Fixed Point Results Using Implicit Mid Point Rule Iterative Process

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ABSTRACT:
The present work is related to the study of fixed point results via Picard-Mann iteration process of implicit midpoint rule for one nonlinear mapping. Moreover we shall show that Picard-Mann iteration process of implicit midpoint rule for one nonlinear mapping has same rate of convergence Picard-Kranoselskii iteration process of implicit midpoint rule for one nonlinear mapping. Our analytical results are supported by suitable numerical examples.

Keywords: Picard-Mann iteration process of implicit midpoint rule, Picard-Kranoselskii iteration process of implicit midpoint rule, contractive condition of rational expression, stability.

1. Introduction and Preliminaries

The aim of present work is to present some convergence and stability results for Picard-Mann iteration process of implicit midpoint rule for one nonlinear mapping. The study of fixed point results for iterative processes is one of the most fascinating areas in analysis. Iterative processes are developed to solve the equations arising in the physical formulation of the mathematical problems. In recent time some notable work is done by [6, 8-10].

We shall now discuss some the iterative schemes that are relevant to our work. Let $H$ be a real normed linear space and $T : H \to H$ be a mapping. A point $\theta$ is called the fixed point if $T(\theta) = \theta$. Throughout the paper $F(T)$ will represent the set of fixed points of mapping $T$. For $x_0 \in H$, the sequence \( \{x_n\}_{n=0}^{\infty} \) generated by

\[
x_{n+1} = Tx_n, x_0 \in H.
\]

(1.1)

is called the Picard iteration process.

Khan in 2013[3], introduced the Picard-Mann hybrid iterative process by the method

\[
x_{n+1} = Ty_n, y_n = (1 - \rho_n)x_n + \rho_nTx_n, \quad (1.2)
\]

where $x_0 \in H$ and $\{\rho_n\}_{n=0}^{\infty} \in [0,1]$.

Khan [3] with suitable examples proved that iteration (1.2) has better rate of convergence than the Picard, Mann and Ishikawa iterative process for contractive type mappings in the sense of Berinde [2].

In this direction Okeke and Abbas [4] introduced the concept of Picard-Kranoselskii hybrid iterative process by

\[
x_{n+1} = T_{\gamma_n}y_n = (1 - \delta)x_n + \delta Tx_n, \quad (1.3)
\]

where $x_0 \in H$ and $\delta \in [0,1]$.

Further Okeke and Abbas [4] claimed the higher rate of convergence of their iterative process than Picard, Mann, Ishikawa and Krasnoselskii iterative process for contractive type mappings in the sense of Berinde [2].

Motivated by the above work Li and Lan [1] in 2019, introduced the notion of Picard-Mann iteration process of implicit midpoint rule for one nonlinear mapping as
\[ x_{n+1} = T\left(\frac{x_n + y_n}{2}\right), \quad y_n = (1 - \alpha_n)x_n + \alpha_n T\left(\frac{x_n + y_n}{2}\right), \quad (1.4) \]

\( x_0 \) is the initial approximation such that \( x_0 \in H \) and \( \{\alpha_n\}_{n=0}^\infty \in [0,1] \).

In the similar way we shall introduce the concept of Picard-Krasnoselskii iteration process of implicit midpoint rule for one nonlinear mapping as

\[ x_{n+1} = T\left(\frac{x_n + y_n}{2}\right), \quad y_n = (1 - \lambda) x_n + AT\left(\frac{x_n + y_n}{2}\right), \quad (1.5) \]

\( x_0 \) is the initial approximation such that \( x_0 \in H \) and \( \lambda \in [0,1] \).

We shall prove the convergence and stability results for the iterative process (1.4) and compare its rate of convergence with (1.5).

**Definition 1.1 [2]:** Let \( H \) be a real normed linear space and \( \{x_n\}_{n=0}^\infty \) and \( \{u_n\}_{n=0}^\infty \) be the sequence converging to \( l_1 \) and \( l_2 \) respectively. Assume that \( \lim_{n \to \infty} \frac{|x_{n+1} - l_1|}{|u_{n+1} - l_2|} = l \). Then

1. If \( l = 0 \) then the sequence \( \{x_n\}_{n=0}^\infty \) converges faster to \( l_1 \) than \( \{u_n\}_{n=0}^\infty \) to \( l_2 \).
2. If \( 0 < l < \infty \), then both the iterative process have same rate of convergence.

**Definition 1.2 [7]:** Let \( \{x_n\}_{n=0}^\infty \) be the sequence in \( X \). Then the iterative process \( x_{n+1} = f(Tx_n) \) which converges to a fixed point \( q \) of \( T \) is said to be stable with respect to \( T \) if for \( \inf_{\|s\| \leq \delta} \|T^s - T\| < \infty \), \( \delta \in (0,1) \) and for all \( x, y \in H \).

**Lemma 1.4 [11]:** If \( l \) be a real number satisfying \( 0 \leq l < \infty \) and \( \{\theta_n\}_{n=0}^\infty \) be the sequence of positive numbers such that \( \lim_{n \to \infty} \theta_n = 0 \) and for \( u_{n+1} \leq l u_n + \theta_n, n = 1, 2, ... \) then \( \lim_{n \to \infty} u_n = 0 \).

### 2. Main Results

**Theorem 2.1:** Let \( H \) be a closed convex subset of a real normed linear space \( X \) and \( T : H \to H \) be a contraction mapping. Let \( \{x_n\}_{n=0}^\infty \) and \( \{u_n\}_{n=0}^\infty \) be the sequence generated by the iterative process (1.4) and (1.5) respectively with sequence \( \{\alpha_n\}_{n=0}^\infty \in [0,1] \) and \( 0 < \alpha_n \leq \alpha_n \). Then the Picard–Mann hybrid iteration process of implicit mid point rule as same rate of convergence as the Picard Krasnoselskii hybrid iteration process of implicit mid point rule.

Proof: Let \( p \) be the fixed point of the mapping \( T \). Then for iteration process (1.4) we have,

\[ \| x_{n+1} - p \| = \| T\left(\frac{x_n + y_n}{2}\right) - p \| \leq \delta \| \frac{x_n + y_n}{2} - p \| + \frac{\delta}{2}(\| x_n - p \| + \| y_n - p \|). \quad (2.1) \]

Now

\[
\| y_n - p \| = \| (1 - \alpha_n)x_n + \alpha_n T\left(\frac{x_n + y_n}{2}\right) - p \|
\leq (1 - \alpha_n) \| x_n - p \| + \alpha_n \| T\left(\frac{x_n + y_n}{2}\right) - p \|
\leq (1 - \alpha_n) \| x_n - p \| + \alpha_n \| \frac{x_n + y_n}{2} - p \|
\leq (1 - \alpha_n) \| x_n - p \| + \alpha_n \| \frac{x_n + y_n}{2} - p \|
\leq (1 - \alpha_n) \| x_n - p \| + \alpha_n \| \frac{x_n - p}{2} + \alpha_n \| y_n - p \|
\]

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\[
\left(1 - \frac{\alpha_n \delta}{2}\right) \| y_n - p \| \leq \left(1 - \alpha_n\right) \| x_n - p \| + \frac{\alpha_n \delta}{2} \| x_n - p \|
\]

\[
\leq \left(1 - \alpha_n + \frac{\alpha_n \delta}{2}\right) \| x_n - p \|
\]

\[
\Rightarrow \| y_n - p \| \leq \frac{\left(1 - \alpha_n + \frac{\alpha_n \delta}{2}\right) \| x_n - p \|}{\left(1 - \frac{\alpha_n \delta}{2}\right)}
\]  (2.2)

Using (2.2) in (2.1) we have,

\[
\| x_{n+1} - p \| \leq \frac{\delta}{2} \left(1 + \frac{\left(1 - \alpha_n + \frac{\alpha_n \delta}{2}\right)}{\left(1 - \frac{\alpha_n \delta}{2}\right)} \right) \| x_n - p \|
\]

\[
\| x_n - p \| \leq \frac{\delta}{2} \left(1 + \frac{\left(1 - \alpha_{n-1} + \frac{\alpha_{n-1} \delta}{2}\right)}{\left(1 - \frac{\alpha_{n-1} \delta}{2}\right)} \right) \| x_{n-1} - p \|
\]

\[
\vdots
\]

\[
\| x_1 - p \| \leq \frac{\delta}{2} \left(1 + \frac{\left(1 - \alpha_0 + \frac{\alpha_0 \delta}{2}\right)}{\left(1 - \frac{\alpha_0 \delta}{2}\right)} \right) \| x_0 - p \|
\]

Combining all the above inequalities we have

\[
\| x_{n+1} - p \| \leq \left(\frac{\delta}{2}\right)^{n+1} \| x_0 - p \| \prod_{k=0}^{n} \left(1 + \frac{\left(1 - \alpha_k + \frac{\alpha_k \delta}{2}\right)}{\left(1 - \frac{\alpha_k \delta}{2}\right)}\right)
\]  (2.3)

Using the fact that \(0 < \lambda < \alpha_n\) in (2.3) we have

\[
\| x_{n+1} - p \| \leq \left(\frac{\delta}{2}\right)^{n+1} \| x_0 - p \| \prod_{k=0}^{n} \left(1 + \frac{\left(1 - \lambda + \frac{\lambda \delta}{2}\right)}{\left(1 - \frac{\lambda \delta}{2}\right)}\right)^{n+1}
\]

\[
\Rightarrow \| x_{n+1} - p \| \leq \left(\frac{\delta}{2}\right)^{n+1} \| x_0 - p \| \left(1 + \frac{\left(1 - \lambda + \frac{\lambda \delta}{2}\right)}{\left(1 - \frac{\lambda \delta}{2}\right)}\right)^{n+1}
\]  (2.4)

Let \(a_n = \left(\frac{\delta}{2}\right)^{n+1} \| x_0 - p \| \left(1 + \frac{\left(1 - \lambda + \frac{\lambda \delta}{2}\right)}{\left(1 - \frac{\lambda \delta}{2}\right)}\right)^{n+1}\)

By similar arguments we have for Picard- Krausoselkii hybrid iteration process of implicit mid point rule

\[
\| u_{n+1} - p \| \leq \left(\frac{\delta}{2}\right)^{n+1} \| u_0 - p \| \left(1 + \frac{\left(1 - \lambda + \frac{\lambda \delta}{2}\right)}{\left(1 - \frac{\lambda \delta}{2}\right)}\right)^{n+1}
\]  (2.5)

Let \(\varphi_n = \left(\frac{\delta}{2}\right)^{n+1} \| u_0 - p \| \left(1 + \frac{\left(1 - \lambda + \frac{\lambda \delta}{2}\right)}{\left(1 - \frac{\lambda \delta}{2}\right)}\right)^{n+1}\)

\[
\text{Now } \varphi_n = \frac{\left(\frac{\delta}{2}\right)^{n+1} \| x_0 - p \| \left(1 + \frac{\left(1 - \lambda + \frac{\lambda \delta}{2}\right)}{\left(1 - \frac{\lambda \delta}{2}\right)}\right)^{n+1}}{\left(\frac{\delta}{2}\right)^{n+1} \| x_0 - p \| \left(1 + \frac{\left(1 - \lambda + \frac{\lambda \delta}{2}\right)}{\left(1 - \frac{\lambda \delta}{2}\right)}\right)^{n+1}}
\]

Since \(u_0 \neq p\) and \(0 < \| x_0 - p \| < \infty\) and \(0 < \| u_0 - p \| < \infty\) so by (2.7) we have \(\lim_{n \to \infty} \frac{\delta_n}{\| x_0 - p \|} = l\), with \(0 < l < \infty\). Hence by definition (1.1) both iterative process have same rate of convergence.
Example 2.2: Let $H$ and $T: H \to H$ be a contraction mapping defined by $\|Tx - Ty\| < \frac{r}{2}$. Then 0 is the only fixed point of the mapping $T$.

Consider the initial approximation $x_0 = 0.1$. Let $\alpha_n = \frac{1}{2}$ and $\delta = \frac{1}{3}$.

Convergence pattern of iterative process (1.4) and (1.5) is shown in the Table 1.

<table>
<thead>
<tr>
<th>$x_n$</th>
<th>Iteration (1.4)</th>
<th>Iteration (1.5)</th>
</tr>
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<tr>
<td>$x_0$</td>
<td>0.1</td>
<td>0.1</td>
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<td>$x_1$</td>
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<td>0.00000000005</td>
</tr>
<tr>
<td>$x_{27}$</td>
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<td>0.00000000002</td>
</tr>
<tr>
<td>$x_{28}$</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 1: Comparison of rate of convergence of among iterative processes

Table 1 shows that both the iterative process has same rate of convergence.

Remark 2.3: In above example the value of parameters $\alpha_n$ and $\delta$ is taken different because if we take $\alpha_n = \delta$ then both the iterative process (1.4) and (1.5) become identical.
**Theorem 2.4:** Let $H$ be a closed convex subset of a real normed linear space $X$ and $T : H \to H$ be a contraction mapping. Let $\{x_n\}_{n=0}^\infty$ be the sequence generated by the iterative process (1.4) with sequence $\{\alpha_n\}_{n=0}^\infty \in [0,1]$ and $\sum \alpha_n = \infty$. Then the sequence $\{x_n\}_{n=0}^\infty$ converges to the fixed point of mapping $T$.

**Proof:** Banach contraction principle guarantees the existence of unique fixed point $p$ of mapping $T$. Now we prove the convergence of iteration (1.4) to $p$. Using (2.2) in (2.1) we have,

$$
\| x_{n+1} - p \| \leq \frac{\delta}{2} \left( 1 + \frac{1 - \alpha_n + \frac{\alpha_n}{\| x_n - p \|}}{1 - \frac{\alpha_n}{\| x_n - p \|}} \right) \| x_n - p \|
$$

$$
\leq \frac{\delta}{2} \left( 1 + \frac{1 - \alpha_n + \frac{\alpha_n}{\| x_n - p \|}}{1 - \frac{\alpha_n}{\| x_n - p \|}} \right) \| x_n - p \|
$$

$$
\leq \frac{2\delta}{2 - \alpha_n \delta} \left( 1 - \frac{\alpha_n}{\| x_n - p \|} \right) \| x_n - p \|
$$

From the above inequality we have the following estimate

$$
\| x_{n+1} - p \| \leq \frac{2\delta}{2 - \alpha_n \delta} \left( 1 - \frac{\alpha_n}{\| x_n - p \|} \right) \| x_n - p \|
$$

$$
\| x_n - p \| \leq \frac{2\delta}{2 - \alpha_n \delta} \left( 1 - \frac{\alpha_{n-1}}{\| x_{n-1} - p \|} \right) \| x_{n-1} - p \|
$$

$$
\| x_1 - p \| \leq \frac{2\delta}{2 - \alpha_1 \delta} \left( 1 - \frac{\alpha_0}{\| x_0 - p \|} \right) \| x_0 - p \|
$$

Combining the above inequalities we have

$$
\| x_{n+1} - p \| \leq (2\delta)^{n+1} \| x_0 - p \| \prod_{i=0}^{n+1} \frac{1}{2 - \alpha_i \delta} \prod_{i=0}^{n+1} \left( 1 - \frac{\alpha_i}{\| x_i - p \|} \right) \quad (2.8)
$$

Since $1 - b \leq e^{-b}$ for all $b \in [0,1]$. Now $\{\alpha_n\}_{n=0}^\infty \in [0,1]$ so we have $0 < 1 - \frac{\alpha_i}{\| x_i - p \|} < 1$, hence (2.8) becomes

$$
\| x_{n+1} - p \| \leq (2\delta)^{n+1} \| x_0 - p \| \prod_{i=0}^{n+1} \frac{1}{2 - \alpha_i \delta} e^{-\sum_{i=0}^{n+1} \frac{\alpha_i}{\| x_i - p \|}} \quad (2.9)
$$

Taking limit $n \to \infty$ in (2.9) we obtain $\lim_{n \to \infty} \| x_n - p \| = 0$. Which is the desired result.

Okabe [5] introduced the following contractive condition of rational expression

$$
\| Tx - Ty \| \leq \frac{\psi(x,y) - \psi(x,y) - \psi(x,y)}{1 + \psi(x,y)} \quad (2.10)
$$

Where $x, y \in H, b \in [0,1], \alpha \geq 0$ and $\psi: H \to H$ be a monotone increasing function such that $\psi(0) = 0$.

Now we prove some fixed point results related to contractive condition (2.10).

**Theorem 2.5:** Let $H$ be a closed convex subset of a real normed linear space $X$ and $T : H \to H$ be a contraction mapping satisfying condition (2.10). Let $\{x_n\}_{n=0}^\infty$ and $\{u_n\}_{n=0}^\infty$ be the sequence generated by the iterative process (1.4) and (1.5) respectively with sequence $\{\alpha_n\}_{n=0}^\infty \in [0,1]$ and $0 < \lambda < \alpha_n$. Then both the iterative processes have same rate of convergence.

**Proof:** Let $q$ be the fixed point of the mapping $T$. Now by (1.4) and (2.10) we have

$$
\| x_{n+1} - q \| = \| T \left( \frac{x_n + y_n}{2} \right) - q \| = \| Tq - T \left( \frac{x_n + y_n}{2} \right) \|
$$
\[
\frac{\phi( (b_1 - q_1) + \sum_{i=1}^{k} a_i y_i )}{1 + \phi( (b_1 - q_1) + \sum_{i=1}^{k} a_i y_i )} \leq \frac{\phi( (b_1 - q_1) + \sum_{i=1}^{k} a_i y_i )}{1 + \phi( (b_1 - q_1) + \sum_{i=1}^{k} a_i y_i )} \leq \frac{b}{2} \| x_n - q \| + \| y_n - q \| \tag{2.11}
\]

Now
\[
\| y_n - p \| = \| (1 - \alpha_n) x_n + \alpha_n \psi \| \left( \frac{x_n + y_n}{2} \right) - q \| \leq (1 - \alpha_n) \| x_n - q \| + \alpha_n \| \left( \frac{x_n + y_n}{2} \right) - q \|
\]
\[
\leq (1 - \alpha_n) \| x_n - q \| + \alpha_n \left| \frac{\phi( (b_1 - q_1) + \sum_{i=1}^{k} a_i y_i )}{1 + \phi( (b_1 - q_1) + \sum_{i=1}^{k} a_i y_i )} \right| \| y_n - q \|
\]
\[
\leq (1 - \alpha_n) \| x_n - q \| + \frac{\alpha_n}{2} \left[ \left( \frac{x_n - q}{2} \right) + \alpha_n \frac{\| y_n - q \|}{2} \right]
\]
\[
\leq (1 - \alpha_n) \| x_n - q \| + \frac{\alpha_n}{2} \| y_n - q \|
\]
\[
\Rightarrow (1 - \alpha_n) \| y_n - q \| \leq \left( 1 - \alpha_n + \frac{\alpha_n}{2} \right) \| x_n - q \|
\]
\[
\| y_n - q \| \leq \frac{(1 - \alpha_n + \frac{\alpha_n}{2})}{(1 - \alpha_n)} \| x_n - q \| \tag{2.12}
\]

Using (2.12) in (2.11) we have
\[
\| x_{n+1} - q \| \leq \frac{b}{2} \| x_n - q \| + \left( 1 - \alpha_n + \frac{\alpha_n}{2} \right) \| x_n - q \| \leq \frac{b}{2} \left[ 1 + \left( 1 - \alpha_n + \frac{\alpha_n}{2} \right) \right] \| x_n - q \| \leq \frac{b}{2} \left[ 1 + \left( 1 - \alpha_n + \frac{\alpha_n}{2} \right) \right] \| x_n - q \| \leq \frac{b}{2} \left[ 1 + \left( 1 - \alpha_n + \frac{\alpha_n}{2} \right) \right] \| x_0 - q \| \tag{2.13}
\]

Combining above all the inequalities we have
\[
\| x_{n+1} - q \| \leq \frac{b}{2} \left[ 1 + \left( 1 - \alpha_n + \frac{\alpha_n}{2} \right) \right] \| x_0 - q \| \tag{2.14}
\]

Using the fact that \( 0 < \lambda < \alpha_n \), we have from (2.13)
\[
\| x_{n+1} - q \| \leq \frac{b}{2} \| x_0 - q \| \left[ 1 + \left( 1 - \alpha_n + \frac{\alpha_n}{2} \right) \right] \| x_0 - q \| \leq \frac{b}{2} \| x_0 - q \| \left[ 1 + \left( 1 - \alpha_n + \frac{\alpha_n}{2} \right) \right] \| x_0 - q \| \tag{2.15}
\]

Similarly by using above arguments we have for (1.5)
\[
\| u_{n+1} - q \| \leq \frac{b}{2} \| u_0 - q \| \left[ 1 + \left( 1 - \alpha_n + \frac{\alpha_n}{2} \right) \right] \| u_0 - q \| \tag{2.16}
\]
Let \( \rho_n = \left( \frac{b_2}{2} \right)^n \| x_0 - q \| \left[ 1 + \frac{(1-\lambda b_2^2)^n}{(1-\frac{b_2^2}{2})} \right] \) and \( \theta_n = \left( \frac{b_2}{2} \right)^n \| u_0 - q \| \left[ 1 + \frac{(1-\lambda b_2^2)^n}{(1-\frac{b_2^2}{2})} \right] \).

Then
\[
\frac{\rho_n}{\theta_n} = \left( \frac{\left( \frac{b_2}{2} \right)^n \| x_0 - q \|}{\left( \frac{b_2}{2} \right)^n \| u_0 - q \|} \right) \left( \frac{1 + \frac{(1-\lambda b_2^2)^n}{(1-\frac{b_2^2}{2})}}{1 + \frac{(1-\lambda b_2^2)^n}{(1-\frac{b_2^2}{2})}} \right).
\]

(2.16)

Since \( u_0 \neq q \) and \( 0 < \| x_0 - p \| < \infty \) and \( 0 < \| u_0 - p \| < \infty \) so by (2.16) we have \( \lim_{n \to \infty} \frac{\rho_n}{\theta_n} = l \), with \( 0 < l < \infty \). Hence by definition (1.1) both iterative process have same rate of convergence.

References: