WWW.IJRPR.COM

**International Journal of Research Publication and Reviews**

Journal homepage: www.ijrpr.com ISSN 2582-7421

# A Modified Differential Transform Method for the Solutions of Nonlinear Differential Equations via the He's Polynomial

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**ABSTRACT**

*The aim of the study is to consider a novel approach for solving nonlinear differential equation. The approach consists of combine use of the differential transform method and the He's polynomials. Though very effective, the differential transform method (DTM) still suffers from some short comings which is lack of a systematic methodology for derivation of the differential transforms for linear and nonlinear equations. In this paper, it is shown that this defect has be overcome via the use of He's polynomials method. The proposed method in handling of linear and nonlinear differential equations is well illustrated by a number of examples. The transformed analogues of some frequent nonlinearity are presented. Numerical experiments demonstrate the effectiveness of the proposed approach for solving linear and nonlinear differential equations and it was in good agreement with exact solution.*

***Keywords:*** *differential transform method (DTM), modified differential transform method (MDTM), He's polynomials, and differential equation.*

## 1.0 INTRODUCTION

In real world, many physical and natural phenomena are formulated as differential equations. Most of these differential equations are nonlinear. So there are difficulties in finding the exact or analytical solutions caused by the nonlinear part Ghorbani (2009). Many methods have been proposed to solve or approximate nonlinear differential equations, such as: Adomian decomposition method (ADM) Tate & Dinde (2019), variational iteration method (VIM)

(Wazwaz, 2009), homotopy perturbation method (HPM) Momani & Odibat (2007), differential transform method (DTM) Rashidi, *et. al*, (2020) and many other authors, although these methods provides some useful solutions, but involve some restrictions, linearization and transformations. Nonlinear phenomena play a crucial role in designing more realistic mathematical models to describe the physical nature, so there is a need for a method that can handle nonlinear terms easily without any form of restrictions, linearization or transformations. The DTM is an effective numerical and analytical method for solving different types of differential equations as well as integral equations. This method converts the differential equations into recurrence relations, and then by Taylor series expansion, with a different approach, obtains the convergent series solutions. The concept of DTM was first introduced by Zhou in 1986 to solve linear and nonlinear initial value problems in electrical circuit analysis Ayaz (2003). In this study, the DTM will be modified to solve nonlinear ODEs. In this paper we modify the DTM through the He's polynomial to provide semi-analytic solution of nonlinear ODEs and the results compared to other methods.

## 2.0 MATERIALS AND METHOD

### *2.1 The Differential Transform Method (DTM) and He's Polynomials*

This subsection is devoted to a quick review over the fundamentals of the DTM and the He's polynomials.

$$Y(k) = \frac{1}{k!}\left[\frac{d^k y(x)}{dx^k}\right] \qquad \ldots(1)$$

where $Y(k)$ is a transformed function. The inverse of $Y(k)$ is express as the inverse

$$y(x) = \sum_{k=0}^{\infty} Y(k)x^k \approx Y_N(x) = \sum_{k=0}^{N} Y(k)x^k \qquad \ldots(2)$$

In general if $y(x) = y^n(x)$, then $Y(K)$

i. If $y(x) = u(x) \pm v(x)$, then $Y(k) = U(k) \pm V(k)$

ii. If $y(x) = \alpha u(x)$, then $Y(k) = \alpha U(k)$ where "$\alpha$" is constant.

iii. If $y(x) = y^{'}(x)$, then $Y(k) = (k+1)Y(k+1)$

iv. If $y(x) = y^{''}(x)$, then $Y(k) = (k+1)(k+2)Y(k+2)$

v. If $y(x) = y^{(n)}(x)$, then $Y(k) = \frac{(k+n)!}{k!}Y(k+n)$

vi. If $y(x) = x^0\ or\ y(x) = 1$, then $Y(k) = \delta(k)$

vii. If $y(x) = x$, then $Y(k) = \delta(k-1)$

viii. If $y(x) = x^n(x)$, then $Y(k) = \delta(k-n) = 1,\ if\ k = n$ or $Y(k) = \delta(k-n) = 0, if\ k \neq n$

ix. If $y(x) = e^{\lambda k}(x)$, then $Y(k) = \frac{\lambda^k}{k!}$

x. If $y(x) = u(x)v(x)$, then $Y(k) = \sum_{n=0}^{k} U(k)V(k-n)$

### *2.2 He's Polynomials and their evaluation*

He's polynomials are indispensable in nonlinear analyses by the HPM. Let N be a nonlinear operator acting upon an unknown function $y$. For treating functional equations including such nonlinear terms like $Nu$, HPM entails decomposition of $Ny$into an infinite summation of the He's polynomials, $H_n s$, corresponding to $N$as:

$$Ny = \sum_{n=0}^{\infty} H_n, \qquad \ldots(3)$$

where $H_n s$are classically suggested in Ghorbani (2009) and to be obtained by

$$H_n(y_0, \ldots, y_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left[F\left(\sum_{i=0}^{n} p^i y_i\right)\right]_{p=0}, \qquad n \geq 0 \qquad \ldots(4)$$

This gives

$$H_0 = F(y_0),$$

$$H_1 = \frac{\partial}{\partial p}\left[F\left(\sum_{i=0}^{1} p^i y_i\right)\right]_{p=0} = y_1 F'(y_0),$$

$$H_2 = \frac{1}{2!}\frac{\partial^2}{\partial p^2}\left[F\left(\sum_{i=0}^{2} p^i y_i\right)\right]_{p=0} = y_2 F'(y_0) + \frac{1}{2!}\ y_1^2 F''(y_0),$$

$$H_3 = \frac{1}{3!}\frac{\partial^3}{\partial p^3}\left[F\left(\sum_{i=0}^{3} p^i y_i\right)\right]_{p=0} = u_3 F'(y_0) + y_1 y_2 F''(y_0) + \frac{1}{3!}\ y_1^3 F'''(y_0).$$

### *2.3 Analysis of the proposed DTM*

We illustrate the steps of our new modification of the DTM by considering the first-order initial value problem (IVP), second-order IVP, third-order IVP, and move to generalize to the $n^{th}$-order IVP.

### *2.4 First-order homogeneous nonlinear IVP*

$$y'(x) + ay(x) + bf[y(x)] = 0, y(0) = y_0 \qquad \ldots(5)$$

By applying subsection **2.1** and **2.2** on Eqn. (5), we get

$$\frac{(k+1)!}{k!}Y_{k+1} + aY_k + b[H_n] = 0 \qquad \ldots(6)$$

Where $k = 0,1,2,\ldots$ and $n = 0,1,2,\ldots$ and

$$y_0 = Y(0) \qquad \ldots(7)$$

$$Y_{(k+1)} = -\frac{k!}{(k+1)!}[aY_k + a_n(H_n)] \qquad \ldots(8)$$

Lastly, the series solution is obtained as follows:

$$Y(x) = Y_0 x^0 + Y_1 x^1 + \ldots + Y_n x^n \qquad \ldots(9)$$

### *2.5 Second-order homogeneous nonlinear IVP*

$$y''(x) + ay'(x) + by(x) + cf[y(x)] = 0, y(0) = y_0, y'(0) = y_1 \qquad \ldots(10)$$

By applying subsection **2.1** and **2.2** on Eqn. (10), we get

$$\frac{(k+2)!}{k!}Y_{k+2} + a\frac{(k+1)!}{k!}Y_{k+1} + bY_k + c[H_n] = 0 \qquad \ldots(11)$$

Where $k = 0,1,2,...$ and $n = 0,1,2,...$ and

$y_0 = Y(0), y_1 = Y'(0)$ ...(12)

$Y_{(k+2)} = -\frac{k!}{(k+2)!}\left[a\frac{(k+1)!}{k!}Y_{k+1} + bY_k + c(H_n)\right]$ ...(13)

Lastly, the series solution is obtained as follows:

$Y(x) = Y_0x^0 + Y_1x^1 + ... + Y_nx^n$ ...(14)

### *2.6 Third-order homogeneous nonlinear IVP*

$y'''(x) + ay''(x) + by'(x) + cy(x) + df[y(x)] = 0, \ y_0 = y(0) \ y_1 = y'(0), \ y_2 = y''(0)$ ...(15)

By applying subsection **2.1** and **2.2** on Eqn. (15), we get

$\frac{(k+3)!}{k!}Y_{k+3} + a\frac{(k+2)!}{k!}Y_{k+2} + b\frac{(k+1)!}{k!}Y_{k+1} + cY_k + d(H_n) = 0$ ...(16)

Where $k = 0,1,2,...$ and $n = 0,1,2,...$ and

$y_0 = Y(0), \ y_1 = Y'(0), \ y_2 = Y''(0)$ ...(17)

$Y_{(k+3)} = -\frac{k!}{(k+3)!}\left[a\frac{(k+2)!}{k!}Y_{k+2} + b\frac{(k+1)!}{k!}Y_{k+1} + cY_k + d(H_n)\right]$ ...(18)

Lastly, the series solution is obtained as follows:

$Y(x) = Y_0x^0 + Y_1x^1 + ... + Y_nx^n$ ...(19)

### *2.7 $n^{th}$- order homogeneous nonlinear IVP*

Consider the equation of $n^{th}$ order ODE as follows

$y^n(x) + a_1y^{n-1}(x) + ... + a_{n-1}y'(x) + a_ny(x) = 0$ ...(20)

where$a_n$, $a_{n-1,...,}$ $a_1$ are real constants with the initial conditions;

$y(0) = y_0, y'(0) = y_1, ..., y^{(n-1)}(0) = y_{(n-1)}$ ...(21)

where $y_0$, $y'_0, ...,$ $y_0^{(n-1)}$ are real constants.

By applying subsection **2.1** and **2.2** on Eqn. (20) and Eqn. (21), we get

$\frac{(k+n)!}{k!}Y_{k+1} + a_1\left[\frac{(k+n-1)!}{k!}Y_{k+n-1}\right] + ... a_{n-1}[(k+1)Y_{k+1}] + a_n[H_n] = 0$ ...(22)

Where $k = 1,2, ..., m$ and $n = 1,2, ..., l$ and

$y_0 = Y(0), \ y_1 = Y(1), ...,$ ...(23)

$Y_{(m+n)} = -\frac{1}{(m+n)!}[a_1[(m+n-1)! \, Y_{m+n-1}] + ... + a_{n-1}[k! \, (m+1)Y_{m+1}] + a_n(k! \, H_n)]$ ...(24)

Lastly, the series solution is obtained as follows:

$Y(x) = Y_0x^0 + Y_1x^1 + ... + Y_nx^n$ ...(25)

## 3.0 RESULTS AND DISCUSSION

### *3.1 Nonlinear Differential Equation*

The effectiveness of MDTM for solving non-linear ordinary differential equations can be illustrated as follows:

**Problem 1:**

Consider the first order nonlinear differential equation [Ogunrinde, (2019)]:

$y'(x) = y^2(x) \ , y(0) = 1$ ...(26)

where the exact solution is

$y(x) = \frac{1}{1-x}$ ...(27)

Now, applying the MDTM, on Eqn. (26), gives:

$\frac{(k+1)!}{k!} Y_{k+1}(x) = Y_k^2(x)$ ...(28)

Thus,

$Y_{k+1}(x) = \frac{k! H_k(x)}{(k+1)!}$, ...(29)

where $H_k = Y_k^2$ ...(30)

and from initial condition we get

$y(0) = 1 = Y_0(x) = 1$ ...(31)

Applying the transform initial condition Eqn. (31) in Eqn. (29), we obtain the following

When $k$ = 0, Eqn. (29) becomes

$Y_1(x) = \frac{0! \times (Y_0)^2}{(0+1)!} = 1$ ...(32)

When $k$ = 1, Eqn. (29) becomes

$Y_2(x) = \frac{1! \times 2Y_0 Y_1}{(1+1)!} = 1$ ...(33)

When $k$ = 2, Eqn. (29) becomes

$Y_3(x) = \frac{2! \times [(Y_1)^2 + 2Y_0 Y_2]}{(2+1)!} = 1$ ...(34)

When $k$ = 3, Eqn. (29) becomes

$Y_4(x) = \frac{3! \times [2Y_1 Y_2 + 2Y_0 Y_3]}{(3+1)!} = 1$ ...(35)

When $k$ = 4, Eqn. (29) becomes

$Y_5(x) = \frac{4! \times [(Y_2)^2 + 2Y_1 Y_3 + 2Y_0 Y_4]}{(4+1)!} = 1$ ..(36)

we finally obtain the series solution up to fifth term as

$Y(x) \approx \sum_{k=0}^{5} Y_k x^k \approx 1 + x + x^2 + x^3 + x^4 + x^5$ ...(37)

**Table 1: Numerical solution of MDTM, DTM and Exact solution**

| x | Exact solution | MDTM (n=5) | DTM (n=7) |
|---|---|---|---|
| **0** | 1 | 1 | 1 |
| **0.1** | 1.111111 | 1.11111099 | 1.11111 |
| **0.2** | 1.249984 | 1.24998398 | 1.24992 |
| **0.3** | 1.428259 | 1.42825897 | 1.42753 |
| **0.4** | 1.663936 | 1.66393596 | 1.65984 |
| **0.5** | 1.984375 | 1.98437495 | 1.96875 |
| **0.6** | 2.430016 | 2.43001594 | 2.38336 |
| **0.7** | 3.058819 | 3.05881893 | 2.94117 |
| **0.8** | 3.951424 | 3.95142392 | 3.68928 |
| **0.9** | 5.217031 | 5.21703091 | 4.68559 |

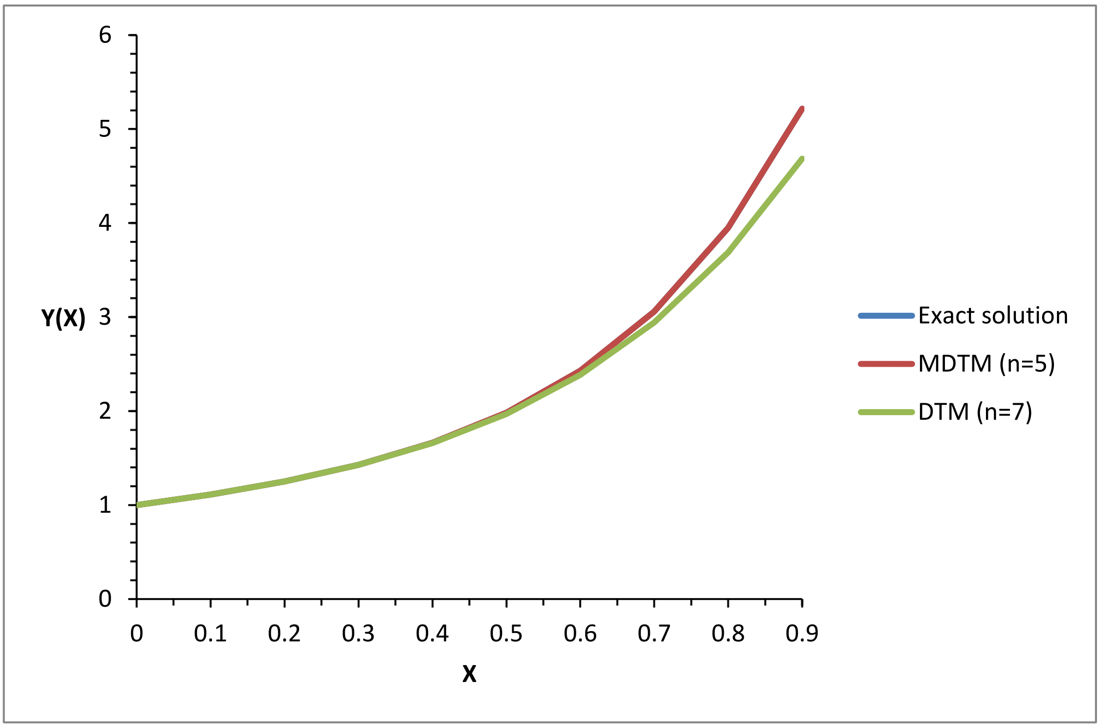


**Figure 1: the graph of numerical solution for exact solution, MDTM and DTM**

Figure 1 shows the numerical solution for the exact solution, MDTM, and DTM obtains by using MAPLE18 software. The trend of the graph for MDTM and DTM is slightly the same as the exact solution. In Table 2, the absolute error for MDTM and DTM shows that the numerical solution for both methods is close to the exact solution. But, by comparing the absolute solution for both methods, the minimum absolute error goes to MDTM. This shows that MDTM is more accurate than DTM.

**Table 2: the absolute error of MDTM, DTM and the minimum absolute error to the exact solution**

| x | Exact solution | MDTM (n=5) | DTM (n=7) | MDTM | DTM | Minimum |
|---|---|---|---|---|---|---|
| **0** | 1 | 1 | 1 | 0 | 0 | 0 |
| **0.1** | 1.111111 | 1.11111099 | 1.11111 | 1E-08 | 1E-06 | 1E-08 |
| **0.2** | 1.249984 | 1.24998398 | 1.24992 | 2E-08 | 6.4E-05 | 2E-08 |
| **0.3** | 1.428259 | 1.42825897 | 1.42753 | 3E-08 | 0.000729 | 3E-08 |
| **0.4** | 1.663936 | 1.66393596 | 1.65984 | 4E-08 | 0.004096 | 4E-08 |
| **0.5** | 1.984375 | 1.98437495 | 1.96875 | 5E-08 | 0.015625 | 5E-08 |
| **0.6** | 2.430016 | 2.43001594 | 2.38336 | 6E-08 | 0.046656 | 6E-08 |
| **0.7** | 3.058819 | 3.05881893 | 2.94117 | 7E-08 | 0.117649 | 7E-08 |
| **0.8** | 3.951424 | 3.95142392 | 3.68928 | 8E-08 | 0.262144 | 8E-08 |
| **0.9** | 5.217031 | 5.21703091 | 4.68559 | 9E-08 | 0.531441 | 9E-08 |

**Problem 2:**

We consider the first order nonlinear inhomogeneous IVP given as:

$y'(x) + y^2(x) = 1\ , y(0) = 0$ [Ogunrinde, (2019)] $\dots(38)$

With the exact solution

$y(x) = tanh(x)$ $\dots(39)$

Now, applying the MDTM, on Eqn. (38), gives:

$\frac{(k+1)!}{k!} Y_{k+1}(x) = 1.\delta(k) - Y_k^2(x)$ $\dots(40)$

This leads to the following recurrence relation

$$Y_{k+1}(x) = \frac{k![1.\delta(k) - H_k(x)]}{(k+1)!} \qquad \ldots(41)$$

Where $H_k = Y_k^2$ and $\delta(k) = \begin{cases} 1, k=0 \\ 0, k \geq 1 \end{cases}$ $\ldots(42)$

and from initial condition we get

$$y(0) = 0 = Y_0(x) = 0 \qquad \ldots(43)$$

Applying the transform initial condition Eqn. (43) in Eqn. (41), we obtain the following

When $k$ = 0, Eqn. (41) becomes

$$Y_1(x) = \frac{0![1.\delta(0) - (Y_0)^2]}{(0+1)!} = 1 \qquad \ldots(44)$$

When $k$ = 1, Eqn. (41) becomes

$$Y_2(x) = \frac{1![1.\delta(1) - 2Y_0Y_1]}{(1+1)!} = 0 \qquad \ldots(45)$$

When $k$ = 2, Eqn. (41) becomes

$$Y_3(x) = \frac{2! \times [1.\delta(2) - ((Y_1)^2 + 2Y_0Y_2)]}{(2+1)!} = -\frac{1}{3} \qquad \ldots(46)$$

When $k$ = 3, Eqn. (41) becomes

$$Y_4(x) = \frac{3! \times [1.\delta(3) - (2Y_1Y_2 + 2Y_0Y_3)]}{(3+1)!} = 0 \qquad \ldots(47)$$

When $k$ = 4, Eqn. (41) becomes

$$Y_5(x) = \frac{4! \times [1.\delta(4) - ((Y_2)^2 + 2Y_1Y_3 + 2Y_0Y_4)]}{(4+1)!} = \frac{2}{15} \qquad \ldots(48)$$

$$Y_6(x) = 0 \qquad \ldots(49)$$

When $k$ = 6, Eqn. (41) becomes

$$Y_7(x) = \frac{6! \times [1.\delta(6) - ((Y_3)^2 + 2Y_0Y_6 + 2Y_1Y_5 + 2Y_2Y_4)]}{(6+1)!} = -\frac{17}{315} \qquad \ldots(50)$$

We finally obtain the series solution up to seventh term as

$$Y(x) \approx \sum_{k=0}^{7} Y_k x^k \approx x - \frac{1}{3}x^3 + \frac{2}{15}x^5 - \frac{17}{315}x^7 \qquad \ldots(51)$$

**Table 3: Numerical solution of MDTM, DTM, ADM and Exact solution**

| x | Exact solution | DTM (k=10) | MDTM (k=7) | ADM (n=8) |
|---|---|---|---|---|
| **0** | 0 | 0 | 0 | 0 |
| **0.1** | 0.099667995 | 0.09966799 | 0.099667995 | 0.099668 |
| **0.2** | 0.19737532 | 0.19737531 | 0.19737532 | 0.1973753 |
| **0.3** | 0.291312612 | 0.2913122 | 0.291312612 | 0.2913122 |
| **0.4** | 0.379948962 | 0.37994358 | 0.37994894 | 0.3799436 |
| **0.5** | 0.462117157 | 0.46207837 | 0.462116759 | 0.4620784 |
| **0.6** | 0.537049567 | 0.53685723 | 0.537045473 | 0.5368572 |
| **0.7** | 0.604367777 | 0.60363148 | 0.60433874 | 0.6036315 |
| **0.8** | 0.66403677 | 0.66170604 | 0.663879964 | 0.661706 |
| **0.9** | 0.71629787 | 0.70991915 | 0.715610462 | 0.7099192 |
| **1** | 0.761594156 | 0.74603175 | 0.759037999 | 0.7460317 |

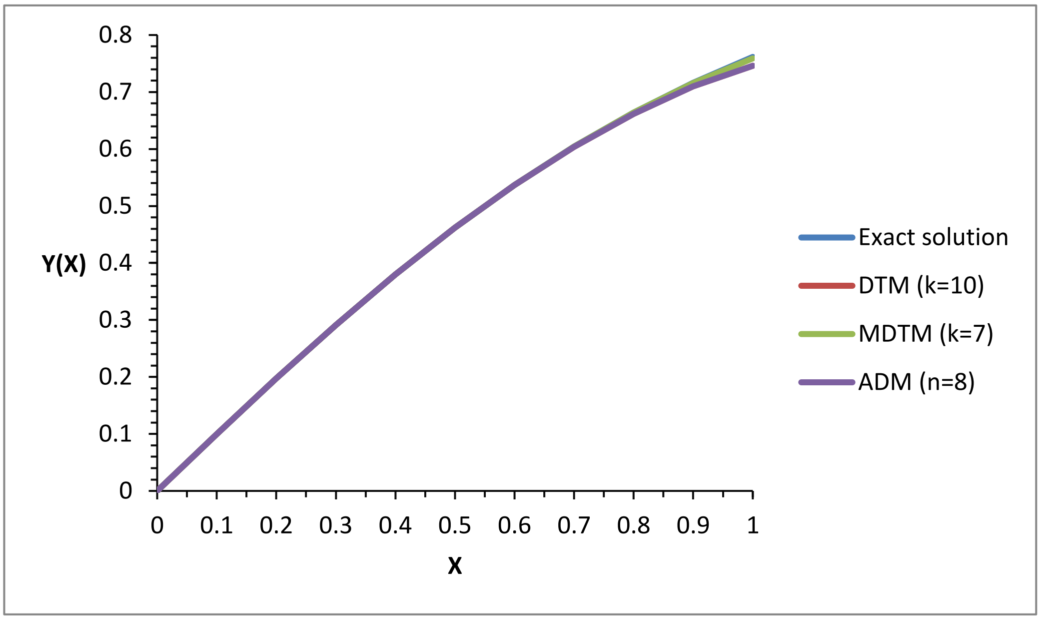


**Figure 2: the graph of numerical solution for exact solution, MDTM, DTM and ADM**

Figure 2 shows the graph for exact solution, MDTM, DTM, and ADM that obtained by using Maple software. The trend of the graph for MDTM and exact solution is similar but graph for DTM and ADM slightly diverges from the exact solution. By comparing the absolute error for both methods in Table 4, the MDTM has the minimum absolute error compared to DTM and ADM. This shows that the numerical solution for MDTM is closer to the exact solution compared to the DTM and ADM.

**Table 4: the absolute error of MDTM, DTM, ADM and the minimum absolute error to the exact solution**

| x | Exact solution | DTM | MDTM | ADM | MDTM Error | DTM Error | ADM Error | Minimum |
|---|---|---|---|---|---|---|---|---|
| **0** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| **0.1** | 0.099667995 | 0.099668 | 0.099668 | 0.099667995 | 4.44089E-16 | 2.178E-11 | 2.178E-11 | 4.44089E-16 |
| **0.2** | 0.19737532 | 0.197375 | 0.197375 | 0.197375309 | 2.89574E-12 | 1.102E-08 | 1.102E-08 | 2.89574E-12 |
| **0.3** | 0.291312612 | 0.291312 | 0.291313 | 0.291312197 | 5.52547E-10 | 4.153E-07 | 4.153E-07 | 5.52547E-10 |
| **0.4** | 0.379948962 | 0.379944 | 0.379949 | 0.379943578 | 2.26384E-08 | 5.384E-06 | 5.384E-06 | 2.26384E-08 |
| **0.5** | 0.462117157 | 0.462078 | 0.462117 | 0.462078373 | 3.98151E-07 | 3.878E-05 | 3.878E-05 | 3.98151E-07 |
| **0.6** | 0.537049567 | 0.536857 | 0.537045 | 0.536857234 | 4.09421E-06 | 0.0001923 | 0.0001923 | 4.09421E-06 |
| **0.7** | 0.604367777 | 0.603631 | 0.604339 | 0.603631482 | 2.90373E-05 | 0.0007363 | 0.0007363 | 2.90373E-05 |
| **0.8** | 0.66403677 | 0.661706 | 0.66388 | 0.661706037 | 0.000156807 | 0.0023307 | 0.0023307 | 0.000156807 |
| **0.9** | 0.71629787 | 0.709919 | 0.71561 | 0.709919151 | 0.000687408 | 0.0063787 | 0.0063787 | 0.000687408 |
| **1** | 0.761594156 | 0.746032 | 0.759038 | 0.746031746 | 0.002556157 | 0.0155624 | 0.0155624 | 0.002556157 |

## 4.0 Conclusion

The differential transform method (DTM) is a straightforward and popular tool for handling many types of functional equations. However, the draw backs of the method lie in handling of linear and nonlinear unknown terms: not a general alternative is available for evaluation of the differential transforms of linear and nonlinearities. In this work, we have overcome this defect by proposing a simple general routine involving the He's polynomials. For illustration, the differential transforms of some famous nonlinear operators are provided. The application of the proposed method is shown in solution of some linear and nonlinear differential equations. We conclude that the findings disclosed in this work will broaden the applicability and popularity of the MDTM considerably. Our modification of the differential Transform Method (DTM) has helped us in successfully solving some nonlinear Differential Equations addressed by this study. A quick inspection of all the problems considered revealed that the MDTM results corresponded with the exact solutions to the problems in series form. Similarly, it could be observed that the MDTM was implemented without any need for transformation of the nonlinear term and also does not require perturbation or linearization. Thus, for ease of solution to any nonlinear Differential equation problems without cumbersome algebraic computations, this study therefore recommends MDTM for future and as an alternative to other semi-analytical method of solution.

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