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Inference of Parameters of Modified Autoregressive Model on Selected Linear and Non-Linear Time Series Models of Different Orders

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ABSTRACT

In this research, stationarity is the most important property of time series. This research examines the inference of parameterization of modified models of different orders. Derivation of parameters of Linear and Nonlinear time series models of first, second and third order were obtained. Inverse Smooth Transition Autoregressive (ISTAR), Exponential Smooth Transition Autoregressive (ESTAR), and Trigonometric Smooth Transition Autoregressive (TSTAR) models were modified from Autoregressive model. The goal of this research is to derive the parameters of the updated models utilizing the maximum likelihood approach, computational features of stationarity, and modified ordinary differential equations of time series models.

Keywords: Modified models, Stationarity, Properties, Smooth Transition Autoregressive and Autoregressive model.

1.0 Introduction

Time series models are utilized to predict future events by analyzing past events that have been observed and collected at regular time intervals. According to Box and Pierce (1970), a time series model typically consists of the mean component and the conditional variance component. Time series refers to a sequence in which data points are measured chronologically throughout time. Time series data specifically consists of an assortment of both temporal and numerical values. A sequence of clearly defined procedures is followed to implement autoregressive, moving average, exponential, seasonal, and autoregressive moving average modelling. The first step is figuring out who the model is. Finding the right structure, such as Smooth Transition Autoregressive, Exponential Smooth Autoregressive, Trigonometric, and Autoregressive (AR) models, depends on whether the model is stationary or non-stationary under distributions. This is the identification process. Analyzing autocorrelation and partial autocorrelation function graphs can be one method of identification. An alternative method for achieving identification is an automated iterative procedure that fits several possible model structures and ranks them. We then use a goodness-of-fit statistic to find the best model.

To represent the persistence or autocorrelation in a time series, autoregressive mathematical models are utilized, such as moving average and smooth transition models. These models are commonly employed in sciences, engineering, econometrics, and other fields. There are several reasons for utilizing autoregressive (AR) models, smooth transition autoregressive (STAR) models, exponential smooth autoregressive models, and trigonometric models for data analysis. Modeling clarifies the underlying mechanisms that lead to the series' perceived persistence, providing important insights into the physical system. Based on past values, these models can also be used to forecast how a time series or set of econometric data will behave. This forecast can be used as a benchmark for evaluating the possible importance of other variables in the system. They are frequently utilized to predict economic and industrial time series. Autoregressive (AR) models, Smooth Transition Autoregressive (STAR) models, Exponential Smooth Autoregressive models, and Trigonometric models are also applicable for simulation purposes. These approaches enable the creation of synthetic series that resemble an observed series' persistence structure. When establishing confidence ranges for statistics and estimated econometric quantities, simulations are especially helpful.

Autoregressive, autoregressive moving average, moving average, seasonal, and exponential modelling follow a clear set of steps. Identifying the model is the initial step. Identification entails determining the appropriate structure for the model, whether it is stationary or non-stationary under various distributions. This can be done using various models such as Autoregressive (AR) models, Smooth Transition Autoregressive models, Exponential Smooth Autoregressive models, and Trigonometric models. By examining the plots of the autocorrelation and partial autocorrelation functions, identification can be achieved. Identification is commonly accomplished by employing an automated iterative procedure that entails fitting different model structures and ordering. Subsequently, A goodness-of-fit statistic is used to determine which model is the most suitable.

2.0 Literature Review

The autoregressive (AR) model is widely employed as the conventional model for the mean component. It is common practice while analyzing time series data to investigate the concept of stationarity, which refers to the pattern that a specific variable displays over time. Stationarity consists of three components. The series demonstrates a stationary mean, suggesting that there is no inherent tendency for the average of the series to change over time. Furthermore, it is assumed that the variability of the series remains constant over time. It is commonly believed that the autocorrelation pattern remains stable throughout the series. Many non-linear time series models have been proposed in the last twenty years. Among these are the random coefficient of AR model by (Ratnasingam,S.,& Ning,W (2020)), the amplitude dependent exponential AR (EX PAR) model by Maulana, A., & Slamet, I. (2020), the threshold AR model by Fan, J.Q. (2019), and the bilinear model by Granger and Andersen (1978), among several others.

Nevertheless, fluctuations in data do not consistently adhere to a linear pattern, hence posing challenges in accurately analysing and forecasting future outcomes using certain time series modelling methodologies. Hence, for such data, it would not be practicable to expect a single, linear model to capture these distinct behaviours. Linear relationships and various combinations of them are frequently insufficient for accurately characterising the behaviour of such data. Time series analysis reveals a multitude of nonlinear features, including cycles, thresholds, bursts, chaos, heteroscedasticity, asymmetries, and various combinations of these. Tong (1990), Granger and Ter¨asvirta (2019), Franse, and van Dijk (2020), Tsay(2020) and Kim and Nelson (2019) have discussed the different models that can be formulated in these forms.

The key assumption in analyzing time series data is stationarity. Stationarity means that the statistical properties that influence the behavior of the process remain constant over time. Essentially, the process is in statistical equilibrium. A process Y_t is considered strictly stationary if the joint distribution of Y_t is identical to that of Y_{t-k} for any values of t and k.

t = 1, 2,… k. In other words, the *Y*'s are (marginally) identically distributed (Tsay and Kung-Sik, (2018). It follows that $E(Y_t) = E(Y_{t-k})$ for all *t* and *k* so that the mean function is constant for all time. Furthermore, $Var(Y_{t)} = Var(Y_{t-k})$ for all *t* and *k* so that the variance is also constant over time. The fundamental premise of a stationary time series is that it exhibits white noise, meaning that the error term of the model must follow a normal distribution with a mean of zero and a specific variance σ^2.

Imagine a collection of time-ordered data points, denoted as $\{y_1, y_2 \cdots\cdots\cdots\cdots y_n\}$, that are derived from observations of a particular phenomenon. Time series data consists of observations that are measured over time, either continuously or at specific time intervals (Kim, 2022). One of the key aspects in model fitting, particularly when dealing with econometrics and time series data, is understanding the most effective methods to employ. AR models, Smooth Transition Autoregressive, Exponential Smooth Autoregressive model and Trigonometric model. The fundamental premise of these models is the need of stationarity, meaning that the data being applied to them must exhibit stationarity.

Autoregressive Smooth transition and moving average models are mathematical models that capture the persistence, or autocorrelation, in a time series. The models are extensively utilized in the fields of econometrics, hydrology, engineering, and other related disciplines. There are multiple reasons for using Autoregressive (AR) models, Smooth Transition Autoregressive (STAR) models, Exponential Smooth Autoregressive models, and Trigonometric models to data. Modelling can enhance comprehension of the physical system by providing insights into the underlying physical mechanism that imparts persistence to the series. The models can also be utilized to forecast the behavior of a time series or econometric data based on historical values. This forecast can serve as a benchmark for assessing the potential significance of other variables in the system. They are extensively utilized for forecasting economic and industrial time series. Autoregressive (AR) models, Smooth Transition Autoregressive (STAR), Exponential Smooth Autoregressive model, and Trigonometric model can be used for simulation. This involves generating synthetic series that have the same persistence structure as an observed series. Simulations are particularly valuable for determining confidence ranges for statistical measures and estimated values in econometrics.

These studies suggest that, in trying to decide by classical methods whether economic data are stationary or integrated, it would be useful to perform tests of the null hypothesis of stationarity as well as tests of the null hypothesis of a unit root. This paper provides a straight forward test of the null hypothesis of stationarity against the alternative of a unit root at different order of autoregressive and moving average and various sample sizes. There have been surprisingly few previous attempts to test the null hypothesis of stationarity. Park and Mahdi (2020) consider a test statistic which is essentially the ***F*** statistic for 'superfluous' deterministic trend variables; this statistic should be close to zero under the stationary null but not under the alternative of a unit root. Zhang and Zhou (2022) consider the Dickey-Fuller test statistics, but estimates both trend-stationery and difference-stationary models and then uses the bootstrap to evaluate the distribution of these statistics.

3.0 Material and Methods

In this section, we modified new models from an existing model (Autoregressive) in order to increase the accuracy of an existing models, the new models are Inverse Smooth Transition Autoregressive model (ISTAR), Exponential Smooth Transition Autoregressive model (ESTAR) and Trigonometric Smooth Transition Autoregressive model (TSTAR). Derivation of Variance of modified models across orders and Autoregressive model were obtained.

$AR_{(p)}$ is $Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \ldots + \phi_p Y_{t-p} + e_t,\ e_t \sim WN(0, \sigma^2)$ .by Terasvirta.T and Granger.C.WS, (2022)

Where e_t is white noise process with zero mean and variance (σ^2) Where Y_t is variable of interest at time (t), are the coefficient that define the unit root.

MODIFIED MODELS are:

$Y_t = \emptyset_1 Y_{t-1} + \emptyset_2 Y_{t-2} + \ldots + \emptyset_p Y_{t-p} + e_t$, --------------------- 3.0 is an existing model, where e_t is white noise process with a mean of zero and variance σ^2. $\emptyset_1, \emptyset_2, \ldots \emptyset_p$ are autoregressive parameters that define the unit root. The mathematical model for existing and proposed models are

$ISTAR_{(p)}$ is $Y_t = \frac{1}{\phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \ldots + \phi_p Y_{t-p} + e_t}$ -------------------- 3.1 is called Inverse Smooth Transition Autoregressive model. Where e_t is white noise process with zero mean and variance(σ^2), Y_t is variable of interest at time (t), $\phi_1, \phi_2, \ldots \phi_p$ are the coefficient that define the unit root.

$ESTAR_{(p)}$ is $Y_t = e^{\phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \ldots + \phi_p Y_{t-p} + e_t}$ ---------------------- 3.2 is called Exponential Smooth Transition Autoregressive model.

$TSTAR_{(p)}$ is $Y_t = Sin\phi_1 Y_{t-1} + Sin\phi_2 Y_{t-2} + \cdots + Sin\phi_p Y_{t-p} + e_t$---------------------- 3.3 is called Trigonometric Smooth Transition Autoregressive model.

Derivation of Parameters of Modified models of different orders

Modified Models are;

1. ISTAR = $Y_t = \frac{1}{\phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \phi_p Y_{t-p} + e_t}$ Is called Inverse Smooth Transition Autoregressive model.
2. ESTAR = $Y_t = e^{\phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \phi_p Y_{t-p} + e_t}$is called Exponential Smooth Transition Autoregressive model.
3. TSTAR = $Y_t = sin\phi_1 Y_{t-1} + sin\phi_2 Y_{t-2} + \cdots + sin\phi_p Y_{t-p} + e_t$ is called Trigonometric Smooth Transition Autoregressive model.

$e_t \sim WN\ (0, \sigma^2)$. Where e_t is white noise process with a constant mean and variance (σ^2). Y_t is variable of Interest and time (t) and $\phi_1, \phi_{2,\ldots} \phi_p$ are the coefficient that define the unit root.

(Bature et al,2024) proposed model under time series model for Linear and Nonlinear time series under stationarity and non-Stationarity.

Derivation of Variance of the First order for the Inverse Smooth Transition Autoregressive Model ($ISTAR_{(1)}$). The model can be written as,

$$Y_t = \frac{1}{\phi_1 Y_{t-1} + e_t}$$

$e_t \sim WN\ (0, \sigma^2)$. Where e_t is white noise.

Assumption of Time series, Since the model formulated were firstly assumed to be stationary, And it follows that;

$$Var\ (Y_t) = V(Y_{t-i}) = (\sigma_{ti}^2), where\ i = 0,1,2 \ldots k, and\ Var\ (e_t) = (\sigma_e^2)$$

We can obtain the parameters of ISTAR model by taking the variance of the model.

$$Var(Y_t) = Var\left[\frac{1}{\phi_1 Y_{t-1} + e_t}\right] \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots 1$$

$$Var(Y_t) = \left[\frac{1}{\phi_1 Var(Y_{t-1}) + Var(e_t)}\right] \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots 2$$

$$Var(Y_t) = \left[\frac{1}{\phi_1^2 \sigma_{t-1}^2 + \sigma_e^2}\right] \ldots \ldots \ldots \ldots \ldots \ldots \ldots \quad \ldots 3$$

Recall that $Var\ (Y_t) = V(Y_{t-i}) = (\sigma_t^2)$

Therefore,

$$Var(Y_t) = \left[\frac{1}{\phi_1^2 \sigma_t^2 + \sigma_e^2}\right] \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots 4$$

$$\sigma_t^2 = \frac{1}{\phi_1^2 \sigma_t^2 + \sigma_e^2} \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots 5$$

$$\sigma_t^2(\phi_1^2 \sigma_t^2 + \sigma_e^2) = 1 \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots 6$$

Differentiate the equation 6 with respect to σ_t^2

Then the equation 6 can be written as;

$$\hat{\sigma}_t^2 = \frac{-\sigma_e^2}{2\phi_1^2}, \phi_1 \neq 0$$

The variance of **ISTAR $_{(1)}$** is $\frac{-\sigma_e^2}{2\phi_1^2}$

Derivation of Variance of the second order for the Inverse Smooth transition Autoregressive ISTAR $_{(2)}$) model.

$$Y_t = \frac{1}{\phi_1 Y_{t-1} + \phi_2 Y_{t-2} + e_t}$$

$e_t \sim WN\ (0, \sigma^2)$. Where e_t is white noise.

taking the variance of the model.

$$Var(Y_t) = Var\left[\frac{1}{\phi_1 Y_{t-1} + \phi_2 Y_{t-2} + e_t}\right] \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots 7$$

Recall that $Var\ (Y_t) = V(Y_{t-i}) = (\sigma_t^2)$

$$\sigma_t^2 = \left[\frac{1}{{\phi_1}^2 Var(Y_{t-1}) + {\phi_2}^2 Var(Y_{t-2)} + Var(e_t)}\right] \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 8$$

Therefore,

$$\sigma_t^2 = \frac{1}{\phi_1^2\sigma_{t-1}^2 + \phi_2^2\sigma_{t-2}^2 + \sigma_e^2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 9$$

$$\sigma_t^2 = \frac{1}{\phi_1^2\sigma_t^2 + \phi_2^2\sigma_t^2 + \sigma_e^2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 10$$

Therefore,

$$\sigma_t^2(\phi_1^2\sigma_t^2 + \phi_2^2\sigma_t^2 + \sigma_e^2) = 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots 11$$

Also, Differentiate the equation 11 with respect to $\sigma_t^2$the equation would be written as:

$$\sigma_t^2 = \frac{-\sigma_e^2}{2(\phi_1^2 + \phi_2^2)} \; where \; \phi_1, \phi_2 \neq 0$$

Hence, the variance of **ISTAR$_{(2)}$** model = $\frac{-\sigma_e^2}{2(\phi_1^2+\phi_2^2)}$.

Derivation of Variance of the third order for the Inverse Smooth transition Autoregressive (ISTAR$_{(3)}$) model.

$$Y_t = \frac{1}{\phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \phi_3 Y_{t-3} + e_t} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 12$$

$e_t \sim WN\ (0, \sigma^2)$. Where e_t is white noise.

taking the variance of the model.

$$Var(Y_t) = Var\left[\frac{1}{\phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \phi_3 Y_{t-3} + e_t}\right] \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 13$$

$$Var(Y_t) = \left[\frac{1}{{\phi_1}^2 Var(Y_{t-1}) + {\phi_2}^2 Var(Y_{t-2)} + {\phi_3}^3 Var(Y_{t-3)} + Var(e_t)}\right]$$

$$Var(Y_t) = \frac{1}{\phi_1^2\sigma_{t-1}^2 + \phi_2^2\sigma_{t-2}^2 + \phi_3^2\sigma_{t-3}^2 + \sigma_e^2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 14$$

Recall from stationarity property, $Var\ (Y_t) = V(Y_{t-1}) = V(Y_{t-2})$

$$V(Y_{t-k}) = \sigma_t^2 and\ V(e_t) = \sigma_e^2$$

Therefore,

$$\sigma_t^2 = \frac{1}{\phi_1^2\sigma_t^2 + \phi_2^2\sigma_t^2 + \phi_3^2\sigma_t^2 + \sigma_e^2}$$

$$\sigma_t^2(\phi_1^2\sigma_t^2 + \phi_2^2\sigma_t^2 + \phi_3^2\sigma_t^2 + \sigma_e^2) = 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots 15$$

Also, Differentiate the equation 15 with respect to $\sigma_t^2$the equation can be written as:

$$\sigma_t^2 = \frac{-\sigma_e^2}{2(\phi_1^2 + \phi_2^2 + \phi_3^2)} \; where \; \phi_1, \phi_2, \phi_3 \neq 0$$

Hence, the variance of ISTAR$_{(3)}$ model =$\frac{-\sigma_e^2}{2(\phi_1^2+\phi_2^2+\phi_3^2)}$.

Derivation of Variance of the First Order for the Exponential Smooth Transition Autoregressive (ESTAR$_{(1)}$) model.

$$Y_t = e^{\phi_1 Y_{t-1} + e_t} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 16$$

Take the variance of the model;

$$Var\ (Y_t) = Var\ [e^{\phi_1 Y_{t-1} + e_t}] \ldots\ldots\ldots\ldots\ldots\ldots .17$$

$$Var\ (Y_t) = e^{\phi_1^2 Var(Y_{t-1}) + Var(e_t)} \ldots\ldots\ldots\ldots\ldots\ldots .18$$

From the assumption of time series, the formulated model assumed to be stationary where;

$Var\ (Y_t) = V(Y_{t-1}) = \ldots\ V(Y_{t-k}) = \sigma_t^2 and\ V(e_t) = \sigma_e^2$

The equation 18 can be written as

$$\sigma_t^2 = e^{\phi_1^2\sigma_t^2 + \sigma_e^2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots . 19$$

Take the natural log of equation 19,

$$\ln\sigma_t^2 = \ln\left[e^{\phi_1^2\sigma_t^2+\sigma_e^2}\right] \dots\dots\dots\dots\dots\dots\dots 20$$

$$\ln\sigma_t^2 = \phi_1^2\sigma_t^2 + \sigma_e^2, Recal\ that \ln e = 1$$

Differentiate the equation 20 with respect to σ_t^2;

$$\frac{1}{\sigma_t^2} = \phi_1^2 \Rightarrow \sigma_t^2 = \frac{1}{\phi_1^2}\ Note\ \phi_1 \neq 0$$

Derivation of Variance of the Second order for the Exponential Smooth Transition Autoregressive ($ESTAR_{(2)}$) model.

$$Y_t = e^{\phi_1 Y_{t-1}+\phi_2 Y_{t-2}+e_t} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots 21$$

where $e_t \sim WN\ (0, \sigma^2)$. and e_t is white noise.

Take the variance of the model;

$$Var\ (Y_t) = Var\ [e^{\phi_1 Y_{t-1}+\phi_2 Y_{t-2}+e_t}] \dots\dots\dots\dots\dots\dots .22$$

From the property of stationarity,

$Var\ (Y_t) = V(Y_{t-1}) = \dots V(Y_{t-k}) = \sigma_t^2\ and\ V(e_t) = \sigma_e^2$

The equation 22 can be written as

$$\sigma_t^2 = e^{\phi_1^2\sigma_t^2+\phi_2^2\sigma_t^2+\sigma_e^2} \dots\dots\dots\dots\dots\dots\dots .23$$

Take the natural log of equation 23,

$$\ln\sigma_t^2 = \ln\left[e^{\phi_1^2\sigma_t^2+\phi_2^2\sigma_t^2+\sigma_e^2}\right] \dots\dots\dots\dots\dots\dots\dots 24$$

Equation 24 can be written as;

$$\ln\sigma_t^2 = \phi_1^2\sigma_t^2 + \phi_2^2\sigma_t^2 + \sigma_e^2, where \ln e = 1$$

Differentiate the above equation with respect to σ_t^2;

$$\frac{1}{\sigma_t^2} = \phi_1^2 + \phi_2^2$$

$$\Rightarrow \sigma_t^2(\phi_1^2 + \phi_2^2) = 1$$

$$\sigma_t^2 = \frac{1}{(\phi_1^2 + \phi_2^2)}\ Where\ \phi_1, \phi_2 \neq 0.\ is\ variance\ of\ ESTAR_{(2)}\ model$$

Derivation of Variance of the Third Order for the Exponential Smooth Transition Autoregressive $(ESTAR)_{(3)}$) model.

The model can be written as;

$$Y_t = e^{\phi_1 Y_{t-1}+\phi_2 Y_{t-2}+\phi_3 Y_{t-3}+e_t}$$

where $e_t \sim WN\ (0, \sigma^2)$. and e_t is white noise.

Take the variance of the model;

$$Var\ (Y_t) = Var\ [e^{\phi_1 Y_{t-1}+\phi_2 Y_{t-2}+\phi_3 Y_{t-3}+e_t}] \dots\dots\dots\dots\dots\dots .25$$

Recall From the property of stationarity,

$$Var\ (Y_t) = V(Y_{t-1}) = V(Y_{t-2}) = V(Y_{t-3}) = \sigma_t^2$$

The equation 25 can be written as

$$\sigma_t^2 = e^{\phi_1^2\sigma_t^2+\phi_2^2\sigma_t^2+\phi_3^2\sigma_t^2+\sigma_e^2} \dots\dots\dots\dots\dots\dots\dots .26$$

Take the natural log of equation 26,

$$\ln\sigma_t^2 = \ln\left[e^{\phi_1^2\sigma_t^2+\phi_2^2\sigma_t^2+\phi_3^2\sigma_t^2+\sigma_e^2}\right] \dots\dots\dots\dots\dots\dots\dots 27$$

Equation 27 can be written as;

$$\ln\sigma_t^2 = \phi_1^2\sigma_t^2 + \phi_2^2\sigma_t^2 + \phi_3^2\sigma_t^2 + \sigma_e^2, where \ln e = 1$$

Differentiate the above equation with respect to σ_t^2;

$$\frac{1}{\sigma_t^2} = \phi_1^2 + \phi_2^2 + \phi_3^2$$

$$\sigma_t^2(\phi_1^2+\phi_2^2+\phi_3^2)=1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots 28$$

Therefore,

$\sigma_t^2=\frac{1}{(\phi_1^2+\phi_2^2+\phi_3^2)}$ *Where* $\phi_1,\phi_2,\phi_3\neq 0$ *is variance of* $ESTAR_{(3)}$

Derivation of Variance of the first order for the Trigonometric Smooth Transition Autoregressive ($TSTAR_{(1)}$) model.

$TSTAR_{(1)}$ model is given below as;

$$Y_t=sin\phi_1Y_{t-1}+e_t \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 29$$

where $e_t\sim WN\,(0,\sigma^2)$. and e_t is white noise.

Take the variance of the model

$$Var(Y_t)=Var[sin\phi_1Y_{t-1}+e_t]\ldots\ldots\ldots\ldots 30$$

From the property of stationarity,

;

$$Var\,(Y_t)=\phi_1^2sin^2Var(Y_{t-1})+Var(e_t)\ldots\ldots\ldots 31$$

$$\sigma_t^2=\phi_1^2sin^2\sigma_{t-1}^2+\sigma_e^2\ldots\ldots\ldots\ldots\ldots\ldots 32$$

Differentiate equation 32 with respect to σ_t^2

$$1=\phi_1^2 2sin\sigma_t^2cos\sigma_t^2,\ \textit{recall that } 2sin\sigma_t^2cos\sigma_t^2=\ sin2\sigma_t^2\textit{from trigonometric identities}$$

$$1=\phi_1^2sin2\sigma_t^2\ldots\ldots\ldots\ldots\ldots 33$$

The equation 33 can be written as;

$$sin2\sigma_t^2=\frac{1}{\phi_1^2}$$

$$\sigma_t^2=arc\ sin\left[\frac{1}{2\phi_1^2}\right]\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 34$$

is variance of **$TSTAR_{(1)}$**

Derivation of Variance of the second order for the Trigonometric Smooth Transition Autoregressive ($TSTAR_{(2)}$) model.

$TSTAR_{(2)}$ model is given below as;

$$Y_t=sin\phi_1Y_{t-1}+sin\phi_2Y_{t-2}+e_t\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 35$$

where $e_t\sim WN\,(0,\sigma^2)$. and e_t is white noise.

Take the variance of the model

$$Var(Y_t)=Var[sin\phi_1Y_{t-1}+sin\phi_2Y_{t-2}+e_t]\ldots\ldots\ldots\ldots 36$$

From the property of stationarity;

$Var\,(Y_t)=V(Y_{t-1})=\ldots V(Y_{t-k})=\sigma_t^2$ *and* $V(e_t)=\sigma_e^2$

Therefore,

$$Var\,(Y_t)=\phi_1^2sin^2V(Y_{t-1})+\phi_2^2sin^2V(Y_{t-2})+Var(e_t)$$

$$\sigma_t^2=\phi_1^2sin^2\sigma_t^2+\phi_2^2sin^2\sigma_t^2+\sigma_e^2\ldots\ldots\ldots\ldots\ldots\ldots\ldots 37$$

Differentiate equation 37 with respect to σ_t^2

$1=\phi_1^2 2sin\sigma_t^2cos\sigma_t^2+\phi_2^2 2sin\sigma_t^2cos\sigma_t^2$, *recall that* $2sin\sigma_t^2cos\sigma_t^2=\ sin2\sigma_t^2$ *from trigonometric identities*

$$1=sin2\sigma_t^2(\phi_1^2+\phi_2^2)\ldots\ldots\ldots\ldots\ldots\ldots 38$$

Hence the variance of **$TSTAR_{(2)}$** is;

$$arc\ sin\left[\frac{1}{2(\phi_1^2+\phi_2^2)}\right]\ .\text{Where } \phi_1 \text{and } \phi_2\neq 0$$

Derivation of Variance of the third order for the Trigonometric Smooth Transition Autoregressive ($TSTAR_{(3)}$) model.

$TSTAR_{(3)}$ model is given below as;

$$Y_t = sin\phi_1 Y_{t-1} + sin\phi_2 Y_{t-2} + sin\phi_3 Y_{t-3} + e_t$$

where $e_t \sim WN\ (0, \sigma^2)$. and e_t is white noise.

Take the variance of the model,

$$Var(Y_t) = Var[sin\phi_1 Y_{t-1} + sin\phi_2 Y_{t-2} + sin\phi_3 Y_{t-3} + e_t] \ldots\ldots\ldots\ldots .39$$

From the property of stationarity,

$Var\ (Y_t) = V(Y_{t-1}) = \ldots V(Y_{t-k}) = \sigma_t^2\ and\ V(e_t) = \sigma_e^2$

Therefore,

$$Var\ (Y_t) = \phi_1^2 sin^2 V(Y_{t-1}) + \phi_2^2 sin^2 V(Y_{t-2}) + \phi_3^2 sin^2 V(Y_{t-3}) + Var(e_t)$$

$$\sigma_t^2 = \phi_1^2 sin^2 \sigma_t^2 + \phi_2^2 sin^2 \sigma_t^2 + \phi_3^2 sin^2 \sigma_t^2 + \sigma_e^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots .40$$

Differentiate equation 40 with respect to σ_t^2

$1 = \phi_1^2 2sin\sigma_t^2 cos\sigma_t^2 + \phi_2^2 2sin\sigma_t^2 cos\sigma_t^2 + \phi_3^2 2sin\sigma_t^2 cos\sigma_t^2, \quad recall\ that\ 2sin\sigma_t^2 cos\sigma_t^2 = \ sin2\sigma_t^2$ *from trigonometric identities*

$$1 = sin2\sigma_t^2(\phi_1^2 + \phi_2^2 + \phi_3^2) \ldots\ldots\ldots\ldots\ldots\ldots .41$$

Hence the variance of $\mathbf{TSTAR_{(3)}}$ is;

$$arc\ sin\left[\frac{1}{2(\phi_1^2 + \phi_2^2 + \phi_3^2)}\right]$$

Where $\phi_1, \phi_2$and $\phi_3 \neq 0$.

RESULTS and DISCUSSION

Estimation of Parameters for the Different Order Forms.

The parameter values were estimated for the first, second and third order of Autoregressive functions selected as follows:

Estimation of Parameter for the First Order.

From the pth- order of autoregressive $[AR_{(p)}]$ given in equation of the first order $[AR_{(1)}]$ were deduced as follows;

$$Y_t = \phi_1 Y_{t-1} + e_t \qquad \ldots 3.0$$

where e_t is white noise process with a mean of zero mean and variance . Where Y_t is variable of interest at time (t), is the coefficient that define the unit root.

Based on the assumptions stated in chapter two that in stationary autoregressive models, the $E(Y_t) = E(Y_{t-k})$ and $Var(Y_t) = Var(Y_{t-k})$ for all t and k so that the mean and variance functions are constants for all time. Since the models formulated were firstly assumed to be stationary, it follows that;

$$V(Y_t)= V(Y_{t-i}) = \sigma_{t,}^2 i = 0, 1, 2, \ldots \text{ and } V(e_t)=\sigma_e^2 \qquad \ldots 3.1$$

Based on these we can easily obtain the parameter of the $AR_{(1)}$ model as follows;

$$V(Y_t) = \phi_1^2 V(Y_{t-1}) + \ V(e_t) \qquad \ldots 3.2$$

$$\sigma_t^2 = \ \phi_1^2 \sigma_{t-1}^2 + \sigma_e^2 \qquad \ldots 3.3$$

And since $V(Y_t)= V(Y_{t-i})$ = , the above equation (1.3) can be written as

$$\sigma_t^2 = \frac{\sigma_e^2}{1-\phi_1^2} \qquad \ldots 3.4$$

When the autoregressive first order is derived from the characteristic equation, it is stationary if and only if the roots of the $AR_{(p)}$ characteristic equation do not exceed 1 in absolute value.

Estimation of Parameter for the Second Order.

From the pth- order of Autoregressive $[AR_{(p)}]$ given in equation of second order $[AR_{(2)}]$ were deduced as follows;

$$Y_t = \ \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + e_t \qquad \ldots 3.5$$

where is white noise process with zero mean and variance . Where Y_t is variable of interest at time (t), are the coefficient that define the unit root.

$V(Y_t) = \phi_1^2 V(Y_{t-1}) + \phi_2^2 V(Y_{t-2}) + V(e_t)$ …3.6

$\sigma_t^2 = \phi_1^2 \sigma_{t-1}^2 + \phi_2^2 \sigma_{t-2}^2 + \sigma_e^2$ …3.7

following the same procedure that and assumptions of time series;

$V(Y_t) = V(Y_{t-1}) = V(Y_{t-2}) = \sigma_t^2$ …3.8

$\sigma_t^2 = \frac{\sigma_e^2}{1-\phi_1^2-\phi_2^2}$ …3.9

For the second order Autoregressive Model we introduced the autoregressive characteristic polynomial $\phi(Y) = 1 - \phi_1(Y) - \phi_2(Y^2)$ …3.2.0

and the corresponding $AR_{(2)}$ characteristic polynomial equation:

$$1 - \phi_1 Y - \phi_2 Y^2 = 0 \; OR \quad \phi_2 Y^2 + \phi_1 Y - 1 = 0 \qquad \text{…3.2.1}$$

It can be demonstrated that, under the condition that e_t is not influenced by Y_{t-1}, Y_{t-2}, Y_{t-3},..., A stationary solution to equation 1.4 is present only if the roots of the AR characteristic equation have an absolute value (modulus) less than or equal to 1. Occasionally, it is said that the roots ought to be situated beyond the boundaries of the unit circle in the complex plane. This assertion will apply universally to the pth-order scenario without any modifications.

In the second-order case, the roots of the quadratic characteristic equation (3.4) are easily found to be

The characteristics solution $= \frac{-\phi_1 \pm \sqrt{(\phi_1^2 + 4\phi_2)}}{2\phi_2}$ …3.2.2

The $AR_{(2)}$ is stationary if the absolute value of (3.2.2) does not exceed 1.

Jonathan and Kung-Sik, (2018) showed that this will be true if and only if three conditions are satisfied.

Estimation of Parameter for the Third Order.

From the pth- order of autoregressive [$AR_{(p)}$] given in equation of the third order [$AR_{(3)}$] were deduced as follows;

$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \phi_3 Y_{t-3} + e_t$ …3.21

where is white noise process with zero mean and variance. Where Y_t is variable of interest at time (t), are the coefficient that define the unit root.

$V(Y_t) = E[(\phi_1^2 Y_{t-1}^2 + \phi_2^2 Y_{t-2}^2 + \phi_3^2 Y_{t-3}^2 + 2\phi_1\phi_2 Y_{t-1}Y_{t-2} + 2\phi_1\phi_3 Y_{t-1}Y_{t-3} + 2\phi_2\phi_3 Y_{t-2}Y_{t-3} + e_t^2)]$
…3.22

following the same procedure that;

$V(Y_t) = \phi_1^2\, E(Y_{t-1}^2) + \phi_2^2 E(Y_{t-2}^2) + \phi_3^2 E(Y_{t-3}^2) + 2\phi_1\phi_2 E(Y_{t-1}Y_{t-2}) + 2\phi_1\phi_3 E(Y_{t-1}Y_{t-3}) + 2\phi_2\phi_3 E(Y_{t-2}Y_{t-3}) +$
$E(e_t^2)$ …3.23

$\gamma_0 - \phi_1^2\gamma_0 - \phi_2^2\gamma_0 - \phi_3^2\gamma_0 = \sigma_e^2$ …3.24

$\text{Var}(Y_t) = \gamma_0 = \frac{\sigma_e^2}{(1-\phi_1^2-\phi_2^2-\phi_3^2)}$ …3.25

The $AR_{(3)}$ is stationary if the absolute value of (3.21) does not exceed 1.

Mahdi. E, Jonathan and Kung-Sik, (2020) showed that this will be true if and only if the following three conditions are satisfied.

$$|\phi_2| < 1, \left| \phi_1 + \phi_{2+\phi_3} \right| < 1, and \; | \phi_1 - \phi_2 - \phi_3| < 1$$

Table 1a Inference of Derivation of Variance of Autoregressive model and Modified models of different orders.

Models	First Order	Second Order	Third Order
$AR_{(p)}$	$\frac{\sigma_e^2}{1-\phi_1^2}$	$\frac{\sigma_e^2}{(1-\phi_1^2-\phi_2^2)}$	$\frac{\sigma_e^2}{(1-\phi_1^2-\phi_2^2-\phi_3^2)}$
$ISTAR_{(p)}$	$\frac{-\sigma_e^2}{2(\phi_1^2)}$	$\frac{-\sigma_e^2}{2(\phi_1^2+\phi_2^2)}$	$\frac{-\sigma_e^2}{2(\phi_1^2+\phi_2^2+\phi_3^2)}$
$ESTAR_{(p)}$	$\frac{1}{(\phi_1^2)}$	$\frac{1}{(\phi_1^2+\phi_2^2)}$	$\frac{1}{(\phi_1^2+\phi_2^2+\phi_3^2)}$
$TSTAR_{(p)}$	$arc\ sin\left[\frac{1}{2(\phi_1^2)}\right]$	$arc\ sin\left[\frac{1}{2(\phi_1^2+\phi_2^2)}\right]$	$arc\ sin\left[\frac{1}{2(\phi_1^2+\phi_2^2+\phi_3^2)}\right]$

CONCLUSION

This study utilised time series' computational characteristics to derive the parametrization of both modified and existing models. The aim was to enhance the accuracy of the existing model by considering the first, second, and third order of stationarity. Also, Autoregressive (AR) model is employed to forecast future values by using historical data. This research modified Autoregressive model in order to improved the accuracy of an existing model.

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