On Non-homogeneous Cubic Equation with Four Unknowns $xy + 2z^2 = 10w^3$

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**ABSTRACT:**

This paper is devoted to obtain non-zero distinct integer solutions to non-homogeneous cubic equation with four unknowns given by $xy + 2z^2 = 10w^3$. Keywords: Non-homogeneous cubic, Cubic with four unknowns, Integer solutions.

**Introduction:**

Diophantine equations theory have been working by numerous mathematicians since ancient time of Diophantus. There is a big gap in the general theory of homogeneous/non homogeneous cubic equations with three or more variables. Also, the cubic diophantine equations are rich in variety and offer an unlimited field for research [1, 2]. For an extensive review of various problems, one may refer [3-30]. This paper concerns with another interesting non-homogeneous cubic diophantine equation with four unknowns $xy + 2z^2 = 10w^3$ for determining its infinitely many non-zero integral solutions.

**Method of Analysis:**

The non-homogeneous cubic equation with four unknowns to be solved is

$$xy + 2z^2 = 10w^3$$

(1)

The process of obtaining different sets of non-zero distinct integer solutions to (1) is illustrated below:

Illustration 1:

The introduction of the linear transformations

$$x = 2p, y = 2q, z = p - q, (p \neq q)$$

(2)

in (1) leads to

$$p^2 + q^2 = 5w^3$$

(3)

Assume

$$w = a^2 + b^2$$

(4)

Express the integer 5 on the R.H.S. of (3) as the product of complex conjugates as

$$5 = (2 + i)(2 - i)$$

(5)
Using (4) and (5) in (3) and employing the method of factorization, define
\[ p + iq = (2 + i) \( a + ib \)^3 \] (6)

On equating the real and imaginary parts in (6), it is seen that
\[ p = 2f(a, b) - g(a, b), \quad q = f(a, b) + 2g(a, b) \] (7)

where
\[ f(a, b) = a^3 - 3ab^2, \quad g(a, b) = 3a^2b - b^3 \] (8)

In view of (2), the integer solutions to (1) are given by
\[ x = 4f(a, b) - 2g(a, b), \quad y = 2f(a, b) + 4g(a, b), \quad z = f(a, b) - 3g(a, b) \]
along with (4).

Note 1:
Apart from (5), the integer 5 on the R.H.S. of (3) is expressed as below:
\[
5 = (1 + 2i)(1 - 2i), \\
5 = \frac{(2 + 11i)(2 - 11i)}{25}, \\
5 = \frac{(2 + 29i)(2 - 29i)}{169}
\]
The repetition of the above process leads to three more sets of integer solutions to (1).

Illustration 2:
Introducing the linear transformations
\[ x = u + z, \quad y = u - z \] (9)
in (1) gives
\[ u^2 + z^2 = 10w^3 \] (10)

which is satisfied by
\[ u = 10^2m(m^2 + n^2) \] (11)
\[ z = 10^2n(m^2 + n^2), \quad w = 10(m^2 + n^2) \] (12)

In view of (9), it is seen that
\[ x = 10^2(m + n)(m^2 + n^2), \quad y = 10^2(m - n)(m^2 + n^2) \] (13)

Thus, (12) and (13) give the integer solutions to (1).

Illustration 3:
Express the integer 10 on the R.H.S. of (10) as the product of complex conjugates as
\[ 10 = (3 + i)(3 - i) \] (14)
Using (4) & (14) in (10) and following the procedure as in Illustration 1, the integer solutions to (1) are found to be

\[ x = 4f(a, b) + 2g(a, b), \quad y = 2f(a, b) - 4g(a, b), \quad z = f(a, b) + 3g(a, b) \]  

(15)

Thus, (4) and (15) represent the integer solutions to (1).

Note 2:

It is to be noted that the integer 10 on the R.H.S. of (10) is also considered as

\[ 10 = (1 + 3i)(1 - 3i) \]

This choice leads to a different set of integer solutions to (1).

Illustration 4:

Introducing the transformations

\[ x = 2p, \quad y = 2w, \quad z = p - w, \quad p \neq w \]  

(16)

in (1), it is written as

\[ p^2 = w^2(5w - 1) \]  

(17)

There are two sets of integer solutions to (17) as shown below:

\[ w = \begin{cases} 5k^2 - 6k + 2, \\ 5k^2 - 4k + 1 \end{cases} \]

\[ p = \begin{cases} (5k^2 - 6k + 2)(5k - 3), \\ (5k^2 - 4k + 1)(5k - 2) \end{cases} \]  

(18)

In view of (16), the corresponding two sets of values of \( x, y, z \) satisfying (1) are respectively given by

\[ x = 2(5k - 3)(5k^2 - 6k + 2), \quad y = 2(5k^2 - 6k + 2), \quad z = (5k - 4)(5k^2 - 6k + 2) \]

\[ x = 2(5k - 2)(5k^2 - 4k + 1), \quad y = 2(5k^2 - 4k + 1), \quad z = (5k - 3)(5k^2 - 4k + 1) \]

It is worth to mention that, apart from the above integer solutions, there are other choices of integer solutions to (1) which are exhibited below in Table 1:

Table 1 – Integer solutions

<table>
<thead>
<tr>
<th>S. No</th>
<th>( x )</th>
<th>( y )</th>
<th>( z )</th>
<th>( w )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( 2a^{3s} )</td>
<td>( 4a^{3s} )</td>
<td>( a^{3s} )</td>
<td>( a^{2s} )</td>
</tr>
<tr>
<td>2</td>
<td>( 2a^{6s} )</td>
<td>( 4 )</td>
<td>( a^{3s} )</td>
<td>( a^{2s} )</td>
</tr>
<tr>
<td>3</td>
<td>( 4a^{6s} )</td>
<td>( 2 )</td>
<td>( a^{3s} )</td>
<td>( a^{2s} )</td>
</tr>
<tr>
<td></td>
<td>5000 $a^6$s</td>
<td>1</td>
<td>50$a^3$s</td>
<td>10$a^5$s</td>
</tr>
<tr>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>5</td>
<td>$10(5a^2+1)^2$</td>
<td>1</td>
<td>$5a(5a^2+1)$</td>
<td>$(5a^2+1)$</td>
</tr>
<tr>
<td>6</td>
<td>$5(5a^2+1)^2$</td>
<td>2</td>
<td>$5a(5a^2+1)$</td>
<td>$(5a^2+1)$</td>
</tr>
<tr>
<td>7</td>
<td>$2(5a^2+1)^2$</td>
<td>5</td>
<td>$5a(5a^2+1)$</td>
<td>$(5a^2+1)$</td>
</tr>
<tr>
<td>8</td>
<td>$5(5a^2+1)$</td>
<td>$2(5a^2+1)$</td>
<td>$5a(5a^2+1)$</td>
<td>$(5a^2+1)$</td>
</tr>
<tr>
<td>9</td>
<td>$2k^3s^2$</td>
<td>$5s^2+k^2$</td>
<td>$k^2s^2$</td>
<td>$ks$</td>
</tr>
<tr>
<td>10</td>
<td>$10k^3s^2$</td>
<td>$s^2-5ks$</td>
<td>$5k^3s^2$</td>
<td>$ks$</td>
</tr>
<tr>
<td>11</td>
<td>$10k^2s$</td>
<td>$ks^2-5s$</td>
<td>$5ks$</td>
<td>$ks$</td>
</tr>
</tbody>
</table>

**Conclusion:**

In this paper, we have made an attempt to determine different patterns of non-zero distinct integer solutions to the non-homogeneous cubic equation with four unknowns given by $xy + z^2 = 10w^3$. As the cubic equations are rich in variety, one may search for other forms of cubic equations with multi-variables to obtain their corresponding solutions.

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