International Journal of Research Publication and Reviews

Journal homepage: www.ijrpr.com ISSN 2582-7421

On the Ternary Non-Homogeneous Quintic Equation $x^2 + 5y^2 = 2z^5$

J. Shanthi[1], M. A. Gopalan[2]

[1]Assistant Professor, Department of Mathematics, Shrimati Indira Gandhi College, Affiliated to Bharathidasan University, Trichy-620 002, Tamil Nadu, India.

Email: shanthivishvaa@gmail.com

[2]Professor, Department of Mathematics, Shrimati Indira Gandhi College, Affiliated to Bharathidasan University, Trichy-620 002, Tamil Nadu, India.

Email: mayilgopalan@gmail.com

ABSTRACT:

The ternary non-homogeneous quintic equation given by $x^2 + 5y^2 = 2z^5$ is analysed for determining its distinct integer solutions.Also,a generation formula for the integer solutions to the given quintic equation ,being given its particular solution, is illustrated.

Key words: Ternary quintic ,Non-homogeneous quintic , Integer solutions

Introduction:

It is well-known that the Diophantine equations ,homogeneous or non-homogeneous ,have aroused the interest of many mathematicians. In particular,one may refer [1-12] for quintic equations with three unknowns .The above problems motivated us to search for the distinct integer solutions to ternary non-homogeneous quintic equation $x^2 + 5y^2 = 2z^5$

Also,a general formula for generating sequence of integer solutions to the considered quintic equation being given its particular solution is illustrated.

Method of analysis:

The ternary non-homogeneous quintic equation under consideration is

$$x^2 + 5y^2 = 2z^5 \quad (1)$$

The process of determining non-zero distinct integer solutions to (1) is illustrated below:

Method 1:

Introduction of the transformations

$$x = 8X, y = 8Y, z = 2w \quad (2)$$

in (1) leads to

$$X^2 + 5Y^2 = w^5 \quad (3)$$

Assume

$$w = a^2 + 5b^2 \quad (4)$$

Using (4) in (3) and employing the method of factorization ,consider

$$X + i\sqrt{5}Y = (a + i\sqrt{5}b)^5$$

Equating the rational & irrational parts in the above equation and using (2) , it is

seen that

$$x = 8f(a,b), y = 8g(a,b), z = 2(a^2 + 5b^2)$$

$$where$$

$$f(a,b) = a^5 - 50a^3b^2 + 125ab^4, g(a,b) = 5a^4b - 50a^2b^3 + 25b^5 \quad (5)$$

Thus,(5) represents the integer solutions to (1).

Method 2:

Rewrite (3) as

$$X^2+5Y^2=w^5*1 \qquad (6)$$

Consider the integer 1 on the R.H.S. of (6) as

$$1=\frac{(2+i\sqrt{5})(2-i\sqrt{5})}{9} \qquad (7)$$

Substituting (4) ,(7) in (6) and employing the method of factorization , consider

$$X+i\sqrt{5}Y=\frac{(2+i\sqrt{5})}{3}[f(a,b)+i\sqrt{5}g(a,b)]$$

Following the analysis as in Method 1,the corresponding integer solutions to (1) are given by

$$x=8*3^4[2f(A,B)-5g(A,B)], y=8*3^4[f(A,B)+2g(A,B)], z=18(A^2+3B^2)$$

Note 1 :

In addition to (7) ,the integer 1 on the R.H.S. of (6) is expressed as

$$1=\frac{(5r^2-s^2+i\sqrt{5}2rs)(5r^2-s^2-i\sqrt{5}2rs)}{(5r^2+s^2)^2}, 1=\frac{(2+i3\sqrt{5})(2-i3\sqrt{5})}{49} \qquad (8)$$

In this case ,the repetition of the above process leads to two more sets of integer solutions to (1).

Method 3:

Taking

$$x=(2k+1)y \qquad (9)$$

in (1) , it simplifies to

$$(2k^2+2k+3)y^2=z^5$$

which is satisfied by

$$y=(2k^2+2k+3)^2\alpha^{5s}, z=(2k^2+2k+3)\alpha^{2s} \qquad (10)$$

In view of (9) ,we have

$$x=(2k+1)(2k^2+2k+3)^2\alpha^{5s} \qquad (11)$$

Thus (10) and (11) represent the integer solutions to (1).

Method 4:

Taking

$$y=(2k+1)x \qquad (12)$$

in (1) , it simplifies to

$$(10k^2+10k+3)x^2=z^5$$

which is satisfied by

$$x=(10k^2+10k+3)^2\alpha^{5s}, z=(10k^2+10k+3)\alpha^{2s} \qquad (13)$$

In view of (12) ,we have

$$y=(2k+1)(10k^2+10k+3)^2\alpha^{5s} \qquad (14)$$

Thus (13) and (14) represent the integer solutions to (1).

Method 5:

The substitution

$$y = (2k+1)z^2 \quad (15)$$

in (1) leads to

$$x^2 = z^4(2z - 5(2k+1)^2) \quad (16)$$

After some algebra ,it is seen that (16) is satisfied by

$$z = (10k^2 + 10k + 3 + 2s^2 + 2s), x = (2s+1)(10k^2 + 10k + 3 + 2s^2 + 2s)^2 \quad (17)$$

In view of (15) ,one has

$$y = (2k+1)(10k^2 + 10k + 3 + 2s^2 + 2s)^2 \quad (18)$$

Thus,(17) and (18) satisfy (1).

Method 6:

The substitution

$$x = 2kz^2 \quad (19)$$

in (1) leads to

$$5y^2 = 2z^4(z - 2k^2) \quad (20)$$

After some algebra ,it is seen that (20) is satisfied by

$$z = 2k^2 + 10s^2, y = 2s(2k^2 + 10s^2)^2 \quad (21)$$

In view of (19) ,one has

$$x = 2k(2k^2 + 10s^2)^2 \quad (22)$$

Thus,(21) and (22) satisfy (1).

Observation: Generation formula

Let (x_0, y_0, z_0) be a particular solution to (1).

Then ,the formula for generating a sequence of integer solutions to (1) is presented below:

$$x_s = \frac{\alpha^s + 5\beta^s}{6} x_0 - \frac{5(\alpha^s - \beta^s)}{6} y_0,$$

$$y_s = \frac{(\alpha^s - \beta^s)}{6} x_0 + \frac{5\alpha^s + \beta^s}{6} y_0,$$

$$z_s = 6^{2s} z_0, s = 1,2,3,\ldots$$

where

$$\alpha = 6^5, \beta = -6^5$$

An example has been given in Table 1 below:

Table 1- Example

S	x_s	y_s	z_s
0	21	3	3
1	$-54 * 6^4$	$54 * 6^4$	$3*6^2$
2	$21*6^{10}$	$3*6^{10}$	$3*6^4$
3	$-54*6^{14}$	$54*6^{14}$	$3*6^6$

Conclusion:

In this paper ,an attempt has been made to obtain non-zero distinct integer solutions to the ternary non-homogeneous quintic equation $x^2 + 5y^2 = 2z^5$ As the quintic equations are rich in variety, one may search for the integer solutions to other choices of quintic equations with three or more unknowns.

References

1. M.A.Gopalan and Sangeetha.G ,Integral solutions of ternary quintic Diophantine equation $x^2 + y^2 = (k^2+1)z^5$, Bulletin of pure and applied sciences. Vol.29 No:1,23-28,(2010).

2. M.A.Gopalan ,A.VijayaSankar,An Interesting Diophantine Problem $x^3 - y^3 = 2z^5$Advances in Mathematics,Scientific Developments and Engineering Applications,Narosa Publishing House,1-6,2010

3. M.A.Gopalan, A.VijayaSankar ,Integral solutions of ternary quintic Diophantine equation$x^2 + (2k+1)y^2 = z^5$, International journal of Mathematical sciences.,Vol.19(1-2), 165-169, (2010).

4. S.Vidhyalakshmi., K.Lakshmi., M.A.Gopalan, , Integral solutions of non-homogeneous ternary quintic Diophantine equation $ax^2 + by^2 = (a+b)z^5, a > b > 0$, Archimedes journal of Mathematics ., vol.3, No:2 , 197-204 (2013).

5. M.A.Gopalan, S.Vidhyalakshmi, E. Premalatha.M. Manjula, M and N.Thiruniraiselvi, On the non-homogeneous ternary quintic equation $2(x^2 + y^2) - 3xy = 7^{2n}z^5$, Bessel J.Math.,Vol.3(3), Pp.249-254, 2013

6. 6.M.A.Gopalan,G.Sumathi and S.Vidhyalakshmi,Integral Solutions of the Non-Homogeneous Ternary Quintic Equation Interms of Pell Sequence $x^3 + y^3 + xy(x+y) = 2z^5$,JAMS , 6(1) ,56-62 ,2013

7. M.A.Gopalan,G.Sumathi and S.Vidhyalakshmi,Integral Solutions of Non-Homogeneous Quintic Equation with Three Unknowns $x^2 + y^2 - xy + x + y + 1 = (k^2 + 3)^n z^5$,IJIRSET , 2(4) ,920-925 ,2013

8. 8.S.Vidhyalakshmi,K. Lakshmi and M.A.Gopalan, Integral solutions of the non-homogeneous ternary quintic equation $ax^2 - by^2 = (a-b)z^5$, International Journal of Computational Engineering Research, 3(4), 45-50, 2013.

9. 9. M.A.Gopalan, N.Thiruniraiselvi, R.Presenna, Quintic with three unknowns $3(x^2 + y^2) - 2xy + 2(x+y) + 1 = 33z^5$, International Journal of Multidisciplinary Research and Modern Engineering, Vol.1(1),Pp.171-173, 2015

10. 10.J. Shanthi and M. Parkavi, “On finding Integer solutions to the Non-Homogeneous ternary Quintic Diophantine equation$x^2 + y^2 - xy = 28z^5$”, International Research Journal of Education and Technology, Volume: 05 Issue: 03 , 463-471,2023

11. 11.T. Mahalakshmi , P.Sowmiya, “A Search on Integer solutions to the Non- Homogeneous Ternary Quintic Equation with three unknowns $2(x^2 + y^2) - 3xy = 8z^5$”, International Research Journal of Education and Technology, Volume: 05 Issue: 03 |, pp 453-4 62 ,March-2023.

12. J.Shanthi ,M.A.Gopalan , On the Ternary Non-homogeneous Quintic Equation $x^2 + 3y^2 = 7z^5$IRJEdT ,Volume 05,Issue 08 ,19-25 ,August 2023.