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Observation on the Non-Homogeneous Binary Quadratic Equation $3x^2 - 2y^2 = 4$

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ABSTRACT:

The non-homogenous binary quadratic equation with two unknowns represented by the Pell-like equation $3x^2 - 2y^2 = 4$ is studied for finding its distinct integer solutions. A few interesting properties between the above solutions are presented.

KEYWORDS: Binary quadratic, Hyperbola, Parabola, Pell equation, Integer solutions.

INTRODUCTION:

The non-homongenous binary quadratic equations of the form $ax^2 - by^2 = N, (a, b, N \neq 0)$ are rich in variety and have been analyzed by many mathematicians for their respective integer solutions for particular values of a,b and N. In this context, one may refer [1-13]. This communication concerns with the problem of obtaining non-zero distinct interger solutions to the binary quadratic equation given by $3x^2 - 2y^2 = 4$ representing hyperbola. A few interesting relations among its solutions are presented. Knowning an integral solutions of the given hyperbola,integer solution for other choices of hyperbolas and parabolas are presented.

METHOD OF ANALYSIS:

The non-homogenous binary quadratic equation under consideration is

$$3x^2 - 2y^2 = 4 \quad (1)$$

It is to be noted that (1) represents a hyberbola

Taking x=X+2T,y=X+3T (2)

in (1), it reduced to the equation

$$X^2 = 6T^2 + 4 \quad (3)$$

The smallest positive integer solution (T_0, X_0) of (3) is

$$T_0 = 4, X_0 = 10$$

To obtain the other solutions of (3), consider the pellian equations

$$X^2 = 6T^2 + 1 \quad (4)$$

whose smallest positive integer solutions is

$$\tilde{T}_0 = 2, \tilde{X}_0 = 5$$

The general solution $(\tilde{T}_n, \tilde{X}_n)$ of (4) is given by

$$\tilde{X}_n + \sqrt{6}\tilde{T}_n = (5+2\sqrt{6})^{n+1}, n = 0,1,2\ldots \quad (5)$$

Since irrational roots occur in pairs, we have

$$\tilde{X}_n - \sqrt{6}\tilde{T}_n = (5-2\sqrt{6})^{n+1}, n = 0,1,2\ldots \quad (6)$$

From (5) and (6), solving for $\tilde{X}_n, \tilde{T}_n,$ we have

$$\tilde{X}_n = \frac{1}{2}[(5+2\sqrt{6})^{n+1} + (5-2\sqrt{6})^{n+1}] = \frac{1}{2} f_n$$

$$\tilde{T}_n = \frac{1}{2\sqrt{6}}[(5+2\sqrt{6})^{n+1} - (5-2\sqrt{6})^{n+1}] = \frac{1}{2\sqrt{6}} g_n$$

Applying Brahmagupta lemma between the solutions (T_0, X_0) and $(\tilde{T}_n, \tilde{X}_n)$, the general solution $(\tilde{T}_{n+1}, \tilde{X}_{n+1})$ of (3) is found to be

$$T_{n+1} = X_0\tilde{T}_n + T_0\tilde{X}_n = \frac{5}{\sqrt{6}} g_n + 2 f_n \quad (7)$$

$$X_{n+1} = X_0\tilde{X}_n + T_0\tilde{T}_n = 2 f_n + \frac{2}{\sqrt{6}} g_n \quad (8)$$

Using (7) and (8) in (2) we have,

$$3x_{n+1} = 27 f_n + 11\sqrt{6} g_n$$

$$2y_{n+1} = 22 f_n + 9\sqrt{6} g_n$$

Thus (9) and (10) represent the integer of the hyperbola (1).

A few numerical values are given in the following Table: 1

Table: 1 Numerical Examples

n	x_{n+1}	y_{n+1}
-1	18	22
0	178	218
1	1762	2158
2	17442	21362

Recurrence relations for x and y are:

$$x_{n+3} - 10x_{n+2} + x_{n+1} = 0, n = -1,0,1\ldots..$$

$$y_{n+3} - 10y_{n+2} + y_{n+1} = 0, n = -1,0,1\ldots$$

A. A few interesting relations among the solutions are given be

- $x_{n+1} - 10x_{n+2} + x_{n+3} = 0$
- $5x_{n+1} - x_{n+2} + 4y_{n+1} = 0$

- $x_{n+1} - 5x_{n+2} + 4y_{n+2} = 0$
- $5x_{n+1} - 49x_{n+2} + 4y_{n+3} = 0$
- $49x_{n+1} - x_{n+3} + 40y_{n+1} = 0$
- $x_{n+1} - x_{n+3} + 8y_{n+2} = 0$
- $x_{n+1} - 49x_{n+3} + 40y_{n+3} = 0$
- $6x_{n+1} + 5y_{n+1} - y_{n+2} = 0$
- $60x_{n+1} + 49y_{n+1} - y_{n+3} = 0$
- $6x_{n+1} + 49y_{n+2} - 5y_{n+3} = 0$
- $4y_{n+1} + 49x_{n+2} - 5x_{n+3} = 0$
- $4y_{n+2} + 5x_{n+2} - x_{n+3} = 0$
- $4y_{n+3} + x_{n+2} - 5x_{n+3} = 0$
- $y_{n+1} + 6x_{n+2} - 5y_{n+2} = 0$
- $y_{n+1} + 12x_{n+2} - y_{n+3} = 0$
- $5y_{n+2} + 6x_{n+2} - y_{n+3} = 0$
- $49y_{n+2} - 5y_{n+1} - 6x_{n+3} = 0$
- $49y_{n+3} - y_{n+1} - 60x_{n+3} = 0$
- $5y_{n+3} - y_{n+2} - 6x_{n+3} = 0$
- $y_{n+3} - 10y_{n+2} + y_{n+1} = 0$

B. Each of following expressions represents a cubic integer

1. $\frac{1}{2}(109x_{3n+3} - 11x_{3n+4}) + 3(109x_{n+1} - 11x_{n+2})$
2. $\frac{1}{20}(1079x_{3n+3} - 11x_{3n+5}) + 3(1079x_{n+1} - 11x_{n+3})$
3. $27x_{3n+3} - 22y_{3n+3} + 3(27x_{n+1} - 22y_{n+1})$
4. $\frac{1}{5}(267x_{3n+3} - 22y_{3n+4}) + 3(267x_{n+1} - 22y_{n+2})$
5. $\frac{1}{49}(2643x_{3n+3} - 22y_{3n+5}) + 3(2643x_{n+1} - 22y_{n+3})$
6. $\frac{1}{2}(1079x_{3n+4} - 109x_{3n+5}) + 3(1079x_{n+2} - 109x_{n+3})$

7. $\frac{1}{5}(27x_{3n+4}-218y_{3n+3})+3(27x_{n+2}-218y_{n+1})$
8. $267x_{3n+4}-218y_{3n+4}+3(267x_{n+2}-218y_{n+2})$
9. $\frac{1}{5}(2643x_{3n+4}-218y_{3n+5})+3(2643x_{n+2}-218y_{n+3})$
10. $\frac{1}{49}(27x_{3n+5}-2158y_{3n+3})+3(27x_{n+3}-2158y_{n+1})$
11. $\frac{1}{5}(267_{3n+5}-2158y_{3n+4})+3(267x_{n+3}-2158y_{n+2})$
12. $2643x_{3n+5}-2158y_{3n+5}+3(2643_{n+3}-2158y_{n+3})$
13. $\frac{1}{4}(18y_{3n+4}-178y_{3n+3})+3(18y_{n+2}-178y_{n+1})$
14. $\frac{1}{40}(18y_{3n+5}-1762y_{3n+3})+3(18y_{n+3}-1762y_{n+1})$
15. $\frac{1}{4}(178y_{3n+5}-1762y_{3n+4})+3(178y_{n+3}-1762y_{n+2})$

C. Each of following expressions represents a Bi-Quatratic integers

- $\frac{1}{2}\left[109x_{4n+4}-11x_{4n+5}+4\left(109x_{2n+2}-11x_{2n+3}\right)\right]+6$
- $\frac{1}{20}\left[1079x_{4n+4}-11x_{4n+6}+4\left(1079x_{2n+2}-11x_{2n+4}\right)\right]+6$
- $\left[27x_{4n+4}-22x_{4n+4}+4\left(27x_{2n+2}-22x_{2n+2}\right)\right]+6$
- $\frac{1}{5}\left[267x_{4n+4}-22x_{4n+5}+4\left(267x_{2n+2}-22x_{2n+3}\right)\right]+6$
- $\frac{1}{49}\left[2643x_{4n+4}-22x_{4n+6}+4\left(2643x_{2n+2}-22x_{2n+4}\right)\right]+6$
- $\frac{1}{2}\left[1079x_{4n+5}-109x_{4n+6}+4\left(1079x_{2n+3}-109x_{2n+4}\right)\right]+6$
- $\left[27x_{4n+5}-218y_{4n+4}+4\left(27x_{2n+2}-11x_{2n+3}\right)\right]+6$
- $\left[267x_{4n+5}-218y_{4n+5}+4\left(267x_{2n+3}-218y_{2n+2}\right)\right]+6$
- $\left[2643x_{4n+5}-218y_{4n+6}+4\left(2643x_{2n+3}-218y_{2n+4}\right)\right]+6$
- $\left[27x_{4n+6}-2158x_{4n+4}+4\left(27x_{2n+4}-2158y_{2n+2}\right)\right]+6$
- $\left[267x_{4n+6}-2158y_{4n+5}+4\left(267x_{2n+4}-2158y_{2n+3}\right)\right]+6$
- $\left[2643x_{4n+6}-2158y_{4n+6}+4\left(2643x_{2n+4}-2158y_{2n+4}\right)\right]+6$

- $[18y_{4n+5} - 178y_{4n+4} + 4(18y_{2n+3} - 178y_{2n+2})] + 6$
- $[18y_{4n+6} - 1762y_{4n+4} + 4(18y_{2n+4} - 1762y_{2n+2})] + 6$
- $[178y_{4n+6} - 1762y_{4n+5} + 4(178y_{2n+4} - 1762y_{2n+3})] + 6$

D. Each of following expressions represents a Quintic Integer

- $\frac{1}{2}[(109x_{5n+5} - 11x_{5n+6} + 5[(109x_{3n+3} - 11x_{3n+4}) + 10(109x_{n+1} - 11x_{n+2})]]$
- $\frac{1}{20}[(1079x_{5n+5} - 11x_{5n+7} + 5[(1079x_{3n+3} - 11x_{3n+5}) + 10(1079x_{n+1} - 11x_{n+3})]]$
- $[(27x_{5n+5} - 22y_{5n+5} + 5[(27x_{3n+3} - 22y_{3n+3}) + 10(27x_{n+1} - 22y_{n+1})]]$
- $\frac{1}{5}[(267x_{5n+5} - 22y_{5n+6} + 5[(267x_{3n+3} - 22y_{3n+4}) + 10(267x_{n+1} - 22y_{n+2})]]$
- $\frac{1}{49}[(2643x_{5n+5} - 22y_{5n+7} + 5[(2643x_{3n+3} - 22y_{3n+5}) + 10(2643x_{n+1} - 22y_{n+3})]]$
- $\frac{1}{2}[(1079x_{5n+6} - 109x_{5n+7} + 5[(1079x_{3n+4} - 109x_{3n+5}) + 10(1079x_{n+2} - 109x_{n+3})]]$
- $[(267x_{5n+6} - 218y_{5n+6} + 5[(267x_{3n+4} - 218y_{3n+4}) + 10(267x_{n+2} - 218y_{n+2})]]$
- $\frac{1}{5}[(2643x_{5n+6} - 218y_{5n+7} + 5[(2643x_{3n+4} - 218y_{3n+5}) + 10(2643x_{n+2} - 218y_{n+3})]]$
- $\frac{1}{49}[(27x_{5n+7} - 2158y_{5n+5} + 5[(27x_{3n+5} - 2158y_{3n+5}) + 10(27x_{n+3} - 2158y_{n+1})]]$
- $\frac{1}{5}[(267x_{5n+7} - 2158y_{5n+6} + 5[(267x_{3n+5} - 2158y_{3n+4}) + 10(267x_{n+3} - 2158y_{n+2})]]$
- $[(2643x_{5n+7} - 2158y_{5n+7} + 5[(2643x_{3n+5} - 2158y_{3n+5}) + 10(2643x_{n+3} - 2158y_{n+3})]]$
- $\frac{1}{4}[(18y_{5n+6} - 178y_{5n+5} + 5[(18y_{3n+4} - 178y_{3n+3}) + 10(18y_{n+2} - 178y_{n+1})]]$
- $\frac{1}{40}[(18y_{5n+7} - 1762y_{5n+5} + 5[(18y_{3n+5} - 1762y_{3n+3}) + 10(18y_{n+3} - 1762y_{n+1})]]$
- $\frac{1}{4}[(178y_{5n+7} - 1762y_{5n+6} + 5[(178y_{3n+5} - 1762y_{3n+4}) + 10(178y_{n+3} - 17621y_{n+2})]]$
- $\frac{1}{5}[(27x_{5n+6} - 218y_{5n+5} + 5[(27x_{3n+4} - 218y_{3n+3}) + 10(27x_{n+2} - 218y_{n+1})]]$

III. REMARKABLE OBSERVATIONS:

C. Employing linear combinations among the solutions of (1), one may generate integers solutions for other choices of hyperbolos which are presented in table : 2 below.

Table: 2 Hyperbolas

S.No	Hyperbolas	(P, Q)
1.	$6P^2 - Q^2 = 96$	$[(109x_{n+1} - 11x_{n+2}), (27x_{n+2} - 267x_{n+1})]$
2.	$6P^2 - Q^2 = 9600$	$[(1079x_{n+1} - 11x_{n+3}), 3(9x_{n+3} - 881x_{n+1})]$
3.	$6P^2 - 2Q^2 = 24$	$[(27x_{n+1} - 22y_{n+1}), (54y_{n+1} - 66x_{n+1})]$
4.	$6P^2 - Q^2 = 600$	$[(267x_{n+1} - 22y_{n+2}), (54y_{n+2} - 654x_{n+1})]$
5.	$6P^2 - Q^2 = 57624$	$[(2643x_{n+1} - 22y_{n+3}), (54y_{n+3} - 6474x_{n+1})]$
6.	$6P^2 - Q^2 = 96$	$[(1079x_{n+2} - 109x_{n+3}), (267x_{n+3} - 2643x_{n+2})]$
7.	$6P^2 - Q^2 = 600$	$[(27x_{n+2} - 218y_{n+1}), (534y_{n+1} - 66x_{n+2})]$
8.	$6P^2 - Q^2 = 24$	$[(267x_{n+2} - 218y_{n+2}), (534y_{n+2} - 654x_{n+2})]$
9.	$6P^2 - Q^2 = 600$	$[(2643x_{n+2} - 218y_{n+3}), (534y_{n+3} - 6474x_{n+2})]$
10.	$6P^2 - Q^2 = 57624$	$[(27x_{n+3} - 2158y_{n+1}), (5286y_{n+1} - 66x_{n+3})]$
11.	$6P^2 - Q^2 = 600$	$[(267x_{n+3} - 2158y_{n+2}), (5286y_{n+2} - 654x_{n+3})]$
12.	$6P^2 - Q^2 = 24$	$[(2643x_{n+3} - 2158y_{n+3}), (5286y_{n+3} - 6474x_{n+3})]$
13.	$6P^2 - Q^2 = 384$	$[(18y_{n+2} - 178y_{n+1}), (436y_{n+1} - 44y_{n+2})]$
14.	$6P^2 - Q^2 = 6400$	$[(18y_{n+3} - 1762y_{n+1}), (4316y_{n+1} - 44y_{n+3})]$
15.	$6P^2 - Q^2 = 384$	$[(178y_{n+3} - 1762y_{n+2}), (4316y_{n+2} - 436y_{n+3})]$

D. Employing linear combination among the solutions of (1), one may generate integer solutions for other choices of parabolas which are presented in Table: 3 below:

Table: 3 Parabolas

S. No	Parabolas	(R, Q)
1.	$24R + 2Q^2 = 192$	$[(109x_{2n+2} - 11x_{2n+3} + 4), (27x_{n+2} - 267x_{n+1})]$
2.	$120R + Q^2 = 9600$	$[(1079x_{2n+2} - 11x_{2n+4} + 40), (3(9x_{n+3} - 881x_{n+1})]$
3.	$6R + Q^2 = 24$	$[(27x_{2n+2} - 22y_{2n+2} + 2), (54y_{n+1} - 66x_{n+1})]$

4.	$30R+Q^2=600$	$[(267x_{2n+2}-22y_{2n+3}+10),(54y_{n+1}-654x_{n+1})]$
5.	$294R+Q^2=57624$	$[(2643x_{2n+2}-22y_{2n+4}+98),(54y_{n+3}-6474x_{n+1})]$
6.	$12R+Q^2=96$	$[(1079x_{2n+3}-109x_{2n+4}+4),(267x_{n+3}-2643x_{n+2})]$
7.	$30R+Q^2=600$	$[(27x_{2n+3}-218y_{2n+2}+10),(534y_{n+1}-66x_{n+2})]$
8.	$6R+Q^2=24$	$[(267x_{2n+3}-218y_{2n+3}+2),(534y_{n+2}-654x_{n+2})]$
9.	$30R+Q^2=600$	$[(2643x_{2n+3}-218y_{2n+4}+10),(534y_{n+3}-6474x_{n+2})]$
10.	$294R+Q^2=57624$	$[(27x_{2n+4}-2158y_{2n+2}+98),(5286y_{n+1}-66x_{n+3})]$
11.	$30R+Q^2=600$	$[(267x_{2n+4}-2158y_{2n+3}+10),(5286y_{n+2}-654x_{n+3})]$
12.	$6R+Q^2=24$	$[(2643x_{2n+4}-2158y_{2n+4}+2),(5286y_{n+3}-6474x_{n+3})]$
13.	$24R+Q^2=384$	$[(18y_{2n+3}-178y_{2n+2}+8),(436y_{n+1}-44y_{n+2})]$
14.	$240R+Q^2=38400$	$[(18y_{2n+4}-1762y_{2n+2}+80),(4316y_{n+1}-44y_{n+3})]$
15.	$24R+Q^2=384$	$[(178y_{2n+4}-1762y_{2n+3}+8),(4316y_{n+2}-436y_{n+3})]$

CONCLUSION:

In this Paper, we have presented infinitely many integer solutions for the Non-homogeneous equations represented by hyperbola given by $5x^2-3y^2=18$. Non-homogeneous binary quadratic equations are rich in variety, one may search for the choices of equations and determine their integer solutions along with suitable properties.

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