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On Cordial Labeling of Copper-Oxide and its Extended Networks

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A B S T R A C T

Let G(V,E) be a simple graph with vertex set V(G) and edge set E(G). Let ϱ be a labeling (binary) from the vertex set of G to {0,1}. The mapping ϱ induces an edge labeling 〖ϱ 〗^*:E(G)→{0,1} defined by 〖ϱ 〗^* (wv)=|ϱ(w)−ϱ(𝑣)| for all edges wv in the graph G. Let V_ϱ (j) be the set of vertices w of G with ϱ(w)=j and E_ϱ (j) be the set of edges wv of E(G) with 〖ϱ 〗^* (wv)=j. The cardinalities of V_ϱ (0), V_ϱ (1), E_ϱ (0) and E_ϱ (1) are denoted by v_ϱ (0), v_ϱ (1), e_ϱ (0) and e_ϱ (1) respectively. A labeling ϱ is called cordial labeling if it satisfies the conditions |V_ϱ (0)-V_ϱ (1)| ⩽ 1 and |E_ϱ (0)-E_ϱ (1)| ⩽ 1. A graph is cordial, then it satisfies the conditions of cordial labeling. In this paper, we have completely presented the cordiality of Copper-Oxide and its extended networks containing m rows and n columns of octagons.

Keywords: labeling cordial labelling copper-oxide extended copper-oxide networks

1. Introduction

Graph labeling was first introduced in the middle of sixties. In the intervening years, dozens of graph labeling techniques have been studied in over thousands and is still getting embellished due to increasing number of application driven concepts. Most graphs labelings trace their origins to labelings presented by Alex Rosa [1] in his paper in 1967. The application of magic labeling is widely studied in [12], [28], [21], [27], [30]. To establish the maximum number of stations in a particular electromagnetic spectrum bandwidth, the concept of radio labeling and its related works were studied in [7], [14], [16], [15], [8]. The epidemic spread related concepts were studied through labeling technique termed as burning number problem in [3], [4], [9], [26]. Recently Kins et.al. [17] studied its application in RFI avoiding for Robotic surgery.

The concept of cordial labeling was first introduced by Cahit [1]. This labeling method assigns labels to the vertices or edges of a graph in a way that balances the number of elements assigned to each label. The formal definition of cordial labeling is given below: Let $G(V,E)$ be a simple graph with vertex set $V(G)$ and edge set $E(G)$. Let ϱ be a labeling (binary) from the vertex set of G to {0,1}. The mapping ϱ induces an edge labeling $\varrho^*: E(G) \rightarrow$ {0,1} defined by $\varrho^*(wv)$=|$\varrho(w)$−$\varrho(v)$| for all edges wv in the graph G. Let $V_\varrho(j)$ be the set of vertices w of G with $\varrho(w) = j$ and $E_\varrho(j)$ be the set of edges wv of $E(G)$ with $\varrho^*(wv) = j$. The cardinalities of $V_\varrho(0)$, $V_\varrho(1)$, $E_\varrho(0)$ and $E_\varrho(1)$ are denoted by $v_\varrho(0)$, $v_\varrho(1)$, $e_\varrho(0)$ and $e_\varrho(1)$ respectively. A labeling ϱ is called cordial labeling if it satisfies the conditions $|V_\varrho(0) - V_\varrho(1)| \leq 1$ and $|E_\varrho(0) - E_\varrho(1)| \leq 1$. If a graph is claimed to be cordial, then it satisfies the conditions of cordial labeling. The dynamic survey of graph labeling by Gallian [3] provides a comprehensive overview of various labeling techniques, including cordial labeling and its applications across different types of graphs. The application of cordial labeling has been extensively investigated in [23], [19], [11], [2]. Various aspects of cordial labeling techniques were explored in [6],[19].[20],[22].

Recently, Yenoke et.al [18] has specifically explored the L(2,1) labeling problem for copper oxide and its extended networks. This study expands on the classical notions of cordial labeling, and shows that Copper-Oxide network $\mathrm{CuO(m,n)}$, Extended Copper-Oxide network $CuO_{EX}(m,n)$ and Enhanced Copper-Oxide network $CuO_{EN}(m,n)$ are cordial.

2. Copper oxide and its extended networks

Copper (II) Oxide ($\boldsymbol{CuO}$) [29],[10] network is structured as follows: The octagon structure of Copper (II) Oxide is joined to each other in rows and columns. The link between two octagons is joined by forming each $\boldsymbol{Cu4}$ bond between two octagons. The obtained network structure called as Copper-Oxide network with $\boldsymbol{m}$ rows and $\boldsymbol{n}$ columns of octagons. It is denoted by $\boldsymbol{CuO(m,n)}$. It is illustrated in figure 1(a). The number of vertices and edges are $\boldsymbol{4mn + m + 3n}$ and $\boldsymbol{2n(m + 1)}$ respectively.

In order to prove the theory, we have named the vertices of Copper-Oxide and its extended networks as follows:

Let the vertex in j^{th} linear row and i^{th} linear column be named as w_i^j, $1 \leq j \leq m+1, 1 \leq i \leq 3n$. The vertices which lie between odd and even or even and odd linear rows are marked by v_i^j, $i = 1,2 \dots n+1$, $j = 1,2 \dots m$. Rest of the vertices contained inside each octagon for $CuO_{EN}(m,n)$ are marked by u_i^j, $i = 1,2 \dots n$, $j = 1,2 \dots m$. This partition of vertex set is visible in Figure 1(a).

The constructions of extended networks given by Kins et.al in [18] are as follows:

2.1 Coper-Oxide Network.

The chemical structure of Copper (II) Oxide has an octagonal structure that is joined together in rows and columns. Cu4 bonds are formed between two octagons to connect them. The resulting network topology, known as the Copper-Oxide network, consists of *m* rows and *n* columns of octagons. It is designated as $CuO\ (m,n)$. The cardinality of vertex and edge sets in CuO is $4mn + m + 3m$ and $2n + m + 1$, respectively.

2.2 Extended Copper-Oxide networks.

In the extended Copper-Oxide network $CuO_{EX}(m,n)$, if we place a vertex in each face of an octagon to form a wheel W_{8+1} then the resulting obtained derived structure is denoted by $CuO_{EN}(m,n)$ and is named as Enhanced Coper-Oxide networks. Further, the cardinality of vertex and edge sets are $5mn + m + 3n$ and $10mn + 2n$ respectively.

3. $Cordial$ Labeling of $CuO(m,n)$, $CuO_{EX}(m,n)$ and $CuO_{EN}(m,n)$ Networks

In this section, we have completely proved that copper oxide and two of its derived networks are cordial.

Theorem 3.1 Let $CuO(m,n)$ be *a Copper-Oxide network with* m *rows and* n *columns of octagons. Then for any m, n* $\in \mathbb{N}$, $CuO(m,n)$ *is cordial.*

Proof: Define a mapping $\varrho: V\big(CuO(m,n)\big) \to \{0,1\}$ as follows:

$$\varrho(w_i^{2j-1}) = \begin{cases} 0 & if\ i \equiv 2,3,4,7\ mod\ 9 \\ 1 & if\ i \equiv 0,1,5,6,8\ mod\ 9 \end{cases} \quad i = 1,2 \dots 3n,\ j = 1,2 \dots \left\lceil \frac{m+1}{2} \right\rceil,$$

$$\varrho(w_i^{2j}) = \begin{cases} 0 & if\ i \equiv 0,1,5,6,8\ mod\ 9 \\ 1 & if\ i \equiv 2,3,4,7\ mod\ 9 \end{cases} \quad i = 1,2 \dots 3n,\ j = 1,2 \dots \left\lfloor \frac{m+1}{2} \right\rfloor,$$

$$\varrho(v_i^{2j-1}) = \begin{cases} 0 & if\ i \equiv 0\ mod\ 2 \\ 1 & if\ i \equiv 1\ mod\ 2 \end{cases} \quad i = 1,2 \dots n+1,\ j = 1,2 \dots \left\lfloor \frac{m+2}{2} \right\rfloor,$$

$$\varrho(v_i^{2j}) = \begin{cases} 0 & if\ i \equiv 1\ mod\ 2 \\ 1 & if\ i \equiv 0\ mod\ 2 \end{cases} \quad i = 1,2 \dots n+1,\ j = 1,2 \dots \left\lfloor \frac{m}{2} \right\rfloor.$$ The mapping is visible in Figure 1(b).

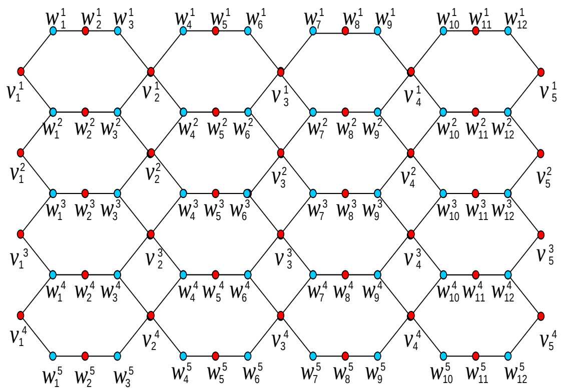

Figure 1(a): Naming of vertices in Copper-Oxide network

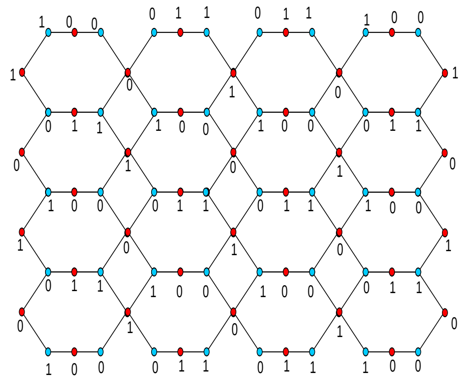

Figure 1(b): $Cordial$ **labeling for** $CuO(4,4)$

Next, we verify the above labeling pattern satisfies vertex cordial condition $|V_\varrho(0) - V_\varrho(1)| \le 1$ and edge cordial condition $|E_\varrho(0) - E_\varrho(1)| \le 1$ for every distinct pair of copper and oxide nodes in $CuO(m,n)$.

Let us dissect the proof of vertex cordial condition in four distinct cases for $CuO(m,n)$.

***Case* 1:** Assume $CuO(m,n)$ network has even *m* rows and odd *n* columns of octagons.

***Case* 1.1**: The number of *'j'* linear rows and *'i'* linear columns for w_i^j is odd. Vertices of these rows are of the form $2y + 1$ where $w_i^j(1)$ and $w_i^j(0)$ complement each other in the even $2y$ rows as seen in the mapping from Figure 1(b). Hence, we only need to check $\varrho(w_i^j)$ for one last row to prove the vertex condition of cordiality. This row contains an odd $2z + 1$ columns of vertices. For the $2z$ columns of w_i^j vertices, $w_i^j(1) = w_i^j(0) = z$, which implies there exists only one vertice of label 1,0 such that $w_i^j(1,0) = w_i^j(0,1) + 1$.

Thus,

$|w_i^j(1) - w_i^j(0)| = 1$-------------------------------(1)

***Case* 1.2**: The number of *'j'* linear rows and *'i'* linear columns for v_i^j is even. For vertices marked by v_i^j, labels of 1,0 alternate for consecutive rows and columns. As the *'i'* linear columns is even, the cardinality of v_i^j vertices is also even. This implies that $v_i^j(1) = v_i^j(0)$ for all $i = 1, 2 \ldots n+1,\ j = 1, 2 \ldots \left\lfloor \frac{m+2}{2} \right\rfloor$.

Therefore,

$\left|v_i^j(1) - v_i^j(0)\right| = 0$ ------------------------------(2)

From (1) and (2),

Vertex condition of cordiality for all vertices $(w_i^j + v_i^j)$ in the graph $CuO(m,n)$ with even *m* rows and odd *n* columns of octagons is given by,

$|w_i^j(1) - w_i^j(0)| + |v_i^j(1) - v_i^j(0)| = 1 + 0$, which implies that $|V_\varrho(0) - V_\varrho(1))| = 1$.

***Case* 2:** Assume $CuO(m,n)$ network has odd *m* rows and even *n* columns of octagons.

***Case* 2.1**: The number of *'j'* linear rows and *'i'* linear columns for w_i^j is even. Vertices of these rows are of the form $2y$ which implies $w_i^j(1)$ and $w_i^j(0)$ complement each other in alternating rows. The cardinality of *'i'* linear columns is moot as the number of *'j'* linear rows is even.

Thus,

$\left|w_i^j(1) - w_i^j(0)\right| = 0$ -------------------------(3)

***Case* 2.2**: The number of *'j'* linear rows and *'i'* linear columns for v_i^j is odd. Vertices of these rows are of the form $2y+1$ where $v_i^j(1)$ and $v_i^j(0)$ complement each other in the even $2y$ rows. Hence, we only need to check $\varrho(v_i^j)$ for one last row to prove the vertex condition of cordiality. This row contains an odd $2z+1$ columns of vertices. For the $2z$ columns of v_i^j vertices, $v_i^j(1) = v_i^j(0) = z$, which implies there exists only one vertice labeled by $\varrho(w_i^j) = 0{,}1$ such that $v_i^j(1{,}0) = v_i^j(0{,}1) + 1$.

Thus,

$|v_i^j(1) - v_i^j(0)| = 1$ -------------------------(4)

From (3) and (4),

Vertex condition of cordiality for all vertices $(w_i^j + v_i^j)$ in the graph $CuO(m,n)$ with odd *m* rows and even *n* columns of octagons is given by

$|w_i^j(1) - w_i^j(0)| + |v_i^j(1) - v_i^j(0)| = 0 + 1$, which satisfies the condition $|V_\varrho(0) - V_\varrho(1))| = 1$.

***Case* 3:** Assume $CuO(m,n)$ network has odd *m* rows and *n* columns of octagons.

***Case* 3.1**. The number of *'j'* linear rows is even and *'i'* linear columns for w_i^j is odd. Vertices of these rows are of the form $2y$ which implies $w_i^j(1)$ and $w_i^j(0)$ complement each other in alternating rows. The cardinality for *'i'* linear columns of w_i^j is moot as the number of 'j' linear rows is even.

Thus,

$|w_i^j(1) - w_i^j(0)| = 0$--------------------(5)

***Case* 3.2**. The number of *'j'* linear rows is odd and *'i'* linear columns is even. For vertices marked by v_i^j, labels of 1,0 alternate for consecutive rows and columns. As the *'i'* linear columns is even, the cardinality of v_i^j vertices is also even. This implies that $v_i^j(1) = v_i^j(0)$ for all $i = 1, 2 \ldots n+1,\ j = 1, 2 \ldots \left\lfloor \frac{m+2}{2} \right\rfloor$.

Therefore,

$\left|v_i^j(1) - v_i^j(0)\right| = 0$ -----------------------(6)

From (5) and (6),

Vertex condition of cordiality for all vertices $(w_i^j + v_i^j)$ in the graph $CuO(m,n)$ with odd *m* rows and *n* columns of octagons is given by

$|w_i^j(1) - w_i^j(0)| + |v_i^j(1) - v_i^j(0)| = 0 + 0$. Thus, the cordial condition $|V_\varrho(0) - V_\varrho(1)| = 0$ holds true.

***Case* 4:** Assume $CuO(m,n)$ network has even *m* rows and *n* columns of octagons.

***Case* 4.1**. The number of *'j'* linear rows is odd and *'i'* linear columns for w_i^j is even. Vertices of these rows are of the form $2y+1$ where $w_i^j(1)$ and $w_i^j(0)$ complement each other in the even $2y$ rows. Hence, we only need to check $\varrho(w_i^j)$ for one last row to prove the vertex condition of cordiality. This row contains an even $2z$ column of vertices which implies that $w_i^j(1) = w_i^j(0) = 0$.

Thus,

$\left|w_i^j(1) - w_i^j(0)\right| = 0$-------------------------(7)

***Case* 4.2**. The number of *'j'* linear rows is even and *'i'* linear columns is odd. For v_i^j vertices, labels of 1,0 alternate for consecutive rows and columns. As the *'j'* linear rows is even, the number of vertices marked by v_i^j is also even. This implies that $v_i^j(1) = v_i^j(0)$ for all $i = 1, 2 \ldots n+1,\ j = 1, 2 \ldots \left\lfloor \frac{m+2}{2} \right\rfloor$.

Therefore,

$$|v_i^j(1) - v_i^j(0)| = 0 \text{------------------------}(8)$$

From (7) and (8),

Vertex condition of cordiality for all vertices $(w_i^j + v_i^j)$ in the graph $CuO(m,n)$ with even *m* rows and *n* columns of octagons is given by

$|w_i^j(1) - w_i^j(0)| + |v_i^j(1) - v_i^j(0)| = 0 + 0$. That is, $|V_\varrho(0) - V_\varrho(1)| = 0$.

As vertex condition of cordiality is satisfied for all four cases, $CuO(m,n)$ is vertex cordial.

Next, we verify the edge cordial conditions. We notice that the edges in $CuO(m,n)$ receive its edge labeling in two different cases. We partitioned these two sets of edges in $CuO(m,n)$ and named as E_1 and E_2 respectively. Now define the edge labeling as follows.

$\varrho^*(e_1) = \varrho(w_i^j) \equiv \varrho(w_{i+1}^j)\ mod\ 2$, when $i \equiv 1,2$ mod 3 $\forall\, e_1 \in E_1$

$\varrho^*(e_2) = \varrho(v_i^j) \equiv \varrho(w_{3i-2}^j,\ w_{3i-3}^j,\ w_{3i-2}^{j+1},\ w_{3i-3}^{j+2})\ mod\ 2$, when $i > 1$ $\forall\, e_2 \in E_2$

In each individual octagon, the edges are labeled in the pattern (1,0,0) or (0,1,1). So, $e_1(1) = e_1(0) = 1$ in every individual octagon. There exists an overlap of e_1 edges in consecutive octagons but it remains moot as $e_1(1) = e_1(0) = 1$.

Hence,

$|E_1(1) - E_1(0)| = 0$ -----------------------------(9)

Edge e_2 is mapped to four distinct w_i^j vertices from every v_i^j. Either $\varrho(v_i^j) = 1$ or $\varrho(v_i^j) = 0$, That is, the four inner edges of the octagon are labelled as alternatively 0 and 1 respectively. Thus, each v_i^j contains the label 1 or 0 is mapped to two w_i^j vertices of label 1 and two w_i^j vertices of label 0 resulting in two e_2 edges with label 1 and two e_2 edges with label 0. Also, there is no overlap of e_2 edges as each v_i^j vertice is mapped to four unique w_i^j vertices.

Hence,

$|E_2(1) - E_2(0)| = 0$ ------------------------(10)

From (9) and (10),

$|E_1(1) - E_1(0)| + |E_2(1) - E_2(0)| = 0 + 0$. That is, $|E_\varrho(0) - E_\varrho(1)| = 0$.

As edge condition of cordiality is satisfied for all possible edges $(e_1 \in E_1,\ e_2 \in E_2)$, $CuO(m,n)$ is edge cordial.

Therefore, the copper-oxide network is cordial.

Theorem 3.2 *Let $CuO_{EX}(m,n)$ be an extended Copper-Oxide network with m octagon rows and n octagon columns. Then, the network $CuO_{EX}(m,n)$ is cordial.*

Proof: As $CuO(m,n)$ is proved to be cordial, vertex condition of cordiality remains true for $CuO_{EX}(m,n)$ as both networks have similar mapping of vertices. Check Figure 2(a) for labeling of $CuO_{EX}(m,n)$.

The edges in $CuO_{EX}(m,n)$ receive its edge labeling in five different cases. We partitioned these five sets of edges in $CuO_{EX}(m,n)$ and named as E_1, E_2, E_3, E_4 *and* E_5 respectively. Now define the edge labeling as follows.

$\varrho^*(e_1) = \varrho(w_i^j) \equiv \varrho(v_{i+1}^j)\ mod\ 2$, when $i \equiv 1,2\ mod\ 3$ $\forall\, e_1 \in E_1$

$\varrho^*(e_2) = \varrho(v_i^j) \equiv \varrho(w_{3i-2}^j,\ w_{3i-3}^j,\ w_{3i-2}^{j+1},\ w_{3i-3}^{j+2})\ mod\ 2$, when $i > 1$ $\forall\, e_2 \in E_2$

$\varrho^*(e_3) = \varrho(w_i^j) \equiv \varrho(w_i^{j+1})\ mod\ 2$, when $i \equiv 2\ mod\ 3$ $\forall\, e_3 \in E_3$

$\varrho^*(e_4) = \varrho(v_i^j) \equiv \varrho(v_{i+1}^j)\ mod\ 2$ $\forall\, e_4 \in E_4$

$\varrho^*(e_5) = \varrho(w_i^j) \equiv \varrho(w_{i+2}^{j+1},\ w_{i+2}^{j-1})\ mod\ 2$, when $i \equiv 1$ mod 3 $\forall\, e_5 \in E_5$

As $CuO(m,n)$ is proved to be cordial, edge condition of cordiality remains true for all the outer edges $(e_1 \in E_1,\ e_2 \in E_2)$. To prove that $CuO_{EX}(m,n)$ is edge cordial, we need to show that E_3, E_4 *and* E_5 satisfy cordiality.

Each octagon in $CuO_{EX}(m,n)$ contains four inner edges (one edge of e_3 and e_4, *two edges of* e_5). For *m* rows and *n* columns of octagons in the network $CuO_{EX}(m,n)$, the number of horizontal and vertical inner edges is given by

$$|E_3| + |E_4| = 2mn$$

As each octagon contains an equal number of e_3, e_4 edges and that these edges do not overlap in any other octagon,

$$|E_3| = |E_4|, \text{ which implies } |E_3| = |E_4| = mn.$$

All e_3 edges are assigned with the label of 1 and e_4 edges with 0. Therefore, $|E_3(1)| = |E_4(0)|$ and $|E_3(1) - E_4(0)| = 0$.

From the four inner edges of $CuO_{EX}(m,n)$, the cardinality of two edges marked by e_5 are given by $2|E_5| = 2mn$ which implies $|E_5| = mn$.

Each octagon contains an edge e_5 with label 1 and an edge e_5 with label 0. In this case, $|E_5(1)| = |E_5(0)|$ which implies $|E_5(1) - E_5(0)| = 0$.

From all possible edge cordial conditions using edge sets E_1, E_2, E_3, E_4 and E_5 in $CuO_{EX}(m,n)$, we conclude that $|E_\varrho(0) - E_\varrho(1)| = 0$.

Thus, the extended copper-oxide network is cordial.

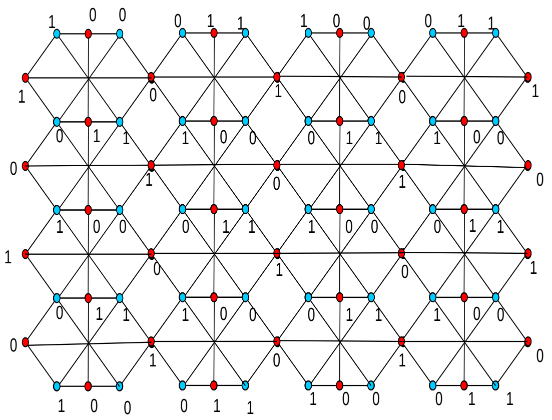

Fig 2. (a) - *Cordial* labeling of $CuO_{EX}(4,4)$

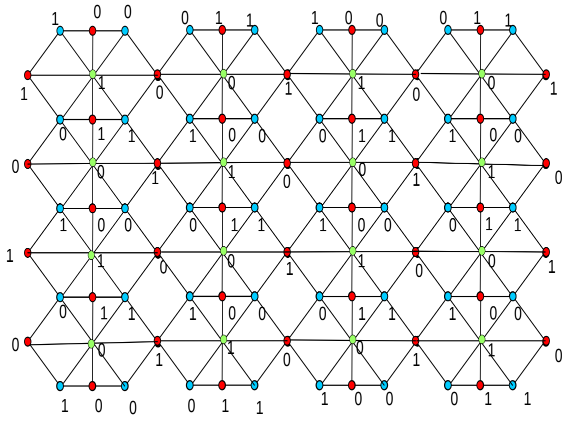

Fig 2. (b) - *Cordial* labeling of $CuO_{EN}(4,4)$

Theorem 3.3 *The enhanced Copper-Oxide network $CuO_{EN}(m,n)$ is cordial.*

Proof: Define a mapping $\varrho: V(CuO_{EN}(m,n)) \to \{0,1\}$ as follows:

$\varrho(w_i^{2j-1}) = \begin{cases} 0 & if\ i \equiv 2,3,4,7\ mod\ 9 \\ 1 & if\ i \equiv 0,1,5,6,8\ mod\ 9 \end{cases}$ $i = 1,2 \dots 3n,\ j = 1,2 \dots \left\lceil \frac{m+1}{2} \right\rceil$,

$\varrho(w_i^{2j}) = \begin{cases} 0 & if\ i \equiv 0,1,5,6,8\ mod\ 9 \\ 1 & if\ i \equiv 2,3,4,7\ mod\ 9 \end{cases}$ $i = 1,2 \dots 3n,\ j = 1,2 \dots \left\lfloor \frac{m+1}{2} \right\rfloor$,

$\varrho(v_i^{2j-1}) = \begin{cases} 0 & if\ i \equiv 0\ mod\ 2 \\ 1 & if\ i \equiv 1\ mod\ 2 \end{cases}$ $i = 1,2 \dots n+1,\ j = 1,2 \dots \left\lceil \frac{m+2}{2} \right\rceil$,

$\varrho(v_i^{2j}) = \begin{cases} 0 & if\ i \equiv 1\ mod\ 2 \\ 1 & if\ i \equiv 0\ mod\ 2 \end{cases}$ $i = 1,2 \dots n+1,\ j = 1,2 \dots \left\lfloor \frac{m}{2} \right\rfloor$,

$\varrho(u_i^{2j-1}) = \begin{cases} 0 & if\ i \equiv 0\ mod\ 2 \\ 1 & if\ i \equiv 1\ mod\ 2 \end{cases}$ $i = 1,2 \dots n,\ j = 1,2 \dots m$,

$\varrho(u_i^{2j}) = \begin{cases} 0 & if\ i \equiv 1\ mod\ 2 \\ 1 & if\ i \equiv 0\ mod\ 2 \end{cases}$ $i = 1,2 \dots n,\ j = 1,2 \dots m$. The mapping is visible in Figure 2(b).

The mapping of vertices in $CuO_{EN}(m,n)$ is similar to that of $CuO(m,n)$ except there exists an inner vertice u_i^j in each octagon. As $CuO(m,n)$ is already proved to be cordial, the vertices w_i^j, v_i^j satisfy vertex cordiality. Cardinality of the inner vertices u_i^j is given by,

$$|u_i^j| = mn$$

Consider the four cases stated for $CuO(m,n)$ to prove vertex cordiality for $CuO_{EX}(m,n)$.

For cases 1, 2 and 4,

$|u_i^j(1)| + |u_i^j(0)| = mn$, which implies $|u_i^j(1)| = |u_i^j(0)| = mn/2$. Hence, $|u_i^j(1) - u_i^j(0)| = 0$.

Vertices w_i^j, v_i^j have already been proved to satisfy cordiality for the above cases. Thus, all the vertices w_i^j, v_i^j, u_i^j for cases 1, 2 and 4 satisfy

$$|V_\varrho(0) - V_\varrho(1)| \leq 1$$

As *m* and *n* are odd in case 3, vertice u_i^j foll`ows

$|u_i^j(1) - u_i^j(0)| = 1$-----------------------(11)

Vertices w_i^j, v_i^j in case 3 have already been proved to satisfy

$|w_i^j, v_i^j(1) - w_i^j, v_i^j(0)| = 0$ ----------------------------(12)

From (11) and (12),

$$|V_\varrho(0) - V_\varrho(1)| = 1$$

As vertex condition of cordiality remains true for all four cases, $CuO_{EN}(m,n)$ is vertex cordial.

The edges in $CuO_{EN}(m,n)$ receive its edge labeling in three different cases. We partitioned these three sets of edges in $CuO_{EN}(m,n)$ and named as E_1, E_2, E_3 respectively. Now, we define the edge labeling as follows.

$\varrho^*(e_1) = \varrho(w_i^j) \equiv \varrho(v_{i+1}^j)\ mod\ 2$, when $i \equiv 1,2\ mod 3\ \ \forall\ e_1 \in E_1$

$\varrho^*(e_2) = \varrho(v_i^j) \equiv \varrho(w_{3i-2}^j,\ w_{3i-3}^j,\ w_{3i-2}^{j+1},\ w_{3i-3}^{j+2})\ mod\ 2\ \ \forall\ e_2 \in E_2$

$\varrho^*(e_3) = \varrho(u_i^j) \equiv \varrho(v_i^j,\ v_i^{j+1},\ w_i^j,\ w_i^{j+1},\ w_i^{j+2},\ w_{i+1}^j,\ w_{i+1}^{j+1},\ w_{i+1}^{j+2})\ mod\ 2\ \ \forall\ e_3 \in E_3$

As $CuO(m,n)$ is proved to be cordial, edge condition of cordiality remains true for all the outer edges ($e_1 \in E_1,\ e_2 \in E_2$). To prove that $CuO_{EN}(m,n)$ is edge cordial, we need to show that all edges $e_3 \in E_3$ satisfy cordiality. Due to the inner vertex u_i^j in $CuO_{EN}(m,n)$, it creates eight inner edges e_3 in each octagon without any overlap in other octagons and so $|E_3| = 8mn$. Whether the inner vertex is labeled by $\varrho(u_i^j) = 1,0$, u_i^j is mapped to six w_i^j and two v_i^j vertices. As these vertices are individually connected to u_i^j, the inner edges would always result in $|e_3(1)| = |e_3(0)| = 4$ for each octagon.

Therefore, $|E_3(1)| = |E_3(0)| = 4mn$, which implies $|E_3(1) - E_3(0)| = 0$.

As all possible edge sets $E_1, E_2\ and\ E_3$ in $CuO_{EN}(m,n)$ satisfy the edge condition of cordiality, $|E_\varrho(0) - E_\varrho(1)| = 0$.

Thus, the enhanced copper-oxide network is cordial.

4. Conclusions

In this research work, we have proved cordiality for different copper-oxide derived networks. The proof, along with mapping of vertices is illustrated above for $CuO(m,n)$, $CuO_{EX}(m,n)$ and $CuO_{EN}(m,n)$. This study can be extended by studying different graph theory problems for the copper-oxide networks. Also, conditions of cordiality can be proved for other chemical structures.

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