to 2 then the reachability Estrada index is bounded as

$$\sqrt{p^2 + 4R} \leq \mathbb{R}EE(G) \leq p - 1 + e^{\sqrt{2R}}.$$

Proof:

From the definition 1.4, $\mathbb{R}EE(G) = \sum_{i=1}^{p} e^{\beta_i}$

$$\mathbb{R}EE^2(G) = \left(\sum_{i=1}^{p} e^{\beta_i}\right)^2 = \sum_{i=1}^{p} e^{2\beta_i} + 2\sum_{1\leq i<j\leq p} e^{\beta_i}e^{\beta_j}$$

Consider the 2nd term of the above equation, by using *Arithmetic-Geometric Mean Inequality*, we have

$$2\sum_{1\leq i<j\leq p} e^{\beta_i}e^{\beta_j} \geq p(p-1).$$

Consider,

$$\sum_{i=1}^{p} e^{2\beta_i} = \sum_{i=1}^{p}\sum_{k\geq 0}\frac{(2\beta_i)^k}{k!} = p + 4R + \sum_{i=1}^{p}\sum_{k\geq 3}\frac{(2\beta_i)^k}{k!}$$

Since we aim to obtain the best possible lower bound, it is reasonable to approximate $\sum_{k\geq 3}\frac{(2\beta_i)^k}{k!}$ by $4\,\frac{(\beta_i)^k}{k!}$. Additionally, we introduce a multiplier $t \in [0,4]$ instead of 4 in above expression. We get,

$$\sum_{i=1}^{p} e^{2\beta_i} \geq p(1-t) + (4-t)R + t.\mathbb{R}EE(G)$$

Then, $\mathbb{R}EE^2(G) \geq p^2 + 4R + t\,[\mathbb{R}EE(G) - R - p]$

For $p \geq 2$, the best lower bound for $\mathbb{R}EE(G)$ is attained when $t = 0$.

$$\mathbb{R}EE(G) \geq \sqrt{p^2 + 4R}.$$

which gives the required lower bound for $\mathbb{R}EE(G)$.

From the definition 1.4,

$$\mathbb{R}EE(G) \leq p + \sum_{i=1}^{p} \sum_{k \geq 1} \frac{|\beta_i|^k}{k!}$$

$$\leq p + \sum_{k \geq 1} \frac{1}{k!} \sum_{i=1}^{p} ((\beta_i)^2)^{\frac{k}{2}} = p + \sum_{k \geq 1} \frac{1}{k!} (2R)^{\frac{k}{2}}$$

$$= p - 1 + \sum_{k \geq 0} \frac{\left(\sqrt{2R}\right)^k}{k!}$$

$$\mathbb{R}EE(G) \leq p - 1 + e^{\sqrt{2R}}.$$

which gives required upper bound for $\mathbb{R}EE(G)$.

$$\therefore \sqrt{p^2 + 4R} \leq \mathbb{R}EE(G) \leq p - 1 + e^{\sqrt{2R}}.$$

Hence the theorem.

Theorem 4.2:

Let G be a connected graph of order p then

$$\frac{\left(\sum_{i=1}^{p} e^{\frac{\beta_i}{2}}\right)^2 - p}{p - 1} \leq \mathbb{R}EE(G) \leq \left(\sum_{i=1}^{p} e^{\frac{\beta_i}{2}}\right)^2 - p(p-1).$$

Proof:

Let a_i, $1 \leq i \leq p$ be any real numbers, then

$$p\left[\frac{1}{p}\sum_{i=1}^{p} a_i - \left(\prod_{i=1}^{p} a_i\right)^{\frac{1}{p}}\right] \leq p\sum_{i=1}^{p} a_i - \left(\sum_{i=1}^{p} \sqrt{a_i}\right)^2 \leq p(p-1)\left[\frac{1}{p}\sum_{i=1}^{p} a_i - \left(\prod_{i=1}^{p} a_i\right)^{\frac{1}{p}}\right]$$

By setting $a_i = e^{\beta_i}$ for $i = 1,2, \ldots, p$, we have

$$p\left[\frac{1}{p}\sum_{i=1}^{p} e^{\beta_i} - \left(\prod_{i=1}^{p} e^{\beta_i}\right)^{\frac{1}{p}}\right] \leq p\sum_{i=1}^{p} e^{\beta_i} - \left(\sum_{i=1}^{p} \sqrt{e^{\beta_i}}\right)^2 \leq p(p-1)\left[\frac{1}{p}\sum_{i=1}^{p} e^{\beta_i} - \left(\prod_{i=1}^{p} e^{\beta_i}\right)^{\frac{1}{p}}\right]$$

Consider,

$$p\left[\frac{1}{p}\sum_{i=1}^{p} e^{\beta_i} - \left(\prod_{i=1}^{p} e^{\beta_i}\right)^{\frac{1}{p}}\right] \leq p\sum_{i=1}^{p} e^{\beta_i} - \left(\sum_{i=1}^{p} \sqrt{e^{\beta_i}}\right)^2$$

$$\sum_{i=1}^{p} e^{\beta_i} - p\left(e^{\sum_{i=1}^{p} \beta_i}\right)^{\frac{1}{p}} \leq p\sum_{i=1}^{p} e^{\beta_i} - \left(\sum_{i=1}^{p} \sqrt{e^{\beta_i}}\right)^2$$

$$\Rightarrow \left(\sum_{i=1}^{p} e^{\frac{\beta_i}{2}}\right)^2 - p \leq (p-1)\sum_{i=1}^{p} e^{\beta_i}$$

$$\sum_{i=1}^{p} e^{\beta_i} \geq \frac{\left(\sum_{i=1}^{p} e^{\frac{\beta_i}{2}}\right)^2 - p}{p-1}$$

$$\mathbb{R}EE(G) \geq \frac{\left(\sum_{i=1}^{p} e^{\frac{\beta_i}{2}}\right)^2 - p}{p-1}.$$

which gives the lower bound for $\mathbb{R}EE(G)$.

Consider,

$$p\sum_{i=1}^{p} e^{\beta_i} - \left(\sum_{i=1}^{p} \sqrt{e^{\beta_i}}\right)^2 \leq p(p-1)\left[\frac{1}{p}\sum_{i=1}^{p} e^{\beta_i} - \left(\prod_{i=1}^{p} e^{\beta_i}\right)^{\frac{1}{p}}\right]$$

$$p\sum_{i=1}^{p} e^{\beta_i} - \left(\sum_{i=1}^{p} \sqrt{e^{\beta_i}}\right)^2 \leq (p-1)\sum_{i=1}^{p} e^{\beta_i} - p(p-1)\left(e^{\sum_{i=1}^{p} \beta_i}\right)^{\frac{1}{p}}$$

$$\Rightarrow \sum_{i=1}^{p} e^{\beta_i} \leq \left(\sum_{i=1}^{p} e^{\frac{\beta_i}{2}}\right)^2 - p(p-1)$$

$$\mathbb{R}EE(G) \leq \left(\sum_{i=1}^{p} e^{\frac{\beta_i}{2}}\right)^2 - p(p-1).$$

which gives an upper bound for $\mathbb{R}EE(G)$.

$$\therefore \frac{\left(\sum_{i=1}^{p} e^{\frac{\beta_i}{2}}\right)^2 - p}{p-1} \leq \mathbb{R}EE(G) \leq \left(\sum_{i=1}^{p} e^{\frac{\beta_i}{2}}\right)^2 - p(p-1).$$

4.1 AN UPPER BOUND ON THE REACHABILITY ESTRADA INDEX RELATING TO REACHABILITY ENERGY

In this section, we will use reachability energy to establish two upper bounds for the reachability Estrada index $\mathbb{R}EE(G)$, where G is a connected graph with a diameter of at most 2.

Theorem 4.1.1:

Let G be a connected graph of diameter not greater than 2 then

$$\mathbb{R}EE(G) - \mathbb{R}E(G) \leq p - 1 - \sqrt{2R} + e^{\sqrt{2R}} \ \& \ \mathbb{R}EE(G) \leq p - 1 + e^{\mathbb{R}E(G)}.$$

Proof:

We have, $\mathbb{R}EE(G) = p + \sum_{i=1}^{p} \sum_{k\geq 1} \frac{(\beta_i)^k}{k!} \leq p + \sum_{i=1}^{p} \sum_{k\geq 1} \frac{|\beta_i|^k}{k!}$

By definition of reachability energy, we have

$$\mathbb{R}EE(G) - \mathbb{R}E(G) \leq p - 1 - \sqrt{2R} + e^{\sqrt{2R}}$$

Another approach to relate $\mathbb{R}EE(G)$ and $\mathbb{R}E(G)$ can be expressed as follows:

$$\mathbb{R}EE(G) \leq p + \sum_{i=1}^{p} \sum_{k\geq 1} \frac{|\beta_i|^k}{k!} \leq p + \sum_{k\geq 1} \frac{1}{k!}\left(\sum_{i=1}^{p} |\beta_i|^k\right) \leq p - 1 + \sum_{k\geq 0} \frac{\left(\mathbb{R}E(G)\right)^k}{k!}$$

$$\mathbb{R}EE(G) \leq p - 1 + e^{\mathbb{R}E(G)}.$$

Hence the result.

Conclusion:

In this paper, we have defined and derived the reachability energy and reachability Estrada index for a connected graph of order p. Additionally, we have established bounds for both the energy and index of the connected graph.

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