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On Non-Homogeneous Sextic Equation with Three Unknowns

$y^2 + 3x^2 = 16z^6$

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ABSTRACT

This paper aims at determining non-zero distinct integer solutions to the non-homogeneous ternary sextic equation $y^2 + 3x^2 = 16z^6$. The process of obtaining different patterns of integer solutions to the above equation is illustrated.

Keywords: non-homogeneous sextic ,ternary sextic ,integer solutions

Introduction

It is well-known that a diophantine equation is an algebraic equation with integer coefficients.

Involving two or more unknowns such that the only solutions focused are integer solutions.

No doubt that diophantine equations are rich in variety [1-4] .There is no universal method available to know whether a diophantine equation has a solution or finding all solutions if it exists.For equations with more than three variables and degree at least three, very little is known.It seems that much work has not been done in solving higher degree diophantine equations.While focusing the attention on solving sextic diophantine equations with variables at least three,the problems illustrated in [5-22] are observed. This paper focuses on finding integer solutions to the sextic equation with three unknowns.

Method of analysis

The non-homogeneous ternary sextic equation to be solved is

$$y^2 + 3x^2 = 16z^6 \qquad (1)$$

Different methods of getting integer solutions to (1) are illustrated below:

Method 1

The introduction of the linear transformations

$$y = 4Y, x = 4X \qquad (2)$$

in (1) leads to

$$Y^2 + 3X^2 = z^6 \qquad (3)$$

Assume

$$z = a^2 + 3b^2 \qquad (4)$$

Substituting (4) in (3) and employing the method of factorization ,define

$$Y + i\sqrt{3}X = (a + i\sqrt{3}b)^6 \qquad (5)$$

Equating the real and imaginary parts in (5) ,note that

$$Y = f(a,b), X = g(a,b) \qquad (6)$$

where

$$f(a,b) = a^6 - 45a^4b^2 + 135a^2b^4 - 27b^6, \quad g(a,b) = 6a^5b - 60a^3b^3 + 54ab^5$$

In view of (2) , we have

$$y = 4f(a,b), x = 4g(a,b) \tag{7}$$

Thus, (4) and (7) represent the integer solutions to (1).

Note 1:

It is worth to observe that (1) reduces to (3) on considering the linear transformations

$$y = 2Y + 6X, x = 2Y - 2X \tag{8}$$

For this choice,the corresponding integer solutions to (1) are given by

$$y = 2f(a,b) + 6g(a,b), x = 2f(a,b) - 2g(a,b)$$

along with (4).

Note 2:

In addition to (8) , one may consider the transformations as

$$y = 2Y - 6X, x = 2Y + 2X \tag{9}$$

In this case ,the corresponding integer solutions to (1) are given by

$$y = 2f(a,b) - 6g(a,b), x = 2f(a,b) + 2g(a,b)$$

along with (4).

Method 2

Write (3) as

$$Y^2 + 3X^2 = z^6 * 1 \tag{10}$$

Write the integer 1 on the R.H.S. of (10) as

$$1 = \frac{(1+i\sqrt{3})(1-i\sqrt{3})}{4} \tag{11}$$

Substituting (4) & (11) in (10) and employing the method of factorization,

define

$$Y + i\sqrt{3}X = \frac{(1+i\sqrt{3})(a+i\sqrt{3}b)^6}{2} \tag{12}$$

Equating the real and imaginary parts in (12) and using (2),note that

$$\left.\begin{aligned} y &= 2[f(a,b) - 3g(a,b)], \\ x &= 2[f(a,b) + g(a,b)] \end{aligned}\right) \tag{13}$$

Thus,(4) and (13) give the required integer solutions to (1).

Note 3 :

Instead of (2),if we consider the transformations given by (8) ,then,the

corresponding integer solutions to (1) are given by

$$y = 4f(a,b), x = -4g(a,b)$$

along with (4).

Note 4 :

Also,if we consider the transformations given by (9) ,then,the respective integer

solutions are seen to be

$$y = -2f(a,b) - 6g(a,b), x = 2f(a,b) - 2g(a,b)$$

along with (4) .

Note 5 :

Apart from (11),the integer 1 on the R.H.S. of (10) may be written as the product of complex conjugates as below :

$$1 = \frac{(3r^2 - s^2 + i\sqrt{3}2rs)(3r^2 - s^2 - i\sqrt{3}2rs)}{(3r^2 + s^2)^2}$$

The repetition of the above process leads to three more sets of integer solutions to (1).

Method 3

Write (3) as

$$Y^2 + 3X^2 = (z^3)^2 \qquad (14)$$

which is satisfied by

$$X = 2rs, Y = r^2 - 3s^2 \qquad (15)$$

and

$$z^3 = r^2 + 3s^2 \qquad (16)$$

Now,(16) is satisfied by

$$r = p(p^2 + 3q^2), s = q(p^2 + 3q^2) \qquad (17)$$

and

$$z = p^2 + 3q^2 \qquad (18)$$

Using (17) in (15) ,we have

$$X = 2pq(p^2 + 3q^2)^2, Y = (p^2 + 3q^2)^2(p^2 - 3q^2) \qquad (19)$$

In view of (2) ,one has

$$x = 8pq(p^2 + 3q^2)^2, y = 4(p^2 + 3q^2)^2(p^2 - 3q^2) \qquad (20)$$

Thus,(18) and (20) represent the integer solutions to (1).

Note 6 :

In view of the transformations (8) , the corresponding integer solutions

to (1) are found to be

$$y = (p^2 + 3q^2)^2(2p^2 - 6q^2 + 12pq), x = (p^2 + 3q^2)^2(2p^2 - 6q^2 - 4pq)$$

along with (18).

Note 7 :

In view of the transformations (9) , the corresponding integer solutions

to (1) are found to be

$$y = (p^2 + 3q^2)^2(2p^2 - 6q^2 - 12pq), x = (p^2 + 3q^2)^2(2p^2 - 6q^2 + 4pq)$$

along with (18).

Note 8 :

It is worth mentioning that (16) is satisfied by

$$r = p^3 - 9pq^2, s = 3p^2q - 3q^3$$

along with (18). In view of (15) ,one has

$$X = 2(p^3 - 9pq^2)(3p^2q - 3q^3), Y = (p^3 - 9pq^2)^2 - 3(3p^2q - 3q^3)^2$$

Employing the transformations (2) ,(8) and (9) in turn ,one obtains three

different sets of integer solutions to (1).

Method 4

Rewrite (3) as

$$Y^2 + 3X^2 = (z^2)^3 \quad (21)$$

which is satisfied by

$$Y = p(p^2 + 3q^2), X = q(p^2 + 3q^2) \quad (22)$$

and

$$z^2 = p^2 + 3q^2 \quad (23)$$

Now, (23) is satisfied by

$$q = 2rs, p = r^2 - 3s^2 \quad (24)$$

and

$$z = r^2 + 3s^2 \quad (25)$$

Using (24) in (22) ,we have

$$Y = (r^2 - 3s^2)(r^2 + 3s^2)^2, X = 2rs(r^2 + 3s^2)^2$$

Employing the transformations (2) ,(8) and (9) in turn ,one obtains three different sets of integer solutions to (1).

Conclusion:

In this paper ,an attempt has been made to determine the non-zero distinct integer solutions to the non-homogeneous ternary sextic diophantine equation given in the title through employing transformations. The researchers in this area may search for other choices of transformations to obtain integer solutions to the ternary sextic diophantine equation under consideration.

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