Delineation of Integer Solutions to Non-Homogeneous Quinary Quintic Diophantine Equation \((x^3 - y^3) - (x^2 + y^2) + (z^3 - w^3) = 2 + 87T^5\)

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ABSTRACT

This paper aims at determining varieties of non-zero distinct integer solutions to non-homogeneous quinary quintic diophantine equation

\[(x^3 - y^3) - (x^2 + y^2) + (z^3 - w^3) = 2 + 87T^5\]

Keywords: quinary quintic, non-homogeneous quintic, integer solutions

Introduction

It is well-known that the Diophantine equations, homogeneous or non-homogeneous, have aroused the interest of many mathematicians. In particular, one may refer [1-18] for quintic equations with three, four and five unknowns.

While collecting problems on fifth degree Diophantine equations, the problem of getting integer solutions to the non-homogeneous quinary quintic diophantine equation given by \((x^3 - y^3) - (x^2 + y^2) + z^3 - w^3 = 2 + 87T^5\) [19] has been noticed. The authors of [19] have presented two sets of integer solutions to the quintic equation considered in [19]. The main thrust of this paper is to exhibit other sets of integer solutions to quintic non-homogeneous quinary equation given by \(x^3 - y^3 - (x^2 + y^2) + z^3 - w^3 = 2 + 87T^5\) in [19] by using elementary algebraic methods. The outstanding results in this study of diophantine equation will be useful for all readers.

Method of analysis

The non-homogeneous quinary quintic diophantine equation to be solved is given by

\[(x^3 - y^3) - (x^2 + y^2) + (z^3 - w^3) = 2 + 87T^5\]  \(1\)

The process of obtaining different sets of integer solutions to (1) is illustrated below:

Illustration 1

Introduction of the linear transformations

\[x = kv + 1, y = kv - 1, z = v + 1, w = v - 1\]  \(2\)

in (1) leads to

\[(4k^2 + 6)v^2 = 87T^5\]  \(3\)

which is satisfied by

\[v = 87(4k^2 + 6)s^5\]  \(4\)

and

\[T = 87(4k^2 + 6)s^2\]  \(5\)

Using (4) in (2), we get
\[ x = 87^3k(4k^2 + 6)^2z^5 + 1, y = 87^3k(4k^2 + 6)^2z^5 - 1, \]
\[ z = 87^3(4k^2 + 6)^2z^5 + 1, w = 87^3(4k^2 + 6)^2z^5 - 1 \]  \hspace{1cm} (6)

Thus, (5) & (6) represent the integer solutions to (1).

**Illustration 2**

Introduction of the linear transformations

\[ x = u + 1, y = u - 1, z = ku + 1, w = ku - 1 \]  \hspace{1cm} (7)

in (1) leads to

\[ (6k^2 + 4)u^2 = 87T^5 \]

which is satisfied by

\[ u = 87^3(6k^2 + 4)^2z^5 \]  \hspace{1cm} (8)

and

\[ T = 87(6k^2 + 4)s^2 \]  \hspace{1cm} (9)

Using (8) in (7), we get

\[ x = 87^3(6k^2 + 4)^2z^5 + 1, y = 87^3(6k^2 + 4)^2z^5 - 1, z = 87^3k(6k^2 + 4)^2z^5 + 1, w = 87^3k(6k^2 + 4)^2z^5 - 1 \]  \hspace{1cm} (10)

Thus, (9) & (10) represent the integer solutions to (1).

**Illustration 3**

Taking

\[ x = u + 1, y = u - 1, z = 32ks^2 + 1, w = 32ks^2 - 1, T = 4s \]  \hspace{1cm} (11)

in (1), it is written as

\[ u^2 = 256s^4(87s - 6k^2) \]  \hspace{1cm} (12)

It is possible to choose the values of \( s \) so that the R.H.S. of (12) is a perfect square and hence the corresponding values of \( u \) are obtained.

Substituting these values of \( s, u \) in (11), the respective integer solutions to (1) are found. The above process is exhibited below:

Let

\[ a^2 = 87s - 6k^2 \]  \hspace{1cm} (13)

which is satisfied by

\[ s_0 = k^2, a_0 = 9k \]

Assume

\[ a_1 = h - a_0, s_1 = h + s_0 \]  \hspace{1cm} (14)

to be the second solution to (13). Substituting (14) in (13) and simplifying, we have

\[ h = 2a_0 + 87 \]

In view of (14), one has

\[ a_1 = a_0 + 87, s_1 = 2a_0 + 87 + s_0 \]

The repetition of the above process leads to the general solution to (13) as

\[ a_n = a_0 + 87n = 9k + 87n, \]
\[ s_n = 2na_0 + 87n^2 + s_0 = 18kn + 87n^2 + k^2 \]

From (12), it is seen that

\[ u_n = 16(87n + 9k)(87n^2 + 18kn + k^2)^2 \]

In view of (11), the integer solutions to (1) are given by
\[ x_n = 16(8n + 9k)(87n^2 + 18kn + k^2)^2 + 1, \]
\[ y_n = 16(8n + 9k)(87n^2 + 18kn + k^2)^2 - 1, \]
\[ z_n = 32k(87n^2 + 18kn + k^2)' - 1, \]
\[ w_n = 32k(87n^2 + 18kn + k^2)' - 1, \]
\[ T_n = 4(87n^2 + 18kn + k^2). \]

**Illustration 4**

Taking
\[ z_n = 36(87n^2 + 9k)(87n^2 + 18kn + k^2)^2 + 1, \]
\[ w_n = 36(87n^2 + 9k)(87n^2 + 18kn + k^2)^2 - 1, \]
in (1), it is written as
\[ x_n = 108k(87n^2 + 18kn + k^2)^2 + 1, \]
\[ y_n = 108k(87n^2 + 18kn + k^2)^2 - 1, \]

Following the analysis as in Illustration 3, the corresponding integer solutions to (1) are given by
\[ T_n = 6(87n^2 + 18kn + k^2). \]

**Conclusion:**

In this paper, we have made an attempt to find infinitely many non-zero distinct integer solutions to the non-homogeneous quintic equation with five unknowns given by \( x^3 - y^3 = (x^2 + y^2 + z^2 - w^2 = 2 + 877^5). \) To conclude, one may search for other choices of solutions to the considered quintic equation with five unknowns and higher degree diophantine equations with multiple variables.

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