International Journal of Research Publication and Reviews

Journal homepage: www.ijrpr.com ISSN 2582-7421

Some Optimisation Problems Related to Right Circular Cones Cylinder Hemisphere Projectile Motion and Minimum Distance between a Point and a Curve

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Abstract

Maximum volume and surface area of a cylinder inscribed in a cone are determined. Minimum distance between a point and a curve whose equation is given in cartesian as well as in polar coordinates is also determined. A spherical ball is inscribed in a right circular cone and the distance between the centre of the ball and the vertex of the cone is found out with analysis of some sort of maxima and minima. Some more problems of optimization, viz, maximization/minimization concerning projectile motion are innovated and solved.

INTRODUCTION

Some problems of optimization ,ie, maximization/minimization of a function with or without

constraints are available in text books[1,2]. Furthermore, SN Maitra[2] published several papers(referenced) innovating some problems and thereof solutions on this line. In this paper the author presented some fresh problems including those of projectile motion in real world together with their solutions.

MAXIMUM VOLUME AND SURFACE AREA OF A CYLINDER INSCRIBED IN A RIGHT CIRCULAR CONE

Let H and R be the height and radius of a cone in which is inscribed a cylinder. If h and r are the height and radius of the cylinder,then by drawing an appropriate figure and considering similar triangles by geometry is obtained

$\frac{H}{R}=\frac{H-h}{r}$ (1)

Or, $h=H(1-\frac{r}{R})$ (2)

Using (2),volume of the cylinder is given by

$V=\pi r^2h=\pi r^2 H(1-\frac{r}{R})$ (3)

For maximum or minimum of V,

$\frac{dV}{dr}=H\pi\left(2r-\frac{3r^2}{R}\right)=0$

Or, $r=\frac{2R}{3}$ (4)

which gives optimum radius of the cylinder for its maximum volume inscribed in the cone which is given by

$V_{max}=\frac{4\pi HR^2}{27}$ (5)

Using equation(2),curved surface area S of the cylinder inscribed in the cone is obtained as

$S=2\pi rh=2\pi rH(1-\frac{r}{R})$ (6)

For maximum/ minimum of S using (6),

$\frac{dS}{dr}=2\pi H(1-\frac{2r}{R})=0$

$r=\frac{R}{2}$ (7)

which gives maximum curved surface area of the cylinder inscribed in the cone and as such because of (6), the same is given by

$S_{max} = \pi\frac{HR}{2}$ (8)

MAXIMUM SURFACE AREA OF THE CLOSED CYLINDER INSCRIBED IN THE CONE

In the light of preceding equation(2) , surface area S of the closed cylinder inscribed in the cone turns out to be

$S=2\pi r H\left(1-\frac{r}{R}\right)+2\pi r^2$ (9)

Or, $S=2\pi r\{H-r(\frac{H}{R}-1)\}$

For maximum or minimum of S,

$\frac{dS}{dr}=2\pi\{H-2r(\frac{H}{R}-1)\}=0$

Or, $2(\frac{H}{R}-1)r=H$ H>R

$r=\frac{H}{2\left(\frac{H}{R}-1\right)}$ (10)

using which in (2) is obtained the optimum value of h:

$h=H\left(1-\frac{r}{R}\right)=H\left(1-\frac{H}{2R\left(\frac{H}{R}-1\right)}\right)$ (10.1)

$=H\left(\frac{H-2R}{2R\left(\frac{H}{R}-1\right)}\right)=H\left(\frac{\frac{H}{R}-2}{2\left(\frac{H}{R}-1\right)}\right)$

(10) and (10.1)denote optimum values of r and h leading to the maximum surface area of the closed cylinder inscribed in the cone:

$S_{max}=2\pi\{\left(\frac{H}{2\left(\frac{H}{R}-1\right)}\right)^2+\frac{H}{2\left(\frac{H}{R}-1\right)}.\frac{H\left(\frac{H}{R}-2\right)}{2\left(\frac{H}{R}-1\right)}\}$

Or, $S_{max}=\frac{\pi H^2}{2\left(\frac{H}{R}-1\right)}$ (11)

MINIMUM DISTANCE BETWEEN A POINT AND A CUBIC CURVE

Let the point be (4,0) lying on the X-axis while the equation the cubic curve is

$y^2=(x-2)^2(x-1)$ (12)

The distance S between the given point and a current point on the curve(12) is:

$S^2=(x-4)^2+y^2$

Or, using (12),

$S^2=(x-4)^2+(x-2)^2(x-1)$ (13)

For maxima/minima of S,

$\frac{dS^2}{dx}=2(x-4)+2(x-2)(x-1)+(x-2)^2=0$

Or, $2x-8+2(x^2-3x+2)+x^2-4x+4$ (14)

$3x^2-8x=0$

Or, $x=0,\frac{8}{3}$ (15)

$\frac{d^2(S^2)}{dx^2}=8$ at $x=\frac{8}{3}$ (16)

Because of (15) and (12),

When $x=\frac{8}{3}, y^2=\frac{20}{27}$ (16.1)

Using (16.1) in (13) is obtained the minimum distance:

$S^2=(\frac{8}{3}-4)^2+\frac{20}{27}=\frac{68}{27}$

$S_{min}=2.5$ (approx.) (17)

which gives the minimum distance between the given point and the cubical curve (12).

MINIMIZATION OF A FUNCTION SUBJECT TO TWO CONSTRAINTS

Let us find the minimum[1] value of the function

S(x,y,z)=$x^2+y^2+z^2$ (18)

Subject to the conditions

x+y+z=1 (19)

xyz=-1 (20)

With (18) and Lagrange's Multiplier we construct a function F as

F=$x^2+y^2+z^2+\lambda(1-x-y-z)+\mu(-1-xyz)$ (21)

$\frac{\delta F}{\delta x}=2x-\lambda-\mu yz=0$ (22)

$\frac{\delta F}{\delta y}=2y-\lambda-\mu xz=0$ (23)

$\frac{\delta F}{\delta y}=2z-\lambda-\mu xy=0$ (24)

Multiplying (22) by x and(23) by y and then subtracting,

$2(x^2-y^2)-\lambda(x-y)$=0

which gives

x=y (25)

Applying (25) in (19) and (20) we get

2x+z=1 (26)

$x^2z=-1$ (27)

Eliminating z between (26) and (27),

$2x^3-x^2$-1=0

$2x^3-2x^2+x^2$-1=0

$2x^2(x-1)+(x-1)(x+1)=0$

Or,(x-1)(2 x^2+x+1) *=0*

x=1, (2 $x^2+x+1=0$ gives imaginary value of x.) (28)

Hence by virtue of (28),(27),(25) and (26) , we get

z=-1, x=y=1 (29)

which leads to maximum value of(18) as

$S_{max=3}$ (30)

Another method without applying Lagrange's Multiplier:

Using (19) and (20), (y^2+z^2) can be expressed in terms of x

$y^2+z^2=(y+z)^2-2yz$ (By use of (19) and(20))

$y^2+z^2=(y+z)^2$-2yz =$(1-x)^2+\frac{2}{x}$ (31)

Using (31) in (18),

S= $2x^2-2x+1+\frac{2}{x}$ (32)

For maxima or minimum of S,

$\frac{dS}{dx}=$ 4x-2-$\frac{2}{x^2}$ =o

0r, $2x^3-x^2$-1=0

The remaining steps are as done previously to arrive at equation (28) to get x=1=y and z=-1, ultimately S_{max}=3.

In this context is suggested another problem:

To find the maximum value of xy+yz+zx subject to the conditions: x+y+z=1, xyz=-1.

MINIMUM DISTANCE BETWEEN A POINT AND CURVE OF POLAR EQUATION

The distance S between a point (b,0) and any point(r, θ) *on* a curve called cardiod

r=a(1+cosθ) (33)

is given by

$S^2=(rcos\theta-b)^2+(rsin\theta)^2=r^2-2brcos\theta+b^2$ (34)

Eliminating $cos\theta$ from (34) by use of (33),

S^2=$r^2 - 2br\frac{r-a}{a} + b^2$ (35)

For maxima/minima of S,

$\frac{dS^2}{dr}$=2r-2$b\left(\frac{2r-a}{a}\right) = 0$

$\frac{d^2(S^2)}{dx^2} > 0$ at r=$\frac{ab}{a-2b}$ (36)

Using (36)in (35)is obtained the minimum distance so that

$S^2 = r^2\left(1 - \frac{2b}{a}\right) + 2br + b^2$

$$S_{max}^2 = \left(\frac{ab}{a-2b}\right)^2\left(1 - \frac{2b}{a}\right) + b^2\left(\frac{3a-2b}{a-2b}) \right) \quad (37)$$

MAXIMUM/MINIMUM LENGTHS OF SUBTANGENT AND SUBNORMAL TO A CURVE

Let us find the maximum length of subtangent to a curve given as an exercise in textbook[1]:

Y=$\frac{a^3}{x^2+a^2}$ (38)

Length of the subtangent[1] at any point (x,y) on the curve (38):

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Logy=3loga-log$(x^2 + a^2)$

Or, $\frac{dy}{ydx} = -\frac{2x}{x^2+a^2}$

Hence length of the subtangent is

$S.T. = \frac{ydx}{dy}$= $\frac{-(x^2+a^2)}{2x}$ (39)

$\frac{d(S.T.)}{dx}$=$\frac{-1}{2}(1 - \frac{a^2}{x^2})$=0 $\frac{d^2S.T.)}{dx^2}$<0

Or, x=$\pm$a for maximum subtangent (40)

which leads (39) to yield maximum subtangent as

$(S.T.)_{max} = a$ (41)

S.N. =$\frac{1}{2}\frac{dy^2}{dx}$=$\frac{1}{2}\frac{d(\frac{a^3}{x^2+a^2})^2}{dx} = -a^6\frac{2x}{(x^2+a^2)^3} = \frac{-2x}{((\frac{x}{a})^2+1)^3}$ (42)

For maxima/ minima of the S.N.,

$\frac{d(S.N.)}{dx} = -2a^6\{\frac{1}{(x^2+a^2)^3} - \frac{6x^2}{(x^2+a^2)^4}$ } =0

Or, a^2-5x^2=0

Or,x=$\pm\frac{a}{\sqrt{5}}$ (43)

Or, $(x^2 + a^2)^4\frac{d(S.N.)}{dx}$=$-a^6(a^2-5x^2)$

Differentiating this equation w.r.t. x,

$(x^2 + a^2)^4\frac{d^2(S.N.)}{dx^2}$+$\frac{d(S.N.)}{dx}\frac{d(x^2+a^2)^4}{dx}$=10$a^6$x=10$a^6\frac{a}{\sqrt{5}}$>0 (44)

At x=$\pm\frac{a}{\sqrt{5}}$, because of (42), we get minimum length of S.N.:

$(S.N.)_{min} = \frac{25\sqrt{5}}{108}a$ (45)

MAXIMUM/MINIMUM LENGTHS OF SUBTANGENT AND SUBNORMAL TO A CISSOID

Equation of the cissoid is

$y^2 = \frac{x^3}{2a-x}$ (46)

Or,2logy=3logx-log(2a-x)

Or, 2$\frac{dy}{ydx} = \frac{3}{x} + \frac{1}{2a-x} = \frac{6a-2x}{x(2a-x)}$

Or,Subtangent at point(x,y) is

S.T.= $\frac{ydx}{dy} = \frac{x(2a-x)}{3a-x}$ *(47)*

For maxima/minima of S.T.(47)

$\frac{d\left(\frac{ydx}{dy}\right)}{dx} = \frac{2(3a-x)(a-x)+(2ax-x^2)}{(3a-x)^2} = \frac{x^2-6ax+6a^2}{(3a-x)^2}$ (48)

$Or, x^2 - 6ax + 6a^2$=0 (49)

x=$\frac{6a-\sqrt{12a^2}}{2}$=$(3-\sqrt{3})a$ (50)

x=$\neq 3+\sqrt{3}$, because this point does not lie on the curve (46)

$\frac{d^2(\frac{ydx}{dy})}{dx^2}$=$\frac{2x-6a}{(3a-x)^2} + \frac{2(x^2-6ax+6a^2)}{(3a-x)^3}$ (51)

= $2(3-\sqrt{3})a - 6a$ <0 (52)

x= $(3-\sqrt{3})a$ (53)

when maximum ST occurs.

Using (53) in (47) is obtained the maximum length of the subtangent:

$(ST)_{max} = \frac{(3-\sqrt{3})}{\sqrt{3}}(\sqrt{3}-1)a$ =4-2$\sqrt{3}$a (54)

Subnormal at (x,y) to the cissoid (46) is given by

S.N.=$\frac{1}{2}\frac{dy^2}{dx} = \frac{3x^2}{2(2a-x)} + \frac{x^3}{2(2a-x)^2}$ (55)

Otherwise, (46) gives on algebraic division,

$y^2 = -x^2 - 2ax - 4a^2 + \frac{8a^3}{2a-x}$ (56),

Hence, S.N.=$\frac{1}{2}\frac{dy^2}{dx}$=-x-a+$\frac{4a^3}{(2a-x)^2}$ (57)

At x=a, S.N.=2a. (58)

For maxima/ minima of S.N. represented by (58)

$\frac{d(S.N)}{dx}$ = -1+$\frac{4a^3}{(2a-x)^3}$ =0 (59)

Or,x=(2-$\sqrt[3]{4}$)a=$\frac{21}{50}a$ (approx.) (60)

at which $\frac{d^2(S.N)}{dx^2}$>0, implying because of (58)

$(S.N.)_{min}$=$\left(\frac{-71}{50} + 4\frac{50^3}{79^3}\right)a$ (61)

NATURE OF LENGTHS OF SUBTANGENT AND SUBNORNAL TO AN ELLIPSE

Lengths of subtangent and subnormal to ellipse

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (62)

are obtained as:

2logy-2logb=log(1-$\frac{x^2}{a^2}$) (63)

Differenting w.r.t. x,

$\frac{2dy}{ydx}$=$\frac{-2x}{a^2-x^2}$

Length of the subtangent at (x,y):

S.T.= $\frac{dx}{ydy}$=$-\frac{a^2}{x}$+x (64)

S.T.= $\frac{3a}{2}$ at x=$\pm\frac{a}{2}$ (65)

Length of the subnormal at (x,y):

S.N.=$\frac{dy^2}{2dx}$ =$-\frac{b^2}{a^2}$x

S.N.= $\frac{b^2}{2a}$ at x=$-\frac{a}{2}$ (66)

The foregoing equations ratify that as x increases through positive and negative values, the lengths of both subtangent and subnormal increase. This can be at ease observed by drawing relevant figures.

MAXIMUM/MINIMUM DISTANCE BETWEEN A POINT AND ANOTHER CURVE

The distance between a point(7,0) and a curve

$27y^2=4(x-2)^3$ (67)

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is obtained as

$S^2=(x-7)^2+\frac{4(x-2)^3}{27}$ (68)

For maxima / minima of S,

$\frac{dS^2}{dx}=2(x-7)+\frac{4(x-2)^2}{9}=0$ (69)

Or, $2x^2+x-55=0$

Or,(x-5)(2x+11)=0

Or, $x=5,-\frac{11}{2}$ (70)

In view of the above equations(67) to (70)

$\frac{d^2S^2}{dx^2}=21>0\ at\ x=5$ implies

$S^2_{min}=8$ (71)

$\frac{d^2S^2}{dx^2}=<0\ at\ x=-\frac{11}{2}$ implies

$S^2_{max}=\frac{875}{4}$

0r, $S_{max}=\frac{5\sqrt{35}}{2}$ (72)

SUBTANGENT(ST) AND SUBNORMAL(SN) TO THE CURVE

$2\log y=3\log(x-2)+\log\frac{4}{27}$

Differentiating w.r.t. x

$\frac{2dy}{ydx}=\frac{3}{x-2}$

ST= $\frac{ydx}{dy}=\frac{2(x-2)}{3}$ (73)

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SN= $\frac{ydy}{dx}=\frac{dy^2}{2dx}=\frac{2(x-2)^2}{9}$ (74)

Both ST and SN increase as x increases

Difference between SN and ST is given by

D= $\frac{2(x-2)^2}{9}-\frac{2(x-2)}{3}$ $x-2\neq 0$ (75)

For maxima/ minima of D

$\frac{dD}{dx}=\frac{4(x-2)}{9}-\frac{2}{3}=0, \frac{d^2D}{dx^2}=\frac{4}{9}>0$

Or, $x=\frac{7}{2}$ gives minimum value of D

$D_{min}=\frac{-1}{2}$

ST=SN when $x=\frac{7}{3}$ (76)

MAXIMUM/ MINIMUM DISTANCE OF AN UNCONVENTIONAL CURVE FROM APOINT LYING ON THE X-AXIS

The distance S between a point(a,0) and a curve

$xy^2=4a^2(2a-x)$

Or, $y^2=4a^2(\frac{2a}{x}-1)$ (77)

is given by

$S^2 = (x-a)^2 + 4a^2(\frac{2a}{x} - 1)$ (78)

For maxima/minima of S,

$\frac{dS^2}{dx}$=2(x-a)-$\frac{8a^3}{x^2}$=0

Or,$(\frac{x}{a})^3 - (\frac{x}{a})^2 - 4 = 0$

Denoting

z=$\frac{x}{a}$ (79)

the above equation becomes

$z^3 - z^2 - 4 = 0$ *(80)*

Or, $z^2(z-2) + z(z-2) + 2(z-2) = 0$

Or, z=2 ie x=2

$\frac{d^2(S^2)}{dx^2}$>0 at x=2a (81)

Using (81) in (78) is obtained

$S_{min} = a$ (82)

SUBTANGENT AND SUBNORMAL TO THE CURVE AS ABOVE

xy^2=4$a^2(2a-x)$

Taking log of both sides and then differentiating w.r.t. x,

2$\frac{dy}{ydx}$=$\frac{-1}{2a-x} - \frac{1}{x}$=$\frac{-2a}{x(2a-x)}$

Or, S.T.= $\frac{ydx}{dy}$=$-x(2a-x)/a$ (83)

For maximum/minimum S.T.

$\frac{d(ST)}{dx}$ =$-2a + 2x = 0$

$\frac{d^2(ST)}{dx^2} = 2 > 0$ at x=a (84)

Hence minimum ST occurs due to (84)and(83):

$At\ x = a,$ $(ST)_{min}$=a

SN=y$\frac{dy}{dx}$=$-\frac{4a^3}{x^2}$ (85)

SN=$-\frac{4a^3}{x^2}$ =-4a , at x=a (86)

We find out the value of x at which (ST)=(SN)

$-x(2a-x)/a = -\frac{4a^3}{x^2}$

Or, $\frac{x^4}{a^4}$-$\frac{2x^3}{a^3}$+4=0 (86.1)

Whose approximate solution by Newton-Raphson Method can be obtained as

x=1.8 (86.2)

Putting in the above equation

z=$\frac{x}{a}$ (87)

we have

z^4-2$z^3 + 4 = 0$ (88)

Or,$(z^2 - z + c)^2 - 2cz^2 - z^2 + 2cz - c^2$+4=0

Or, $(z^2 - z - c)^2$=$(2c+1)z^2$-2cz+c^2-4

Or,$(z^2 - z - c)^2$ = (2c+1){ z^2-$\frac{2cz}{2c+1} + \left(\frac{c}{2c+1}\right)^2\Big\} - \frac{c^2}{2c+1} + c^2 - 4$

To find the value of c

$(z^2 - z - c)^2 = (2c+1)(z - \frac{c}{2c+1})^2$

Or, $z^2 - z - c = \pm\sqrt{2c+1}(z - \frac{c}{2c+1})$ (89)

Solving the quadratic equations we get four values of z of which at least one is real whereas the value of c is determined by solving the cubic equation

$c^2 - \frac{c^2}{2c+1} - 4 = 0$ (90)

Now we can find the minimum /maximum value of

ST-SN=f(z)= z^4-2$z^3 + 4$ (91)

For maxima/ minima of f(z)

$\frac{df(z)}{dz} = 4z^3 - 3z^2 = 0$

z=$\frac{3}{4}$ ie x=$\frac{3}{4}a$ at which

$\frac{d^2f(z)}{dz^2} = 12z^2 - 4z$ >0

which suggests minimum value of (f(z):

Minimum f(z)=$(\frac{3}{4})^4$-2$(\frac{3}{4})^3 + 4$ =2.33 (92)

MAXIMUM/MINIMUM SUBTANGENT AND SUBNORMAL TO TWO UNCONVENTIONAL CURVES

Equation of one curve is

y^2=$\frac{x}{3-x}$ (93)

Or,2logy=logx-log(3-x)

Or,$\frac{2dy}{ydx} = \frac{1}{x} + \frac{1}{3-x} = \frac{3}{x(3-x)}$

Or,ST= $\frac{ydx}{dy} = \frac{2x(3-x)}{3}$ *(94)*

For maximum/minimum ST,

$\frac{d(ST)}{dx} = \frac{2(3-2x)}{3}$=0

$\frac{d^2(ST)}{dx^2} = -\frac{2}{3}$ at $x = \frac{3}{2}$ (95)

From(95) and (94),

$(ST)_{max} = \frac{3}{2}$ (96)

$y^2 = \frac{3-(3-x)}{(3-x)} = \frac{3}{3-x} - 1$

SN =$\frac{dy^2}{2dx}$=$\frac{3}{2(3-x)^2}$ = $\frac{2}{3}$ at x= $\frac{3}{2}$ (97)

Equation of the second curve is

y= $\frac{x}{3-x}$

$y = \frac{3}{3-x} - 1$ (98)

$y^2 = \frac{9}{(3-x)^2}$-$\frac{6}{3-x} + 1$ (99)

SHORTEST DISTANCE BETWEEN A POINT AND THE ABOVE CURVE

Distance S between a point(2,0) and curve(98)is given by

$S^2 = (x-2)^2 + \frac{9}{(3-x)^2}$-$\frac{6}{3-x}$ +1 (100)

For maxima/minima of S,

$\frac{dS^2}{dx}$=f(x)=2(x-2)+ $\frac{18}{(3-x)^3}$-$\frac{6}{(3-x)^2}$=0 (101)

f'(x)= 2 +$\frac{54}{(3-x)^4}$- $\frac{12}{(3-x)^3}$ *(*102)

Eq(101) can be solved approximately by Newton-Raphson Method.Putting x=x_1 =1 in (101),it gives $f(x_1)$ =2 which can be regarded as the first approximate solution. By the above method the second approximate root of (101) is obtained as x_2

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$ (102)

=2-$\frac{f(2)}{f'(2)}$=2-$\frac{12}{44}$=$\frac{19}{11}$ (103)

Substituting x= $\frac{19}{11}$ in (100) we can find $the\ shortest\ distance\ S_{min}$.

DETERMINATION OF SUBTANGENT AND SUBNORMAL TO THE CURVE

logy=logx-log(3-x)

Or, $\frac{dy}{ydx} = \frac{1}{x} + \frac{1}{3-x} = \frac{3}{x(3-x)}$

Or, ST= $\frac{x(3-x)}{3}$ (104)

For maximum/minimum ST

$\frac{d(ST)}{dx} = \frac{(3-2x)}{3}$=0

x=$\frac{3}{2}$

$(ST)_{max}$=$\frac{3}{4}$ (105)

SN=y$\frac{dy}{dx} = \frac{d\left(\frac{9}{(3-x)^2}-\frac{6}{3-x}+1\right)}{2dx}$ (by use of (98))

For maximum/ minimum of SN

SN=y$\frac{d(y)}{dx} = \frac{9}{(3-x)^3}$-$\frac{3}{(3-x)^2}$ (106)

$\frac{d(SN)}{dx}$=$\frac{27}{(3-x)^4}$-$\frac{6}{(3-x)^3} = 0$

Or, 9+6x=0

x=-$\frac{3}{2}$ (107)

$(SN)_{min}$ =$\frac{4}{81}$ (108)

MINIMUM DISTANCE BETEEN TWO RUNNING PERSONS

Let us consider two roads at right angles to each other at junction 0. One person runs along one road starting from

rest with a unform cacceleration f at a distance a from the junction towards the same and the other person begins to run along the other road with a uniform velocity u at the same instant of time away from the junction.

Distance S between the two persons at any instant of time t is given by

$S^2 = (a - \frac{1}{2}ft^2)^2 + (ut)^2$ (109)

$\frac{dS^2}{dt^2} = -\left(a - \frac{1}{2}ft^2\right)f + u^2 = 0$ (differentiating w.r.t. t^2)

$$\frac{d^2(S^2)}{d(t^2)^2} > 0$$

0r, $a - \frac{1}{2}ft^2 = \frac{u^2}{f}$

$t^2 = \frac{2}{f}(a - \frac{u^2}{f})$ (110)

Using (110) in (109) we get the relevant shortest distance

$S_{min}^2 = (\frac{u^2}{f})^2 + \frac{2u^2}{f}(a - \frac{u^2}{f})$ (111)

$=\frac{u^2}{f}(2a - \frac{u^2}{f})$

SHORTEST DISTANCE BETWEEN TWO PERSONS MOVING ALONG TWO PATHS RESPECTIVELY INCLINED AT AN ANGLE

If the angle between the two paths be α instead of a right angle,all other parameters being the same, the distance between them at any instant of time t in consequence of geometry is given by

$S^2 = (a - \frac{1}{2}ft^2)^2 + (ut)^2 - 2(a - \frac{1}{2}ft^2)(ut)cos\alpha$ (112)

$\frac{dS^2}{dt} = -\left(a - \frac{1}{2}ft^2\right)2ft + 2u^2t - 2u(a - \frac{3}{2}ft^2)cos\alpha = 0$ (113)

Solving the cubic equation (113)is determined the real value of t which is again substituted in (112) to find out the aforesaid minimum distance.

Suppose in the above problem the first person moves with a uniform velocity v instead of uniform acceleration f and the task is to find the time of acquiring the shortest distance between the two moving persons and the shortest distance.

In the light of the above, distance S between them at any instant of time t is represented by

$S^2 = (a - vt)^2 + (ut)^2 - 2(a - vt)(ut)cos\alpha$ (114)

For maxima/minima of S,

$\frac{dS^2}{dt} = -2(a - vt)v + 2u^2t - 2u(a - 2vt)cos\alpha = 0$

Or, $(u^2 + v^2 + 2uvcos\alpha)t = a(u + v)$

Or, $t = \frac{a(u+v)}{(u^2+v^2+2uvcos\alpha)}$ (115)

$\frac{d^2(S^2)}{dt^2} > 0$

Substituting (115) in (114) is obtained the minimum distance between the two moving persons.

SOME UNCONVENTIONAL OPTIMISATION PROBLEMS RELATED TO PROJECTILE MOTION AND THEIR SOLUTIONS

Suppose a target enemy vehicle is travelling with a uniform velocity v along a straight line path inclined at an angle θ with another straight line path. An attacker gun position is stationed at a distance a on the latter path from the junction O.Find the shortest distance between the target and the gun position.A perpendicular drawn from the gun position to the above path becomes the shortest distance which is by geometry $atan\theta$. As soon as the enemy target is sighted at the end of the shortest path, a shell is fired with velocity u at an angle α to the horizontal and hits the moving target.The time of flight, viz, the time to hit the target is given by

$T = \frac{2usin\alpha}{g}$ (116)

Considering the projectile motion,

$a^2\tan^2\theta + v^2T^2 = u^2(cos^2\alpha)T^2$

Using (116) in this equation,

$a^2\tan^2\theta = \left(\frac{2usin\alpha}{g}\right)^2 \{u^2(cos^2\alpha) - v^2\}$

Or, $u^4(cos^2\alpha sin^2\alpha) - u^2v^2sin^2\alpha - \frac{a^2g^2\tan^2\theta}{4} = 0$

Or, $u^2 = \frac{v^2sin^2\alpha + \sqrt{v^4sin^4\alpha + a^2g^2\tan^2\theta cos^2\alpha sin^2\alpha}}{2cos^2\alpha sin^2\alpha}$ *(117)*

which gives the projection velocity at an angle α to hit the target.From(117), for minimum value of u

$\frac{2d(u^2cos^2\alpha sin^2\alpha)}{d(sin^2\alpha)} = v^2 + \frac{2v^4sin^2\alpha + a^2g^2\tan^2\theta(1-2sin^2\alpha)}{2\sqrt{v^4sin^4\alpha + a^2g^2\tan^2\theta cos^2\alpha sin^2\alpha}}$

Or, $\frac{cos^2\alpha sin^2\alpha du^2}{d(sin^2\alpha)} + 2(1-2sin^2\alpha)u^2 = v^2 + \frac{2v^4sin^2\alpha + a^2g^2\tan^2\theta(1-2sin^2\alpha)}{2\sqrt{v^4sin^4\alpha + a^2g^2\tan^2\theta cos^2\alpha sin^2\alpha}}$ while $\frac{2du^2}{d(sin^2\alpha)} = 0$, so that

$2(1-2sin^2\alpha)u^2 = v^2 + \frac{2v^4sin^2\alpha + a^2g^2\tan^2\theta(1-2sin^2\alpha)}{2\sqrt{v^4sin^4\alpha + a^2g^2\tan^2\theta cos^2\alpha sin^2\alpha}}$ (118)

Due to (117) and (118)we get an equation containing α:

$2(1-2sin^2\alpha)\frac{v^2sin^2\alpha + \sqrt{v^4sin^4\alpha + a^2g^2\tan^2\theta cos^2\alpha sin^2\alpha}}{2cos^2\alpha sin^2\alpha}$

$= v^2 + \frac{2v^4sin^2\alpha + a^2g^2\tan^2\theta(1-2sin^2\alpha)}{2\sqrt{v^4sin^4\alpha + a^2g^2tau^2n^2\theta cos^2\alpha sin^2\alpha}}$ (119)

Substituting the value of $\alpha\ from$ (119) in (117) is obtained the minimum value of projection velocity u_{min}.

A target is moving vertically upwards with a velocity v.After a timet_1,a shell is fired with a velocity u at angle α to the

horizontal from the gun position distant x from the point of projection of the former and strikes the target at height h after time t of projection of the latter. Then the equations of motion are given by

$h = usin\alpha .t - \frac{1}{2}gt^2 = v(t + t_1) - \frac{1}{2}g(t + t_1)^2$ (120)

$x = utcos\alpha$ (121)

Eliminating t between (120) and (121) and simplifying ,

$xtan\alpha = v(\frac{x}{ucos\alpha} + t_1) - \frac{1}{2}gt_1^2 - \frac{gxt_1}{ucos\alpha}$

Or, $\frac{x}{ucos\alpha}(v - gt_1) = (xtan\alpha - vt_1) + \frac{1}{2}gt_1^2$

$u = \frac{x(v-gt_1)sec\alpha}{xtan\alpha - vt_1 + \frac{1}{2}gt_1^2}$ (122)

For minimum of u,

$\frac{du}{d\alpha}=\frac{x\{(v-gt_1)\{\left(x\tan\alpha-vt_1+\frac{1}{2}gt_1^2\right)\sec\alpha\tan\alpha-x\sec^3\alpha\}}{(x\tan\alpha-vt_1+\frac{1}{2}gt_1^2)^2}=0$

Or,$(x\tan\alpha-vt_1+\frac{1}{2}gt_1^2)\tan\alpha-x\sec^2\alpha=0$ (122.1)

$\tan\alpha=\frac{x}{\frac{1}{2}gt_1^2-vt_1}$ (123)

which gives the optimum angle of projection with minimum velocity of projection as:

$u=\frac{x(v-gt_1)\sec\alpha}{x\tan\alpha-vt_1+\frac{1}{2}gt_1^2}$ (By use of (122.1 in (122))

$=(v-gt_1)\sin\alpha=\frac{(v-gt_1)\tan\alpha}{\sqrt{1+\tan^2\alpha}}$ (By use of (123)

$=\frac{(v-gt_1)\frac{x}{\frac{1}{2}gt_1^2-vt_1}}{\sqrt{1+(\frac{x}{\frac{1}{2}gt_1^2-vt_1})^2}}$

Hence, $u_{min}=\frac{x(v-gt_1)}{\sqrt{x^2+(\frac{1}{2}gt_1^2-vt_1)^2}}$ (124)

TWO PROJECTILES FIRED SIMULTANEOUSELY: A C0LLISION

As soon as an enemy aircraft flying horizontally at height h with a uniform velocity v is sighted by a gun position situated at a horizontal distance d from the former,a shell is fired by the latter with velocity u at angle α to the horizontal and hits the former in time t. The equations of motion are given by

$(vt)^2+d^2=(u\cos\alpha.t)^2$ (125)

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h=$u\sin\alpha.t-\frac{1}{2}gt^2$ (126)

h+$\frac{1}{2}gt^2=u\sin\alpha t$

Eliminating α between (125) and (126),we get

$(ut)^2=\left(h+\frac{1}{2}gt^2\right)^2+(vt)^2+d^2$

Or, $(ut)^2=\left(\frac{1}{2}gt^2\right)^2+(v^2+gh)t^2+(h^2+d^2)$ (127)

Or, $t^2=\frac{(u^2-v^2-gh)\pm\sqrt{(u^2-v^2-gh)^2-(h^2+d^2)g^2}}{\frac{1}{2}g^2}$

$=\frac{(u^2-v^2-gh)\pm\sqrt{(u^2-v^2)^2-2gh(u^2-v^2)-d^2g^2}}{\frac{1}{2}g^2}$ (128)

which gives two times of flight to hit the target,one while the cannon is moving upwards and the other time while it is moving downwards. From (127) u^2 can be expressed as

$(ut)^2=\left(\frac{1}{2}gt^2\right)^2+(v^2+gh)t^2+(h^2+d^2)$ (129)

$u^2=\left(\frac{1}{2}g\right)^2t^2+v^2+gh+\frac{(h^2+d^2)}{t^2}$

For minimum of projection velocity u,

$\frac{du^2}{dt}=-\frac{2(h^2+d^2)}{t^3}+\frac{1}{2}g^2t=0$

$t^4=\frac{4}{g^2}(h^2+d^2)$

Or, $t^2=\frac{2}{g}\sqrt{(h^2+d^2)}$ (130)

Using (130) in (129) is obtained the minimum velocity of projection

$u_{min}^2=\frac{g}{2}\sqrt{(h^2+d^2)}+v^2+gh+\frac{g\sqrt{(h^2+d^2)}}{2}$

Or, $u_{min}^2=g\sqrt{(h^2+d^2)}+v^2+gh$ (131)

The optimum angle α_{opt} of projection is obtained by combining(130),(131) and (126):

h+$\frac{1}{2}gt^2=u\sin\alpha t$

Or, $\sin\alpha_{opt}=\frac{1}{u_{min}}(\frac{h}{t}+\frac{1}{2}gt)$

$$=\frac{1}{\sqrt{g\sqrt{(h^2+d^2)}+v^2+\text{gh}}}\left(\frac{h}{\sqrt{\frac{2}{g}\sqrt{(h^2+d^2)}}}+\frac{1}{2}g\sqrt{\frac{2}{g}\sqrt{(h^2+d^2)}}\right. \quad (132)$$

DISTANCE BETWEEN THE CENTRE OF A SPHERE AND VERTEX OF A CONE

Let H and R be the height of the cone in which is inscribed a sphere of radius r. If h is the distance between the centre of the sphere and the vertex of the cone,we can observe drawing a relevant figure, sides of the two similar triangles in front of the right angles and radii of the sphere and of the cone. Hence by virtue of the two similar triangles, we get

$$\frac{\sqrt{H^2+R^2}}{h}=\frac{R}{r}$$

$$h=\frac{r\sqrt{H^2+R^2}}{R}=r\sqrt{1+\frac{H^2}{R^2}} \qquad (133)$$

which suggests that the greater is the value of of h,the greater is the value of r , and also when greater the value of the ratio $\frac{H}{R}$.Further h increases with the decrease in R and increases with increase in H.

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