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Binary Not-Homogeneous Third-Degree Diophantine Equation $x^2 + xy = y^3 + 2y^2$

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ABSTRACT

The article empathize on finding non-zero different solutions in integers to binary third degree diophantine equation $x^2 + xy = y^3 + 2y^2$. Different sets of solutions in integers are presented. Some fascinating relations from the solutions are obtained. The method to get second order Ramanujan numbers is exhibited..

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Notations :

$$P_s^5 = \frac{s^2(s+1)}{2} \quad , \quad t_{m,s} = \frac{s[2+(s-1)(m-2)]}{2}$$

$$P_s^3 = \frac{s\,(s+1)\,(s+2)}{6} \quad , \quad PEN_s = \frac{s(s+1)(s+2)(s+3)}{24}$$

1. Introduction

The third degree Diophantine equations are enormous in variety and they have contributed to expansion of research in this filed[1,2]. For an extensive approach of these types of problems , one may refer [3-28]. In this article a search is made to get solutions in integers for the considered problem through different methods and also the method of getting second order Ramanujan numbers from the obtained solution is discussed. Some fascinating relations from the solutions are presented.

2. Methods for finding solutions

Consider

$$x^2 + xy = y^3 + 2y^2 \tag{2.1}$$

Method 2.1

Treating (2.1) as a quadratic in x and solving for the same ,

we have

$$x = \frac{y[-1 \pm \sqrt{4y+9}]}{2} \tag{2.2}$$

The square-root on the R.H.S. of (2.2) is removed when

$$y = y(a) = a(a+3) \tag{2.3}$$

and from (2.2) , taking the positive sign before the square-root ,

it is obtained that

$$x = x(a) = a\ (a+1)(a+3) \tag{2.4}$$

It is seen that (2.3) and (2.4) satisfy (2.1).

After performing some algebra , a few solutions in integers are given below

in Table 2.1:

Table 2.1 Solutions in integers

a	$x(a)$	$y(a)$
1	8	4
2	30	10
3	72	18
4	140	28
5	240	40

3. Fascinating relations

(i) $y(a+2) - 2y(a+1) + y(a) = 2$

(ii) $y(a+3) + 3y(a+1) = y(a) + 3y(a+2)$

(iii) $x(a+3) - 3x(a+2) + 3x(a+1) - x(a) = 6$

(iv) $x(a+2) - 2x(a+1) + x(a) \cong 2\,\mathrm{mod}(6)$

(v) $\sum_{a=1}^{s} y(a) = \frac{2[P_s^5 + 5t_{3,s}]}{3}$

(vi) $\sum_{a=1}^{s} x(a) = 6PEN_s + 2P_s^3$

(vii) $x(a) = 2P_a^5 + 6t_{3,a}$

(viii) $x(a) = 6P_a^3 + 2t_{3,a}$

(ix) $x(a) = 9P_a^3 - P_a^5$

(x) $3[x(a) - y(a) - 2P_a^5]$ is a nasty number

(xi) $3[3x(a) - y(a) - 18P_a^3]$ is a square multiple of 6

(xii) $y(2s^2+4s+1)-y(2s^2+4s)$ is a perfect square

(xiii) $y(5s-3)$ is a multiple of 10

(xiv) $y(4s^3-1)-y(4s^3-2)$ is a perfect cube

(xv) $x(a)-y(a)$ is a perfect square when $a=s^2\pm 2s-2$

(xvi) $x(a)-3y(a)+6a=2P_a^5$

4.Formulation of second order Ramanujan numbers (R_2 numbers)

From each of the solutions of (2.1) given by (2.3) &(2.4), we can find R_2 numbers having base numbers as real integers as well as gaussian integers.

Illustration 4.1

Consider (2.3) as

$$\begin{aligned} y(a) &= (a^2+3a) \\ &= (a^2+3a)*1 = (a+3)*a \\ &= \alpha*\beta = \gamma*\delta \ , say \end{aligned}$$

It is observed that

$$(\alpha+\beta)^2+(\gamma-\delta)^2=(\alpha-\beta)^2+(\gamma+\delta)^2$$
$$\Rightarrow (a^2+3a+1)^2+3^2=(a^2+3a-1)^2+(2a+3)^2=a^4+6a^3+11a^2+6a+1$$

Thus, $a^4+6a^3+11a^2+6a+1$ represents the second order Ramanujan number as it is written as sum of two squares in two different ways. Here , the base numbers are real integers.

Illustration 4.2

Consider (2.4) as

$$\begin{aligned} x(a) &= a(a+1)(a+3) \\ &= a(a+1)*(a+3) = a(a+3)*(a+1) \\ &= \mu*\eta = \lambda*\sigma \ , say \end{aligned}$$

It is seen that

$$(\mu+i\eta)^2+(\lambda-i\sigma)^2=(\mu-i\eta)^2+(\lambda+i\sigma)^2=\mu^2-\eta^2+\lambda^2-\sigma^2$$
$$\Rightarrow [a(a+1)+i(a+3)]^2+[a(a+3)-i(a+1)]^2=[a(a+1)-i(a+3)]^2+[a(a+3)+i(a+1)]^2$$
$$=2a^4+8a^3+8a^2-8a-10$$

Thus, $2a^4+8a^3+8a^2-8a-10$ represents the second order Ramanujan number as it is written as sum of two squares in two different ways. Here , the base numbers are gaussian integers

5.Remarks

Remark 5.1

In addition to the solutions (2.3) &(2.4) , we have an another set of solutions in integers to (2.1) by taking the negative sign before the square-root of (2.2) given as

$$x=x(s)=-s(s+2)(s+3)\ ,\ y=y(s)=s(s+3)$$

Remark 5.2

Albeit tacitly , we have two more sets of solutions in integers to (2.1) shown below

$$x = x(s) = s(s-2)(s-3)\ ,\ y = y(s) = s(s-3)\ ,$$
$$x = x(s) = s(1-s)(s-3)\ ,\ y = y(s) = s(s-3)$$

6.Conclusion

This article gives an approach to solve third degree equation with two unknowns though different methods to get solutions in integers. The researchers in this field may attempt to find various other methods to solve binary cubic equation and also approach to get second order Ramanujan numbers and find various other relation from the obtained solutions.

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