International Journal of Research Publication and Reviews, Vol (5), Issue (8), August (2024), Page – 2834-2840

International Journal of Research Publication and Reviews

Journal homepage: www.ijrpr.com ISSN 2582-7421

On Generalized Birecurrent Finsler Space of Mixed Covariant Derivatives in Cartan Sense

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ABSTRACT :

In this paper, we introduce a Finsler space which Cartan's third curvature tensor R^i_{jkh} satisfies the generalized birecurrence property by using the first and second kind of covariant derivatives simultaneously in Cartan sense. Further, we prove that some tensors are non-vanishing. Certain identities belong to main space have been studied.

Keywords: Generalized R^{hv} – mixed birecurrent space, hv −covariant derivative of mixed second order, Cartan's third curvature tensor R^i_{jkh} .

Introduction and Preliminaries

Various curvature tensors that satisfy the generalized birecurrent by using two kinds of covariant derivatives in sense of Cartan introduced by [6, 9, 16]. Qasem and Hadi [11-13] discussed Cartan's third curvature tensor R^i_{jkh}, Cartan's fourth curvature tensor K^i_{jkh} and Berwald curvature tensor H^i_{jkh} which are generalized birecurrent in sense of Cartan. Also, Assallal [8] introduced the generalized P^h −birecurrent space.

Let F_n be an n –dimensional Finsler space equipped with the metric function $F(x,y)$ satisfying the request conditions [4, 14, 17]. The vectors y_i and y^i defined by

(1.1) $y_i = g_{ij}(x,y)y^j$.

The metric tensor g_{ij} and its associative g^{ij} are connected by

(1.2) $g_{ij}g^{ik} = \delta^k_j = \begin{cases} 1 & if\ j = k\,, \\ 0 & if\ j \neq k\,. \end{cases}$

In view of (1.1) and (1.2), we have

(1.3) a) $\delta^i_j y_i = y_j$, b) $\delta^i_j\, y^j = y^i$, c) $\delta^i_j g_{ir} = g_{jr}$, d) $\delta^i_j\, g^{jk} = g^{ik}$ and d) $y_i\, y^i = F^2$.

The $(h)hv-$ torsion tensor which is positively homogeneous of degree -1 in y^i and symmetric in all its indices introduced and defined by [5, 15]

$$C_{ijk} = \tfrac{1}{2}\dot\partial_i\, g_{jk} = \tfrac{1}{4}\dot\partial_i\, \dot\partial_j\, \dot\partial_k\, F^2 .$$

And satisfies

(1.4) a) $C_{ijk}\, y^i = C_{kij}\, y^i = C_{jki}\, y^i = 0$ and b) $C^h_{jk} g_{ih} = C_{ijk}$.

$\acute{E}$. Cartan [7] deduced the covariant derivatives of an arbitrary vector field X^i with respect to x^k which given by

(1.5) $X^i\big|_k = \dot\partial_k X^i + X^r\, C^{\ i}_{rk}$

and

(1.6) $X^i_{|k} = \partial_k\, X^i - \left(\dot\partial_r\, X^i\right) G^{\ r}_{k} + X^r \Gamma^{*\,i}_{rk}$,

where the function $\Gamma^{*\,i}_{rk}$ is defined by $\Gamma^{*i}_{rk} = \Gamma^{\ i}_{rk} - C^{\ i}_{mr}\, \Gamma^{\ m}_{sk}\, y^s$. The functions Γ^{*i}_{rk} and G^r_k are connected by $G^{\ r}_{k} = \Gamma^{*r}_{sk}\, y^s$ where $\partial_j \equiv \frac{\partial}{\partial x^j}$, $\dot\partial_j \equiv \frac{\partial}{\partial y^j}$, $G^i_j = \dot\partial_j G^i$.

The equations (1.5) and (1.6) give two kinds of covariant differentiations which are called v −covariant differentiation (Cartan's first kind covariant derivative) and h −covariant differentiation (Cartan's second kind covariant derivative), respectively. So $X^i|_k$ and $X^i_{|k}$ are v −covariant derivative and h −covariant derivative of the vector field X^i. Therefore, v −covariant derivative and h −covariant derivative of the vectors y^i , y_i and metric tensors g_{ij} and its associative g^{ij} are satisfied [14]

(1.7) a) $g_{ij}\big|_k = 0$, b) $g_{ij|k} = 0$, c) $g^{ij}\big|_k = 0$, d) $g^{ij}_{|k} = 0$,

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e) $y^i_{|k} = 0$, f) $y^i|_k = \delta^i_k$, g) $y_{j|k} = 0$ and h) $y_j|_k = g_{jk}$.

The tensor R^i_{jkh} called *Cartan's third curvature tensor* is positively homogeneous of degree zero in y^i and defined by [1, 14]

$$R^i_{jkh} = \partial_h \Gamma^{*i}_{jk} + (\dot{\partial}_\ell \Gamma^{*i}_{jh}) G^\ell_k + G^i_{jm}(\partial_h G^m_k - G^{\ m}_{h\ell} G^{\ \ell}_k) + \Gamma^{*i}_{mh}\Gamma^{*m}_{jk} - h/k ,$$

This tensor satisfies

(1.8) a) $R^{\ i}_{jkh}\, y^{\,j} = H^{\ i}_{kh} = K^{\ i}_{jkh}\, y^{\,j}$, b) $R_{ijkh} = g_{rj} R^r_{ikh}$ and c) $R^i_{jkh} = g^{ri} R_{rjkh}$,

where $R_{\,ijkh}$ is the associative tensor of $R^{\ i}_{jkh}$.

R –Ricci tensor R_{jk}, deviation tensor R^r_h and curvature scalar R of the curvature tensor R^i_{jkh} are given by [14]

(1.9) a) $R^i_{jki} = R_{\,jk}$, b) $R_{jk}\, y^j = R_k$, c) $R^r_{ikh}\, g^{ik} = R^r_h$,

d) $R_{jk}\, g^{ij} = R^i_k$ and e) $g^{jk} R_{jk} = R$.

The tensor P^i_{jkh} called hv *–curvature tensor* (*Cartan's second curvature tensor*) is positively homogeneous of degree -1 in y^i and defined by [2, 3, 14]

$$P^i_{jkh} = \dot{\partial}_h \Gamma^{*i}_{jk} + C^i_{jr} P^{\ r}_{kh} - C^i_{jh|k} .$$ The $v\ (hv)$ –torsion tensor P^i_{kh} is given

(1.10) $P^i_{jkh}\, y^j = \Gamma^{*i}_{jkh} y^j = P^i_{kh} = C^{\ i}_{kh|r} y^r$,

The associate tensors $P_{\,ijkh}$ and $P_{\,jkh}$ of the hv –curvature tensor P^r_{jkh} and $v(hv)$ – torsion tensor P^r_{kh} are given by [14]

(1.11) a) $P_{ijkh} = g_{ir} P^r_{jkh}$ and b) $P_{jkh} = g_{jr} P^r_{kh}$.

P – Ricci tensor P_{jk}, curvature vector P_k and curvature scalar P of Cartan's second curvature tensor P^i_{jkh} are given by

(1.12) a) $P_{jk} = P^i_{jki}$, b) $P_k = P^i_{ki}$ and c) $P = P_k\, y^k$.

The tensor K^i_{jkh} called *Cartan's fourth curvature tensor* is positively homogeneous of degree zero in y^i and defined by [14]

$$K^{\ i}_{jkh} = \partial_h \Gamma^{*i}_{kj} + (\dot{\partial}_\ell \Gamma^{*i}_{jh}) G^{\ \ell}_k + \Gamma^{*i}_{mh}\Gamma^{*m}_{kj} - h/k^* .$$

The associate tensor K_{ijkh}, K –Ricci tensor K_{jk} , curvature scalar K and deviation tensor K^i_j of the curvature tensor $K^{\ i}_{jkh}$ are given by

(1.13) a) $K_{\,ijkh} = g_{\,rj}\, K^{\ r}_{ikh}$, b) $K^i_{jki} = K_{jk}$, c) $K_{jk} g^{\,ij} = K^i_k$,

d) $K_{jk} g^{\,jk} = K$ and e) $K_{jk} y^k = K_j$.

The Berwald curvature tensor $H^{\ i}_{jkh}$ is defined by [14]

$$H^{\ i}_{jkh} = \partial_h G^{\ i}_{jk} + G^{\ s}_{jk} G^{\ i}_{sh} + G^{\ i}_{sjh} G^{\ s}_k - h/k.$$

The torsion tensor $H^{\ i}_{kh}$ and deviation tensor $H^{\ i}_k$ are satisfied

(1.14) a) $H^{\ i}_{jkh}\, y^{\,j} = H^{\ i}_{kh}$ and b) $H^i_{jk}\, y^{\,j} = H^{\ i}_k$.

H – Ricci tensor H_{jk} , curvature vector H_k and scalar curvature H are given by

(1.15) a) $H^{\ i}_{jki} = H_{jk}$, b) $H^{\ i}_{ki} = H_k$ and c) $H^{\ i}_i = (n-1)\, H$.

A Generalized R^{hv} – mixed birecurrent space

This section introduces a Finsler space which R^i_{jkh} satisfies the generalized birecurrence property by using two kinds of covariant derivatives in Cartan sense. Qasem and AL-Qashbari [10] introduced Finsler space F_n which Cartan's third curvature tensor R^i_{jkh} satisfies the generalized recurrence property with respect to Cartan's second kind covariant derivative, i.e. characterized by the condition

(2.1) $R^i_{jkh|l} = \lambda_l R^i_{jkh} + \mu_l(\delta^i_k g_{jh} - \delta^i_h g_{jk})$, $R^i_{jkh} \neq 0$,

where $|l$ is the h –covariant derivative operator of first order with respect to x^l, also λ_l and μ_l are called recurrence covariant vectors. They called this space and tensor, the *generalized* R^h *–recurrent space* and *generalized* h *–recurrent tensor* and denoted them briefly by $GR^h - RF_n$ and $Gh - R$, respectively.

In the same vein, let us consider that the Cartan's third curvature tensor R^i_{jkh} satisfies the generalized recurrence property with respect to Cartan's first kind covariant derivative, i.e. characterized by the condition

(2.2) $R^i_{jkh}|_m = \lambda_m R^i_{jkh} + \mu_m(\delta^i_k g_{jh} - \delta^i_h g_{jk})$, $R^i_{jkh} \neq 0$,

where $|m$ is the v – covariant derivative operator of first order with respect to x^m. A Finsler space F_n which R^i_{jkh} satisfies the condition (2.2) will be called a *generalized* R^v *–recurrent space* and the tensor will be called a *generalized* v *–recurrent tensor*. This space and tensor denote them briefly by $GR^v - RF_n$ and $Gv - R$, respectively.

Now, taking v – covariant derivative for (2.1) and using (1.7a) with respect to x^m, we get

$$R^i_{jkh|l|m} = \lambda_{l|m} R^i_{jkh} + \lambda_l R^i_{jkh|m} + \mu_{l|m}(\delta^i_k g_{jh} - \delta^i_h g_{jk}).$$

Using the condition (2.2) in above condition, we get

$$R^i_{jkh|l|m} = \lambda_{l|m} R^i_{jkh} + \lambda_l\left[\lambda_m R^i_{jkh} + \mu_m(\delta^i_k g_{jh} - \delta^i_h g_{jk})\right] + \mu_{l|m}\left(\delta^i_k g_{jh} - \delta^i_h g_{jk}\right)$$

Or

$$R^i_{jkh|l|m} = (\lambda_{l|m} + \lambda_l\lambda_m) R^i_{jkh} + (\lambda_l\mu_m + \mu_{l|m})\, (\delta^i_k g_{jh} - \delta^i_h g_{jk}).$$

The above equation can be written as

(2.3) $R^i_{jkh|l|m} = a_{lm} R^i_{jkh} + b_{lm}\left(\delta^i_k g_{jh} - \delta^i_h g_{jk}\right), \quad R^i_{jkh} \neq 0$,

where $|l\big| m$ is $hv-$covariant differential operator of second mixed order with respect to x^l and x^m, respectively. Also, $a_{lm} = \lambda_{l|m} + \lambda_l\lambda_m$ and $b_{lm} = \mu_{l|m} + \lambda_l\mu_m$ are non $-$ zero covariant tensors field of second order.

Definition 2.1. A Finsler space F_n which Cartan's third curvature tensor R^i_{jkh} satisfies the condition (2.3) will be called a *generalized R^{hv} – mixed birecurrent space* and the tensor will be called a *generalized hv – mixed birecurrent tensor*. This space and tensor denote them briefly by $GR^{hv} - (M)BRF_n$ and $Ghv - (M)BR$, respectively.

Remark 2.1. The condition $R^i_{jkh|l|m} = a_{lm} R^i_{jkh},\ R^i_{jkh} \neq 0$, looks as a mixed birecurrent. Also considers as a particular case of the condition (2.3), but it does not do.

Now, transvecting the condition (2.3) by g_{ip}, using (1.8b), (1.7a), (1.7b) and (1.3c), we get

(2.4) $R_{jpkh|l|m} = a_{lm} R_{jpkh} + b_{lm}(g_{kp} g_{jh} - g_{hp} g_{jk}).$

Remark 2.2. Conversely, transvecting (2.4) by the associate metric tensor g^{ip} and using (1.8c), (1.7c), (1.7d) and (1.2), yields the condition (2.3) i.e. the condition (2.3) is equivalent (2.4), therefore $GR^{hv} - (M)BRF_n$ can represent by the condition (2.4). Thus, we conclude.

Theorem 2.1. $GR^{hv} - (M)BRF_n$ may characterize by (2.4).

Transvecting (2.3) by g^{jk} , using (1.9c), (1.7c), (1.7d) and (1.2), we get

(2.5) $R^i_{h|l|m} = a_{lm} R^i_h.$

Thus, we conclude

Corollary 2.1. *The behavior of deviation tensor R^i_h as mixed birecurrent in $GR^{hv} - (M)BRF_n$.*

Contracting the indices i and h in the condition (2.3), using (1.9a), (1.3c) and (1.2), we get

(2.6) $R_{jk|l|m} = a_{lm} R_{jk} + b_{lm}(1-n) g_{jk}.$

Contracting the indices i and h in the condition (2.1), using (1.9a), (1.3c) and (1.2), we get

(2.7) $R_{jk|l} = \lambda_l R_{jk} + \mu_l (1-n) g_{jk}.$

Now, transvecting (2.6) by y^k , then using (2.7), (1.9b), (1.7e), (1.7f) and (1.1), we get

$$R_{j|l|m} = a_{lm} R_j + b_{lm}(1-n) y_j + \delta^k_m\left[\lambda_l R_{jk} + \mu_l(1-n) g_{jk}\right]$$

which can written as

(2.8) $R_{j|l|m} = a_{lm} R_j + b_{lm}(1-n) y_j + \lambda_l R_{jm} + \mu_l (1-n) g_{jm}.$

Transvecting (2.6) by g^{jk}, using (1.9e), (1.7c), (1.7d) and (1.2), we get

(2.9) $R_{|l|m} = a_{lm} R + b_{lm}(1-n).$

From equations (2.6), (2.8) and (2.9), we conclude

Theorem 2.2. *In $GR^{hv} - (M)BRF_n$, $R-$Ricci tensor R_{jk} , curvature vector R_j and curvature scalar R of Cartan's third curvature tensor R^i_{jkh} are non- vanishing .*

Now, transvecting (2.3) by y^j, we get

$$[R^i_{jkh|l|m}] y^j = [a_{lm} R^i_{jkh} + b_{lm}(\delta^i_k g_{jh} - \delta^i_h g_{jk})] y^j$$

Or

$$(R^i_{jkh} y^j)_{|l|m} - R^i_{jkh|l}\, y^j\big|_m = [a_{lm} R^i_{jkh} + b_{lm}(\delta^i_k g_{jh} - \delta^i_h g_{jk})] y^j.$$

Using the condition (2.1) in above equation, then using (1.7e), (1.7f), (1.8a), (1.1) and (1.3c), we get

(2.10) $H^i_{kh|l|m} = a_{lm}H^i_{kh} + b_{lm}\left(\delta^i_k y_h - \delta^i_h y_k\right) + \lambda_l R^i_{mkh} + \mu_l\left(\delta^i_k g_{mh} - \delta^i_h g_{mk}\right)$

Transvecting the condition (2.1) by y^j, using (1.8a) and (1.1), we get

(2.11) $H^i_{kh|l} = \lambda_l H^i_{kh} + \mu_l(\delta^i_k y_h - \delta^i_h y_k)$.

Now, transvecting (2.10) by y^k, we get

$$[H^i_{kh|l|m}]y^k = [a_{lm}H^i_{kh} + b_{lm}(\delta^i_k y_h - \delta^i_h y_k) + \lambda_l R^i_{mkh} + \mu_l(\delta^i_k g_{mh} - \delta^i_h g_{mk})]y^k$$

Or

$$(H^i_{kh}y^k)_{|l|m} - H^i_{kh|l}\, y^k_{|m} = [a_{lm}H^i_{kh} + b_{lm}(\delta^i_k y_h - \delta^i_h y_k) + \lambda_l R^i_{mkh} + \mu_l(\delta^i_k g_{mh} - \delta^i_h g_{mk})]y^k.$$

Using (2.11), (1.14b), (1.7e), (1.7f), (1.3b), (1.3d), (1.1) and put $(R^i_{mkh}y^k = R^i_{mh})$ in above equation, we get

(2.12) $H^i_{h|l|m} = a_{lm}H^i_h + b_{lm}\left(y^i y_h - F^2\delta^i_h\right) + \lambda_l R^i_{mh} + \lambda_l H^i_{mh} + \mu_l\left(y^i g_{mh} + \delta^i_m y_h - \delta^i_h y_m\right)$.

Contracting the indices i and h in the condition (2.10), using (1.15b), (1.3a), (1.9a), (1.3c) and (1.2), we get

(2.13) $H_{k|l|m} = a_{lm}H_k + b_{lm}(1-n)\boldsymbol{y_k} + \lambda_l R_{mk} + \mu_l(1-n)g_{mk}$.

Contracting the indices i and h in the condition (2.12), using (1.15c), (1.3d), (1.1), (1.2), (1.15b) and put $(R^i_{mi} = R_m)$, we get

(2.14) $H_{|l|m} = a_{lm}H + b_{lm}F^2 + \frac{(2-n)}{n-1}\mu_l y_m + \frac{\lambda_l}{n-1}(R_m + H_m)$.

From equations (2.10), (2.12), (2.13) and (2.14), we conclude

Theorem 2.3. *In $GR^{hv} - (M)BRF_n$, the torsion tensor H^i_{kh} , deviation tensor H^i_h , curvature vector H_k as mixed birecurrent in $GR^{hv} - (M)BRF_n$ and curvature scalar H of Berwald curvature tensor H^i_{jkh} is non - vanishing .*

Necessary and sufficient condition for some tensors to be $GR^{hv} - (M)BRF_n$

This section concentrates on the necessary and sufficient condition for some tensors to be generalized R^{hv} – mixed birecurrent. We know that Cartan's third curvature tensor R^i_{jkh} and Cartan's fourth curvature tensor K^i_{jkh} are connected by [10]

(3.1) $R^i_{jkh} = K^i_{jkh} + C^i_{js}H^s_{kh}$.

Taking hv −covariant derivative of mixed second order for (3.1) with respect to x^l and x^m, successively, we get

$$R^i_{jkh|l|m} = K^i_{jkh|l|m} + (C^i_{js}H^s_{kh})_{|l|m} .$$

Using the condition (2.3) in above equation, then using (3.1) in the resulting equation, we get

(3.2) $a_{lm}\,(K^i_{jkh} + C^i_{js}H^s_{kh}) + b_{lm}\left(\delta^i_k g_{jh} - \delta^i_h g_{jk}\right) = K^i_{jkh|l|m} + (C^i_{js}H^s_{kh})_{|l|m}$.

This show that

(3.3) $K^i_{jkh|l|m} = a_{lm}K^i_{jkh} + b_{lm}\left(\delta^i_k g_{jh} - \delta^i_h g_{jk}\right)$.

if and only if

(3.4) $(C^i_{js}H^s_{kh})_{|l|m} = a_{lm}(C^i_{js}H^s_{kh})$.

Transvecting (3.2) by g_{ip} , using (1.13a), (1.4b), (1.3c), (1.7a) and (1.7b), we get

(3.5) $a_{lm}\left(K_{jpkh} + C_{jps}H^s_{kh}\right) + b_{lm}\left(g_{kp}g_{jh} - g_{hp}g_{jk}\right) = K_{jpkh|l|m} + (C_{jps}H^s_{kh})_{|l|m}$

This show that

(3.6) $K_{jpkh|l|m} = a_{lm}\,K_{jpkh} + b_{lm}\left(g_{kp}g_{jh} - g_{hp}g_{jk}\right)$.

if and only if

(3.7) $(C_{jps}H^s_{kh})_{|l|m} = a_{lm}\,C_{jps}H^s_{kh}$.

From equations (3.3) and (3.6), we obtain

Theorem 3.1. *In $GR^{hv} - (M)BRF_n$, Cartan's four curvature tensor K^i_{jkh} and its associative tensor K_{jpkh} are generalized hv – mixed birecurrent if and only if the tensors $(C^i_{js}H^s_{kh})$ and $(C_{jps}H^s_{kh})$ behave as mixed birecurrent, respectively.*

Contracting the indices i and h in the condition (3.2), using (1.13b) and (1.3c), we get

(3.8) $a_{lm}\left(K_{jk} + C^r_{js}H^s_{kr}\right) + b_{lm}(1-n)g_{jk} = K_{jk|l|m} + (C^r_{js}H^s_{kr})_{|l|m}$.

This show that

(3.9) $K_{jk|l|m} = a_{lm}K_{jk} + b_{lm}(1-n)g_{jk}$.

if and only if

(3.10) $(C^r_{js}H^s_{kr})_{|l|m} = a_{lm}(C^r_{js}H^s_{kr})$.

Theorem 3.2. *In $GR^{hv} - (M)BRF_n$, the K–Ricci tensor K_{jk} of Cartan's fourth curvature tensor K^i_{jkh} is non – vanishing if and only if the tensor $(C^r_{js}H^s_{kr})$ behaves as mixed birecurrent.*

Transvecting (3.8) by y^k, using(1.13e),(1.14b) (1.1), (1.7e) and (1.7f), we get

(3.11) $K_{j|l|m} + [(C^r_{js}H^s_{kr})_{|l|m}]y^k = a_{lm}K_j + b_{lm}(1-n)y_j + K_{jk|l}\delta^k_m + a_{lm}C^r_{js}H^s_r$

This show that

(3.12) $(K_{j|l|m}) = a_{lm}K_j + b_{lm}(1-n)y_j$.

if and only if

(3.13) $[(C^r_{js}H^s_{kr})_{|l|m}]y^k = a_{lm}C^r_{js}H^s_r + K_{jm|l}$.

Theorem 3.3. *In $GR^{hv} - (M)BRF_n$, the curvature vector K_j of Cartan's fourth curvature tensor K^i_{jkh} is non – vanishing if and only if (3.13) holds.*

Transvecting (3.8) by g^{ij}, using (1.13c), (1.2), (1.7c) and (1.7d), we get

(3.14) $a_{lm}\big(K^i_k + g^{ij}C^r_{js}H^s_{kr}\big) + b_{lm}(1-n)\delta^i_k = K^i_{k|l|m} + (g^{ij}C^r_{js}H^s_{kr})_{|l|m}$.

This show that

(3.15) $K^i_{k|l|m} = a_{lm}K^i_k + b_{lm}(1-n)\delta^i_k$.

if and only if

(3.16) $(g^{ij}C^r_{js}H^s_{kr})_{|l|m} = a_{lm}(g^{ij}C^r_{js}H^s_{kr})$.

Transvecting (3.8) by g^{jk}, using (1.13d), (1.2), (1.7c) and (1.7d), we get

(3.17) $a_{lm}\big(K + g^{jk}C^r_{js}H^s_{kr}\big) + (1-n)b_{lm} = K_{|l|m} + (g^{jk}C^r_{js}H^s_{kr})_{|l|m}$.

This show that

(3.18) $K_{|l|m} = a_{lm}K + (1-n)b_{lm}$.

if and only if

(3.19) $(g^{jk}C^r_{js}H^s_{kr})_{|l|m} = a_{lm}(g^{jk}C^r_{js}H^s_{kr})$.

From equations (3.15) and (3.18), we conclude

Theorem 3.4. *In $GR^{hv} - (M)BRF_n$, the deviation tensor K^i_k and curvature scalar K of Cartan's fourth curvature tensor K^i_{jkh} are is non – vanishing if and only if the tensors $(g^{ij}C^r_{js}H^s_{kr})$ and $(g^{jk}C^r_{js}H^s_{kr})$ behave as mixed birecurrent.*

For a Riemannian space V_4, Cartan's second curvature tensor P^i_{jkh} and Cartan's third curvature tensor R^i_{jkh} can be expressed as follows [18]

(3.20) $P^i_{jkh} = R^i_{jkh} - \frac{1}{3}\big(R_{jk}\delta^i_h - R_{jh}\delta^i_k\big)$.

Taking hv – covariant derivative of mixed second order for (3.20) with respect to x^l and x^m, successively, we get

$$P^i_{jkh|l|m} = R^i_{jkh|l|m} - \frac{1}{3}\big(R_{jk}\delta^i_h - R_{jh}\delta^i_k\big)_{|l|m} .$$

Using the condition (2.3) in above equation, then using (3.20) in the resulting equation, we get

(3.21) $P^i_{jkh|l|m} = a_{lm}[P^i_{jkh} + \frac{1}{3}\big(R_{jk}\delta^i_h - R_{jh}\delta^i_k\big)] + b_{lm}\left(\delta^i_k g_{jh} - \delta^i_h g_{jk}\right) - \frac{1}{3}\big(R_{jk}\delta^i_h - R_{jh}\delta^i_k\big)_{|l|m}$.

This show that

(3.22) $P^i_{jkh|l|m} = a_{lm}P^i_{jkh} + b_{lm}\left(\delta^i_k g_{jh} - \delta^i_h g_{jk}\right)$

if and only if

(3.23) $\big(R_{jk}\delta^i_h - R_{jh}\delta^i_k\big)_{|l|m} = a_{lm}(R_{jk}\delta^i_h - R_{jh}\delta^i_k)$.

Transvecting (3.21) by g_{ir} , using (1.11a), (1.3c), (1.7a) and (1.7b), we get

(3.24) $P_{jrkh|l|m} = a_{lm}P_{jrkh} + \frac{1}{3}a_{lm}\big(R_{jk}g_{rh} - R_{jh}g_{rk}\big) + b_{lm}\left(g_{rk}g_{jh} - g_{rh}g_{jk}\right) - \frac{1}{3}\big(R_{jk}g_{rh} - R_{jh}g_{rk}\big)_{|l|m}$.

This show that

(3.25) $P_{jrkh|l|m} = a_{lm}P_{jrkh} + b_{lm}\left(g_{rk}g_{jh} - g_{rh}g_{jk}\right)$

if and only if

(3.26) $\big(R_{jk}g_{rh} - R_{jh}g_{rk}\big)_{|l|m} = a_{lm}(R_{jk}g_{rh} - R_{jh}g_{rk})$.

From equations (3.22) and (3.25), we obtain

Theorem 3.5. *In $GR^{hv}-(M)BRF_n$, Cartan's second curvature tensor P^i_{jkh} and its associative tensor P_{jrkh} are generalized hv –mixed birecurrent if and only if the tensors $\left(R_{jk}\delta^i_h - R_{jh}\delta^i_k\right)$ and $\left(R_{jk}g_{rh} - R_{jh}g_{rk}\right)$ behave as mixed birecurrent, respectively.*

Transvecting (3.21) by y^j, we get

$$[P^i_{jkh|l|m}]y^j = \{a_{lm}\left[P^i_{jkh} + \tfrac{1}{3}\left(R_{jk}\delta^i_h - R_{jh}\delta^i_k\right)\right] + b_{lm}\left(\delta^i_k g_{jh} - \delta^i_h g_{jk}\right) - \tfrac{1}{3}\left(R_{jk}\delta^i_h - R_{jh}\delta^i_k\right)_{|l|m}\}y^j$$

Or

$$\left(P^i_{jkh}y^j\right)_{|l|m} - P^i_{jkh|l}\, y^j\big|_m = \left\{a_{lm}\left[P^i_{jkh} + \tfrac{1}{3}\left(R_{jk}\delta^i_h - R_{jh}\delta^i_k\right)\right] + b_{lm}\left(\delta^i_k g_{jh} - \delta^i_h g_{jk}\right) - \tfrac{1}{3}\left(R_{jk}\delta^i_h - R_{jh}\delta^i_k\right)_{|l|m}\right\}y^j.$$

Using the (1.10), (1.9b), (1.1), (1.7e) and (1.7f), we get

(3.27) $P^i_{kh|l|m} = a_{lm}\left[P^i_{kh} + \tfrac{1}{3}\left(R_k\delta^i_h - R_h\delta^i_k\right)\right] + b_{lm}\left(\delta^i_k y_h - \delta^i_h y_k\right) + P^i_{jkh|l}\delta^j_m - \tfrac{1}{3}\left(R_k\delta^i_h - R_h\delta^i_k\right)_{|l|m}.$

This show that

(3.28) $P^i_{kh|l|m} = a_{lm}P^i_{kh} + b_{lm}\left(\delta^i_k y_h - \delta^i_h y_k\right)$

if and only if

(3.29) $\left(R_k\delta^i_h - R_h\delta^i_k\right)_{|l|m} = a_{lm}\left(R_k\delta^i_h - R_h\delta^i_k\right) + 3P^i_{mkh|l}$.

Transvecting (3.27) by g_{ir}, using (1.11a), (1.11b), (1.3c), (1.7a) and (1.7b), we get

$$P_{rkh|l|m} = a_{lm}\left[P_{rkh} + \tfrac{1}{3}(R_k g_{rh} - R_h g_{rk})\right] + b_{lm}\left(g_{rk}y_h - g_{rh}y_k\right) + P_{rmkh|l} - \tfrac{1}{3}(R_k g_{rh} - R_h g_{rk})_{|l|m}\,.$$

This show that

(3.30) $P_{rkh|l|m} = a_{lm}P_{rkh} + b_{lm}\left(g_{rk}y_h - g_{rh}y_k\right).$

if and only if

(3.31) $(R_k g_{rh} - R_h g_{rk})_{|l|m} = a_{lm}(R_k g_{rh} - R_h g_{rk}) + 3P_{rmkh|l}$.

Theorem 3.6. *In $GR^{hv}-(M)BRF_n$, Cartan's covariant derivative of mixed second order for the $(v)hv$ –torsion tensor P^i_{kh} and its associative tensor P_{rkh} are given by (3.28) and (3.30) if and only if (3.29) and (3.31) hold, respectively.*

Contracting the indices i and h in (3.27), using (1.12a), (1.12b), (1.3a), (1.9b) and (1.2), we get

$$P_{k|l|m} = a_{lm}P_k + \tfrac{1}{3}a_{lm}(n-1)R_k + b_{lm}(1-n)y_k + P_{mk|l} - \tfrac{1}{3}(n-1)R_{k|l|m}$$

This show that

(3.32) $P_{k|l|m} = a_{lm}P_k + b_{lm}(1-n)y_k.$

if and only if

(3.33) $(n-1)R_{k|l|m} = a_{lm}(n-1)R_k + 3P_{mk|l}$.

Theorem 3.7. *In $GR^{hv}-(M)BRF_n$, the curvature vector P_k of Cartan's second curvature tensor P^i_{jkh} is non – vanishing if and only if (3.33) holds.*

Contracting the indices i and h in (3.21), using (1.12a), (1.3c) and (1.2), we get

$$P_{jk|l|m} = a_{lm}P_{jk} + \tfrac{1}{3}a_{lm}(n-1)R_{jk} + b_{lm}(1-n)g_{jk} - \tfrac{1}{3}(n-1)R_{jk|l|m}.$$

This show that

(3.34) $P_{jk|l|m} = a_{lm}P_{jk} + b_{lm}(1-n)g_{jk}.$

if and only if

(3.37) $R_{jk|l|m} = a_{lm}R_{jk}$.

Corollary 3.1. *In $GR^{hv}-(M)BRF_n$, the behavior of R –Ricci tensor R_{jk} as mixed birecurrent if and only if P –Ricci tensor P_{jk} is non – vanishing.*

Conclusion :

We discussed the necessary and sufficient condition for different tensors which satisfy the generalized R^{hv} – mixed birecurrent space. Also, some tensors behave as mixed birecurrent. Furthermore, we concluded that the R –Ricci tensor R_{jk} , curvature scalar R , torsion tensor H^i_{kh} , deviation tensor

H_h^i , curvature vector H_k , curvature scalar H , K–Ricci tensor K_{jk} , deviation tensor K_k^i , curvature vector K_j and curvature scalar K are non-vanishing in $GR^{hv}-(M)BRF_n$.

REFERENCES :

1. A. A. Abdallah, Study on the relationship between two curvature tensors in Finsler spaces, Journal of Mathematical Analysis and Modeling, 4(2), (2023), 112-120.
    A. Abdallah, A. A. Navlekar & K. P. Ghadle, Analysis for Cartan’s second curvature tensor in Finsler space, International Journal of Advanced Research in Science, Communication and Technology, 2(3), (2022), 1-5.
2. Abdallah, A. A. Navlekar, K. P. Ghadle & A. A. Hamoud, Decomposition for Cartan’s second curvature tensor of different order in Finsler spaces, Nonlinear Functional Analysis and Applications, 27(2), (2022), 433-448.
3. Abdallah, A. A. Navlekar, K. P. Ghadle & B. Hardan, Fundamentals and recent studies of Finsler geometry, International Journal of Advances in Applied Mathematics and Mechanics, 10(2), (2022), 27-38.
4. Abdallah, A. A. Navlekar, K. P. Ghadle & B. Hardan, Several forms of $h(hv)-$torsion tensor C_{jkh} generalize $\beta P-$birecurrent space, International Journal of Advanced Research in Science, Engineering and Technology, 9(7), (2022), 19505-19510.
5. M. Al – Qashbari, Certain types of generalized recurrent in Finsler space, Ph.D. Thesis, University of Aden (2016), Yemen.
6. É. Cartan, Les espaces de Finsler, Actualités, Paris, 79 (1934), 2nd edit. (1971).
7. F. A. Assallal, On certain generalized $h-$birecurrent of curvature tensor, Master’s Thesis Aden University (2018), Yemen.
8. F. Y. Qasem, On transformations in Finsler spaces, D. Phil. Thesis, University of Allahabad (2000), India.
9. F. Y. Qasem & A. M. Al – Qashbari, Certain identities in generalized R^h-recurrent Finsler space, International Journal of Innovations Science and Mathematics, 4(2), (2016), 66-69.
10. F. Y. Qasem & W. H. Hadi, On a generalized R^h-birecurrent Finsler space, International Journal of Mathematics and physical Sciences Research, 3(2), (2016), 93-99.
11. F. Y. Qasem & W. H. Hadi, On a generalized K^h-birecurrent Finsler space, International Journal of Mathematics And its Applications, 4(1), (2016), 33-40.
12. F. Y. Qasem & W. H. Hadi, On a generalized H^h-birecurrent Finsler space, International Journal of Sciences Basic and Applied Research, 25(2), (2016), 207-217.
13. H. Rund, The differential geometry of Finsler space, Spring $-$Verlag, Berlin Gottingen $-$ Heidelberg, (1959); 2nd edit. (in Russian), Nauka, (Moscow), (1981)
14. M. Matsumoto, On $h-$ Isotropic and C^h- recurrent Finsler, J. Math. Kyoto Univ., 11 (1971), 1-9.
15. P. N. Pandey, Some problems in Finsler spaces, D.Sc. Thesis, University of Allahabad (1993), India.
16. S. I. Ohta, Comparison Finsler geometry, Springer International Publishing, license to Springer Nature Switzerland, (2021).
17. Zafar and A. Musavvir, On some properties of W–curvature tensor, Palestine Journal of Mathematices, 3(1), (2014), 61–69.