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Modify Conditions of Hardy-Rogers' Fixed Point Theorem

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ABSTRACT.

A basic version of the Hardy-Rogers fixed point theorem is presented in this study.

Keywords: Banach theory; generalized space; fixed point theorem; contractive mapping.

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1. Introduction

It is well known that the Banach contraction principle plays an important role in various fields of science, especially in functional analysis and applied mathematics. Banach [2] proved the exists and uniqueness of a point $x \in X$, such that $T: X \to X$ is a contraction mapping, i.e. for all $x, y \in X, \lambda \in [0,1)$,

$$d(Tx, Ty) \leq \lambda d(x, y) \quad (1.1)$$

In [10] Kannan generalized (1.1), for all $x, y \in X$ where $\alpha \in (0, \frac{1}{2})$.

$$d(Tx, Ty) \leq \alpha[d(Tx, x) + d(Ty, y)], \quad (1.2)$$

Then many attempts were made that expanded and developed to (1.1), Reich [17] obtained his results in turn generalized to (1.2). In [9] Hardy and Rogers introduced a generalization of a fixed point theorem of Reich, as the following:

Theorem 1.1 Let (X, d) be a metric space and T a self-mapping on$X$, for all $x, y \in X$ satisfying the condition

1. $d(Tx, Ty) \leq ad(x, Tx) + bd(y, Ty) + cd(x, Ty) + ed(y, Tx) + fd(x, y)$ (1.3)

Where, a, b, c, e, f are nonnegative and we set $\alpha = a + b + c + d + f$. Then

a. If X is a complete and $\alpha < 1$, T has a unique fixed point.

b. If (1.3) is modified to the condition $x \neq y$ implies

1. $d(Tx, Ty) < ad(x, Tx) + bd(y, Ty) + cd(x, Ty) + ed(y, Tx) + fd(x, y)$,

in this case we assume X is compact, T is continuous and $\alpha = 1$, then T has a unique fixed point.

Recently, many researchers have extensively studied these types of fixed point theorems ([4-8], [12-15]). Many of the concepts have been introduced recently in the Hardy-Rogers theory from those studies we mention, Rangamma [16] proved Hardy and Rogers type common fixed point theorem for a family of self-maps in cone 2-metric spaces, in the same way. Chifu [3] present some fixed point results in b-metric spaces using a contractive condition of Hardy-Rogers type concerning the functional H. A new direction to the literature of common fixed point theorems related to T-Hardy-Rogers contraction mappings, Banach pair of mappings, and cone metric spaces due to Rhymend in [18]. A modified class of Hardy-Rogers p-proximate cyclic contraction in uniform spaces was introduced by Olisama [11]. Abbas [1] proved some fixed point theorems for a T-Hardy-Rogers contraction in the setting of partially ordered partial metric spaces. Some fixed point theorems for a generalized almost Hardy-Rogers type F-contraction in metric-like space were introduced by Saipara in [19]. The mean result of this paper is to study the modified conditions of Hardy-Rogers fixed point theorem and reprove.

Modify The Hardy-Rogers Fixed Point Theorem.

Theorem 2.1 Let X be a Complete metric space and let $f: X \to X$ a continuous self-mapping on X, suppose f satisfying the condition (1.3) for all $x, y \in X$, $x \neq y$ and for some $m_1, m_2, m_3, m_4, m_5 \in [0,1)$ such that $\sum_{i=1}^{5} m_i < 1$. Then f has a unique fixed point.

Proof. Let x_0 be an arbitrary point in X and $\{X_{n-1}\}_{n=1}^{\infty}$ be the sequence of iterations for f at x_0, such that

$$f(x_{n-1}) = x_n. \qquad (2.1)$$

We let that $x_{n-1} \neq x_n$ for all $n \in N$. Therefore $d(x_{n-1}, x_n) = d(f(x_{n-2}), f(x_{n-1}))$, by (1.3) we get

$$d(x_{n-1}, x_n) \leq m_1 d(x_{n-2}, f(x_{n-2})) + m_2 d(x_{n-1}, f(x_{n-1})) + m_3 d(x_{n-2}, f(x_{n-1})) + \quad + m_4 d(x_{n-1}, f(x_{n-2})) + m_5 d(x_{n-2}, x_{n-1}).$$

By (2.1) we get

$$d(x_{n-1}, x_n) \leq m_1 d(x_{n-2}, x_{n-1}) + m_2 d(x_{n-1}, x_n) + m_3 d(x_{n-2}, x_n) + m_4 d(x_{n-1}, x_{n-1}) + \quad + m_5 d(x_{n-2}, x_{n-1}).$$

By triangle inequality for some $x_{n-2} \leq x_{n-1} \leq x_n$, we obtained

$$d(x_{n-1}, x_n) \leq m_1 d(x_{n-2}, x_{n-1}) + m_2 d(x_{n-1}, x_n) + m_3 d(x_{n-2}, x_{n-1}) + m_3 d(x_{n-1}, x_n) + m_5 d(x_{n-2}, x_{n-1}).$$

$$= \left(\frac{m_1 + m_3 + m_5}{1 - m_2 - m_3}\right) d(x_{n-2}, x_{n-1}) \qquad (2.2)$$

And,

$$d(x_{n-1}, x_n) \leq \left(\frac{m_1 + m_3 + m_5}{1 - m_2 - m_3}\right)^2 d(x_{n-3}, x_{n-2}).$$

So, if we repeat this work we obtained

$$d(x_{n-1}, x_n) \leq \left(\frac{m_1 + m_3 + m_5}{1 - m_2 - m_3}\right)^n d(x_0, x_1) \qquad (2.3)$$

For some $s \geq n - 1$, we have

$d(x_{n-1}, x_s) \leq d(x_{n-1}, x_n) + d(x_n, x_{n+1}) + \cdots + d(x_{s-1}, x_s)$ by (4) we conclusion that

$d(x_{n-1}, x_s) \leq \{\beta^n + \beta^{n+1} + \cdots + \beta^s\} d(x_0, x_1)$, where $\beta = \left(\frac{m_1 + m_3 + m_5}{1 - m_2 - m_3}\right)$, and since $\sum_{i=1}^{5} m_i < 1$. Therefore $\beta^n \to 0 \ as \ n \to \infty$.

Then, $d(x_{n-1}, x_s) \to 0 \ as \ s \to \infty$.

Every Cauchy sequence $\{x_{n-1}\}_{n=1}^{\infty}$ in X is convergence, since X is Banach space. i.e. There exist $z_1 \in X$ such that $x_n \to z_1$, also we have a continuous map, then

$$f\left(\lim_{n\to\infty} x_n\right) = f(z_1), \quad \lim_{n\to\infty} x_n = z_1.$$

Hence, z_1 is a fixed point of f in X.

We need to prove that z_1 is a unique fixed point of f in X, for that we let there exist another fixed point $z_2 \in X$, such that $z_1 \neq z_2$ and $f(z_1) = z_1$, $f(z_2) = z_2$, therefore by (1.3)

$$d(z_1, z_2) = d(f(z_1), f(z_2)) \leq m_1 d(z_1, f(z_1)) + m_2 d(z_2, f(z_2)) + m_3 d(z_1, f(z_2)) + \quad + m_4 d(z_2, f(z_1)) + m_5 d(z_1, z_2).$$

$$d(z_1, z_2) \leq (m_3 + m_4 + m_5) d(z_1, z_2), \quad (m_3 + m_4 + m_5) < 1.$$

Then $d(z_1, z_2) \leq 0$, which implies that $d(z_1, z_2) = 0$, so $z_1 = z_2$ and z_1 is a unique fixed point of f in X.

Theorem 2.2 Let X be a Banach space and let f_1, f_2 two continuous self-mappings on, satisfying the condition

$$d(f_1(x), f_2(y)) \leq m_1 d(x, f_1(x)) + m_2 d(y, f_2(y)) + m_3 d(x, f_2(y)) + m_4 d(y, f_1(x)) + \quad + m_5 d(x, y)$$

(2.4) for all $x, y \in X$, $x \neq y$ and for some $m_1, m_2, m_3, m_4, m_5 \in [0,1)$ such that $\sum_{i=1}^{5} m_i < 1$. Then f_1 and f_2 have a unique common fixed point.

Proof. For $x_0 \in X, y_0 \in X$ take $f_1\ (x_{n-1}) = x_n$, $f_2\ (y_{n-1}) = y_n$. So,

$$d(x_k, y_k) = d\left(f_1\ (x_{k-1}), f_2\ (y_{k-1})\right) \leq m_1 d(x_{k-1}, f_1(x_{k-1})) + m_2 d(y_{k-1}, f_2(y_{k-1})) + m_3 d(x_{k-1}, f_2(y_{k-1})) +$$

$$+ m_4 d(y_{k-1}, f_1(x_{k-1})) + m_5 d(x_{k-1}, y_{k-1}).$$

$$= m_1 d(x_{k-1}, x_k) + m_2 d(y_{k-1}, y_k) + m_3 d(x_{k-1}, y_k) + m_4 d(y_{k-1}, x_k) + \quad + m_5 d(x_{k-1}, y_{k-1}).$$

Also we have

$$\sum_{k=1}^{n} d(x_k, y_k) = \sum_{k=1}^{n} d\left(f_1\ (x_{k-1}), f_2\ (y_{k-1})\right) \leq$$

$$\sum_{k=1}^{n}[m_1 d(x_{k-1},x_k)+m_2 d(y_{k-1},y_k)+m_3 d(x_{k-1},y_k)+m_4 d(y_{k-1},x_k)+ \quad +m_5 d(x_{k-1},y_{k-1})].$$

$$\sum_{k=1}^{n} d(x_k,y_k)\le [m_1 d(x_0,x_n)+m_2 d(y_0,y_n)+m_3\sum_{k=1}^{n} d(x_{k-1},y_k)+$$

$$+m_4\sum_{k=1}^{n} d(y_{k-1},x_k)+m_5\sum_{k=1}^{n} d(x_{k-1},y_{k-1})].$$

And,

$$\sum_{k=1}^{n} d(x_{k+1},y_k)\le [m_1 d(x_1,x_n)+m_2 d(y_0,y_n)+m_3\sum_{k=1}^{n} d(x_k,y_k)+$$

$$+m_4\sum_{k=1}^{n} d(y_{k-1},x_{k+1})+m_5\sum_{k=1}^{n} d(x_k,y_{k-1})].$$

Also,

$$\sum_{k=1}^{n} d(x_k,x_{k+1})\le (m_1+m_5)d(x_0,x_n)+(m_2+m_3)d(x_1,x_{n+1}).$$

We get that

$$\sum_{k=1}^{n} d(x_k,x_{k+1})\le \sum_{k=1}^{n} d(x_k,y_k)+\sum_{k=1}^{n} d(y_k,x_{k+1}).$$

Hence,

$$\sum_{k=1}^{n} d(x_k,x_{k+1})<\infty. \qquad (2.5)$$

This implies $d(x_k,x_{k+1})\to 0$ as $k\to\infty$, we see that $\{x_k\}$ is a

Cauchy sequence in$X$. Similarly we can show that $\{y_k\}$ is a Cauchy sequence in X, and since X is a Banach space, then X is complete, so there exist a common fixed point in X,

Let $z_1=\lim_{n\to\infty} x_n$, $z_2=\lim_{n\to\infty} y_n$, for all $z_1,z_2\in X$.

This implies that

$$d(x_n,z_1)\to 0 \ , \quad n\to\infty$$

$$d(y_n,z_2)\to 0 \ , \quad n\to\infty$$

As f_1,f_2 are continuous mapping we get

$$d(f_1(x_n),f_1(z_1))\to 0 \ , \quad n\to\infty$$

$$d(f_2(y_n),f_2(z_2))\to 0 \ , \quad n\to\infty$$

That is meaning

$$d\big(z_1,f_1(z_1)\big)=d(f_1^{-1}(f_1(z_1),f_1(z_1))\le m_1 d(f_1(z_1),f_1^{-1}\big(f_1(z_1)\big)+m_2 d(z_1,f_1(z_1))+$$

$$+m_3 d(f_1(z_1),f_1(z_1))+m_4 d(z_1,f_1^{-1}\big(f_1(z_1)\big)+m_5 d(f_1(z_1),z_1) \qquad (2.6)$$

$$=m_1 d(f_1(z_1),z_1)+m_2 d\big(z_1,f_1(z_1)\big)+m_5 d(f_1(z_1),z_1).$$

$$=(m_1+m_2+m_5)\,d\big(z_1,f_1(z_1)\big)\,.$$

Hence,

$$f_1(z_1)=z_1.$$

Similarly we can show that $f_2(z_2)=z_2$.

Now, for each $m_1,m_2,m_3,m_4,m_5\in[0,1)$ and $z_1,z_2\in X$, we get

$$d(z_1,z_2)=d\big(f_1(z_1),f_2(z_2)\big)\le m_1 d\big(z_1,f_1(z_1)\big)+m_2 d\big(z_2,f_2(z_2)\big)+m_3 d\big(z_1,f_2(z_2)\big)+$$

$$+m_4 d(z_2,f_1(z_1)+m_5 d(z_1,z_2).$$

$$d(z_1,z_2)\le m_1 d(z_1,z_1)+m_2 d(z_2,z_2)+m_3 d(z_1,z_2)+m_4 d(z_2,z_1)+m_5 d(z_1,z_2).$$

$$=(m_3+m_4+m_5)\,d(z_1,z_2)$$

Therefore,

$$d(z_1,z_2)\le 0.$$

Hence,

$z_1=z_2$. This implies that z_1 is common fixed point of f_1 and f_2 in X.

Now, let $z_3 \in X$ be another fixed point of f_1 and f_2 in X such that

$f_1(z_3) = z_3$ and $f_2(z_3) = z_3$.

Therefore,

$$d(z_1, z_3) = d\big(f_1(z_1), f_2(z_3)\big) \leq m_1 d\big(z_1, f_1(z_1)\big) + m_2 d\big(z_3, f_2(z_3)\big) + m_3 d\big(z_1, f_2(z_3)\big) +$$

$$+ m_4 d(z_3, f_1(z_1)) + m_5 d(z_1, z_3).$$

$$d(z_1, z_3) \leq m_1 d(z_1, z_1) + m_2 d(z_3, z_3) + m_3 d(z_1, z_3) + m_4 d(z_3, z_1) + m_5 d(z_1, z_3).$$

$$= (m_3 + m_4 + m_5) d(z_1, z_3).$$

Then, $d(z_1, z_3) \leq 0$.

This implies that $z_1 = z_2 = z_3$. Thus z_1 is the unique common fixed point of f_1 and f_2 in X.

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