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On the Uniformly Starlikenss of the Exponential Function

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ABSTRACT.

We introduce and study a new subclass of uniformly starlike functions by the means of the convolution involving the cosine function in the unit disk.

Keywords. Hadamard Product; Linear Operator; Uniformly starlike functions; Cosine function.

1. Introduction

Let A be the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

(1)

Which are analytic and univalent in the open disk $U = \{z: |z| < 1\}$

Also denote by T the subclass of A consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, a_n \geq 0$$

(2)

A function $f \in A$ is said to be in the class of uniformly convex functions of order α, denoted by $UCV(\alpha)$ if

And is said to be in a corresponding subclass of $UCV(\alpha)$ denoted by $S_p(\alpha)$ if

$$\Re\left\{\frac{zf'(z)}{f(z)} - \alpha\right\} \geq \beta \left|\frac{zf'(z)}{f(z)} - 1\right|,$$

$-1 \leq \alpha \leq 1$ and $z \in U$ The class of uniformly convex and uniformly starlike functions has been studied by Goodman, see [1-4] and Ma and Minda [5]. If f of the form (1) and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$,are two functions in A, Then the Hadamard product of f and g is denoted by $f * g$ and is given by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n$$

Definition 1 we consider the following linear operator

$$Jf(z) = f(z) * \cos\left(\frac{z}{2}\right) = z + \sum_{n=2}^{\infty} \frac{(-1)^{2n}}{(2n)!} a_n z^n$$

(3)

where, $f(z)$ ∈A, and has the form (1).

Now using the operator introduced in (3) we can define the following subclass of analytic function, $J * f(z)$

$$\Re\left\{\frac{z(Jf(z))'}{Jf(z)} - \alpha\right\} \geq \left|\frac{z(Jf(z))'}{Jf(z)} - 1\right|, z \in U$$

Now let's write $JTf(z) = J * f(z) \cap T$

The origin of such classes is introduced and studied by various authors including [10-15], [16-20], and [20-34].

2. Characterization Property

Theorem 1. A function f defined by (2) is in the class $JTf(z)$if and only if

$$\sum_{n=2}^{\infty}\frac{1}{(2n)!}\cdot\frac{2n-1-\alpha}{1-\alpha}|a_n|\leq 1$$

Proof. It suffices to show that

$$\left|\frac{z(Jf(z))'}{Jf(z)}-1\right|\leq \Re\left\{\frac{z(Jf(z))'}{Jf(z)}-\alpha\right\}$$

and we have

$$\left|\frac{z(Jf(z))'}{Jf(z)}-1\right|\leq \Re\left\{\frac{z(Jf(z))'}{Jf(z)}-1\right\}+(1-\alpha)$$

that is

$$\left|\frac{z(Jf(z))'}{Jf(z)}-1\right|-\Re\left\{\frac{z(Jf(z))'}{Jf(z)}-1\right\}\leq 2\left|\frac{z(Jf(z))'}{Jf(z)}-1\right|\leq\frac{\sum_{n=2}^{\infty}(n-1)\,1/2n!\,|a_n|}{1-\sum_{n=2}^{\infty}1/2n!\,|a_n|}$$

The above expression is bounded by $(1-\alpha)$ and hence the assertion of the result

Now we want to show that $f\in JTf(z)$satisfies (3)

if $f\in JTf(z)$ then (3) yields

$$\frac{1-\sum_{n=2}^{\infty}n\,/2n!\,a_n z^{n-1}}{1-\sum_{n=2}^{\infty}1/2n!\,a_n z^{n-1}}-\alpha\geq\frac{1-\sum_{n=2}^{\infty}(n-1)\,1/2n!\,a_n z^{n-1}}{1-\sum_{n=2}^{\infty}1/2n!\,a_n z^{n-1}}$$

Letting $z\to 1$ along the real axis leads to the inequality

$$\sum_{n=2}^{\infty}(2n-1-\alpha)\,1/n!\,a_n\leq 1-\alpha$$

Corollary 1. let a function f, defined by (2) belongs to the class $JTf(z)$then

$a_n\leq\frac{1}{2n!}\cdot\frac{1-\alpha}{2n-1-\alpha}$, for $n\geq 2$

Theorem 2 Let the function f, defined by (2) be in the class $JTf(z)$then

$$|z|-|z|^2\frac{1}{2(6-\alpha)}\leq|Jf(z)|\leq|z|+|z|^2\frac{1}{2(6-\alpha)}$$

$$1-|z|\frac{1}{(6-\alpha)}\leq|(Jf(z))'|\leq 1+|z|\frac{1}{(6-\alpha)}$$

The bounds above are attained for the functions given by

$$f(z)=z-\frac{1}{4(6-\alpha)}z^2$$

Theorem 3. Let a function f, be defined by (2) and

$$g(z)=z-\sum_{n=2}^{\infty}b_n z^n$$

be in the class $JTf(z)$. then the function h, defined by

$$h(z)=(1-\beta)f(z)+\beta g(z)=z-\sum_{n=2}^{\infty}c_n z^n$$

Where $c_n=(1-\beta)a_n+\beta b_n$,and $0\leq\beta\leq 1$,is also in the class $JTf(z)$

Now we define the following functions $f_j(z)$,$(j=1,2,3,\ldots\ldots m)$

of the form

$$f_j(z)=z-\sum_{n=2}^{\infty}a_{n,j}z^n,a_{n,j}\geq 0,z\in U$$

(4)

Theorem 4 *(closure theorem).* Let the functions $f_j(z)$ $(j = 1,2,3,\ldots\ldots m)$ defined by (4), be in the class $Jf_j(z)(j = 1,2,3,\ldots\ldots m)$ respectively. Then the function $h(z)$ defined by

$$h(z) = z - \frac{1}{m}\sum_{n=2}^{\infty}\left(\sum_{j=1}^{m} a_{n,j}\right)z^n$$

is in the class $Jf_\xi(z)$where

$\xi = \underset{1\le j\le m}{Max}\{\alpha_j\} with \ \ 0 \le \alpha_j < 1$

3. Results involving convolution

Theorem 5. for functions $f_j(z)$ $(j = 1,2)$ defined by (4) let $f_1(z) \in JTf_1(z)$and $f_2(z) \in JTf_2(z)$. then $f_1 * f_2 \in TR(\eta, \lambda, \gamma)$, where

$\gamma \le 1 - \frac{1}{(2n-1-\alpha)(2n-1-\beta)1/2n! - (1-\alpha)(1-\beta)},$

Theorem 6 Let the functions $f_j(z), (j = 1,2)$ defined by (4) be in the class $JTf(z)$Then $(f_1 * f_2)(z) \in JT_{\rho f(z)}$

4. The Integral Transform

We define the integral transform

$$V_\mu(f)(z) = \int_0^1 \mu\,(t)\frac{f(tz)}{t}dt$$

Where $\mu(t)$ is a real valued, non-negative weight function normalized so that $\int_0^1 \mu\,(t)dt = 1$.

Special case of $\mu(t)$ is $\mu(t) = \frac{(c+1)^\delta}{\mu(\delta)}t^c\left(\log\frac{1}{t}\right)^{\delta-1}$, $c > -1, \delta \ge 0$

Which gives the Komatu operator.

Theorem 7 Let $f \in JTf(z)$Then $V_\mu(f) \in JTf(z)$.

Theorem 8. *(radius of starlikeness*) Let $f \in JTf(z)$then $V_\mu(f)$is starlike of order $0 \le \gamma < 1$ in $|z| \prec R_1$, where

$$R_1 = \min_n\left[\left(\frac{c+n}{c+1}\right)^\delta \cdot \frac{1-\gamma(2n-1-\alpha)}{(n-\gamma)(1-\alpha)}.\frac{1}{n!}\right]^{\frac{1}{n-1}}$$

Theorem 9. $f \in JTf(z)$then $V_\mu(f)$is convex of order $0 \le \gamma < 1$,in $|z| < R_2$,where

$$R_2 = \min_n\left[\left(\frac{c+n}{c+1}\right)^\delta \frac{(1-\gamma)(2n-1-\alpha)}{n(n-\gamma)(1-\alpha)}.\frac{1}{n!}\right]^{\frac{1}{n-1}}$$

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