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A Search on Integer Solutions to the Homogeneous Quadratic Equation with Three Unknowns $x^2+17y^2=21z^2$

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ABSTRACT:

This paper focuses on finding non-zero distinct integer solutions to the Homogeneous Quadratic Diophantine Equation with three unknowns given by $x^2+17y^2=21z^2$

Various sets of integer solutions are obtained. A few interesting properties among the solutions are given. Also, knowing a solution of the given equation, formulas for obtaining sequence of integer solutions based on the given solution are presented

KEYWORDS: Ternary quadratic, Homogeneous quadratic, Integer solutions, Legendre equation

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INTRODUCTION:

The Ternary quadratic Diophantine equations offer an unlimited field for research because of their variety [1-3]. For an extensive review of various problems, one may refer [4-19] for quadratic equations with two and three unknowns. This communication concerns with yet another interesting ternary quadratic equation $x^2+17y^2=21z^2$ representing a cone for determining its infinitely many non- zero integral solutions. Also a few interesting properties among the solutions have been presented.

METHOD OF ANALYSIS

The Ternary Quadratic Diophantine equation representing homogeneous cone under consideration is

$$x^2+17y^2=21z^2 \tag{1}$$

We present below different methods of solving (1).

METHOD I:

Equation (1) is written in the form of ratio as

$$\frac{x+2z}{z+y}=\frac{17(z-y)}{x-2z}=\frac{\alpha}{\beta},\quad \beta\neq 0 \tag{2}$$

which is equivalent to the system of double equations

$\beta x-\alpha y+(2\beta-\alpha)z=0$

$\alpha x+17\beta y-(17\beta+2\alpha)z=0$

Applying the method of cross multiplication, the corresponding values of x, y, z satisfying (1) are given by

$x(\alpha,\beta)=2\alpha^2-34\beta^2+34\alpha\beta$

$y(\alpha,\beta)=-\alpha^2+17\beta^2+4\alpha\beta$

$z(\alpha,\beta)=\alpha^2+17\beta^2$

Note 1: Apart from (2), (1) is also written in the form of ratios as presented below

$(i)\,\frac{x-2z}{(z+y)}=\frac{17(z-y)}{x+2z}=\frac{\alpha}{\beta}$

$(ii)\,\frac{x+2z}{(z-y)}=\frac{17(z+y)}{x-2z}=\frac{\alpha}{\beta}$

$(iii)\,\frac{x-2z}{(z-y)}=\frac{17(z+y)}{x+2z}=\frac{\alpha}{\beta}$

Following the above procedure, the solutions of (1) for choices (i), (ii), (iii) are presented below

Solutions for choice (i):

$x(\alpha,\beta)=-2\alpha^2+34\beta^2+34\alpha\beta$
$y(\alpha,\beta)=-\alpha^2+17\beta^2-4\alpha\beta$
$z(\alpha,\beta)=\alpha^2+17\beta^2$

Solutions for choice (ii):

$x(\alpha,\beta)=-2\alpha^2+34\beta^2-34\alpha\beta$
$y(\alpha,\beta)=-\alpha^2+17\beta^2+4\alpha\beta$
$z(\alpha,\beta)=-\alpha^2-17\beta^2$

Solutions for choice (iii):

$x(\alpha,\beta)=2\alpha^2-34\beta^2-34\alpha\beta$
$y(\alpha,\beta)=-\alpha^2+17\beta^2-4\alpha\beta$
$z(\alpha,\beta)=-\alpha^2-17\beta^2$

METHOD II:

Introduction of the linear transformations

$x=2P,\quad y=X+21T,\quad z=X+17T$ (3)

in (1) leads to

$X^2=357T^2+P^2$ (4)

which is satisfied by

$T=2rs, P=357r^2-s^2, X=357r^2+s^2$
In view of (3), the corresponding integer solutions to (1) are given by

$x=714r^2-2s^2$
$y=357r^2+s^2+42rs$
$z=357r^2+s^2+34rs$

Also, (4) is written as the system of double equations as presented below in Table 1

Table 1: System of double equations

System	1	2	3	4	5	6
$X+P$	$357T^2$	$3T^2$	$7T^2$	$17T^2$	$21T^2$	$51T^2$
$X-P$	1	119	51	21	17	7

System	7	8	9
X+P	17T	51T	3T
X-P	21T	7T	119T

Solving each of the above systems, the value of X, P and T are obtained. Substituting these in (3), the corresponding solutions to (1) are found. For simplicity, we present the solutions below

Solutions for system 1:

$x = 1428K^2 + 1428K + 356$

$y = 714K^2 + 756K + 200$

$z = 714K^2 + 748K + 196$

Solutions for system 2:

$x = 12K^2 + 12K - 116$

$y = 6K^2 + 48K + 82$

$z = 6K^2 + 40K + 78$

Solutions for system 3:

$x = 28K^2 + 28K - 44$

$y = 14K^2 + 56K + 50$

$z = 14K^2 + 48K + 46$

Solutions for system 4:

$x = 68K^2 + 68K - 4$

$y = 34K^2 + 76K + 40$

$z = 34K^2 + 68K + 36$

Solutions for system 5:

$x = 84K^2 + 84K + 4$

$y = 42K^2 + 84K + 40$

$z = 42K^2 + 76K + 36$

Solutions for system 6:

$x = 204K^2 + 204K + 44$

$y = 102K^2 + 144K + 50$

$z = 102K^2 + 136K + 46$

Solutions for system 7:

$x = 476K^2 + 476K + 232$

$y = 238K^2 + 280K + 82$

$z = 238K^2 + 272K + 78$

Solutions for system 8:

$$x = -8K - 4$$
$$y = 80K + 40$$
$$z = 72K + 36$$

Solutions for system 9:

$$x = 88K + 44$$
$$y = 100K + 50$$
$$z = 92K + 46$$

Solutions for system 10:

$$x = -232K - 116$$
$$y = 164K + 82$$
$$z = 156K + 78$$

METHOD III:

Assume

$$z(a,b) = a^2 + 17b^2 \qquad (5)$$

Case (i):

Write 21 as

$$21 = \left(2 + i\sqrt{17}\right)\left(2 - i\sqrt{17}\right) \qquad (6)$$

Using (5) and (6) in (1) and employing the method of factorization, consider

$$x + i\sqrt{17}y = (2 + i\sqrt{17})(a + i\sqrt{17}b)^2$$

Equating real and imaginary parts, we have

$$x = 2a^2 - 34b^2 - 34ab$$
$$y = 4ab + a^2 - 17b^2$$
$$z = a^2 + 17b^2$$

Case(ii):

Write 21 as

$$21 = \frac{(11 + i2\sqrt{17})(11 - i2\sqrt{17})}{3^2} \qquad (7)$$

Using (5) and (7) in (1) and employing the method of factorization, consider

$$x + i\sqrt{17}y = \frac{(11 + i2\sqrt{17})}{3}(a + i\sqrt{17}b)^2$$

Equating real and imaginary parts and replacing a by 3A, b by 3B, we have

$$\left.\begin{aligned} x(A,B) &= 33A^2 - 561B^2 - 204AB \\ y(A,B) &= 6A^2 - 102B^2 + 66AB \end{aligned}\right\} \quad (8)$$

and from we have

$$z(A,B) = 9A^2 + 153B^2 \quad (9)$$

Thus (8) and (9) represent the integer solutions to (1)

Note :1

It is seen that 21 is also represented as follows

$$(iv) \quad 21 = \frac{(31 + i2\sqrt{17})(31 - i2\sqrt{17})}{49}(a^2 + 17b^2)$$

Following the above procedure, the solutions of are obtained.

$$x(A,B) = 217A^2 - 3689B^2 - 476AB$$
$$y(A,B) = 14A^2 - 238B^2 - 434AB$$
$$z(A,B) = 49A^2 + 833B^2$$

METHOD IV:

Equation (1) is written as

$$x^2 + 17y^2 = 21z^2 * 1 \quad (10)$$

Write 1 as

$$1 = \frac{(8 + i\sqrt{17})(8 - i\sqrt{17})}{81} \quad (11)$$

Substituting (5), (6) and (11) in (10) and following the procedure as above and replacing a by $9A$ and b by $9B$, the corresponding solutions to (1) are given by

$$x(A,B) = -9A^2 + 153B^2 - 3060AB$$
$$y(A,B) = 90A^2 - 1530B^2 - 18AB$$
$$z(A,B) = 81A^2 + 1377B^2$$

METHOD V:

Write (1) as

$$21z^2 - 17y^2 = x^2 * 1 \quad (12)$$

Let

$$x = 21a^2 - 17b^2 \quad (13)$$

Consider 1 as

$$1=\frac{\left(\sqrt{21}+\sqrt{17}\right)\left(\sqrt{21}-\sqrt{17}\right)}{4} \tag{14}$$

Using (13) & (14) in (12) and employing the method of factorization, consider

$$\sqrt{21}z+\sqrt{17}y=\frac{1}{4}\left(\sqrt{22}+\sqrt{17}\right)\left(\sqrt{21}a+\sqrt{17}b\right)^2$$

Equating the coefficients of corresponding terms, we have

$$\left.\begin{aligned} z&=\frac{1}{2}\left(21\sqrt{21}a^2+17b^2+34\sqrt{21}ab\right)\\ y&=\frac{1}{2}\left(21\sqrt{17}a^2+17\sqrt{17}b^2+42\sqrt{17}ab\right)\end{aligned}\right\} \tag{15}$$

Replacing a by 3A and b by 3B in (13) & (15) the corresponding integer

$$\left.\begin{aligned} x&=84A^2-68B^2\\ y&=42A^2+34B^2+84AB\\ z&=42A^2+34B^2+68AB\end{aligned}\right\} \tag{16}$$

Then (16) gives the integer solution to (1).

GENERATION OF INTEGER SOLUTIONS

Let (x_0, y_0, z_0) be any given integer solution to (1). We illustrate below the method of obtaining a general formula for generating sequence of integer solutions based on the given solution.

Case (i)

Let

$$\begin{aligned} x_1&=-x_0+9h\\ y_1&=y_0 \qquad h\neq 0\\ z_1&=-z_0+2h\end{aligned} \tag{17}$$

be the second solution of (1).

Substituting (17) in (1) & performing a few calculations, we have

$$h=6x_0+28z_0$$

and then

$$x_1=53x_0+252z_0$$
$$z_1=12x_0+55z_0$$

This is written in the form of matrix as

$$\begin{pmatrix}x_1\\ z_1\end{pmatrix}=M\begin{pmatrix}x_0\\ z_0\end{pmatrix} \tag{18}$$

where

$$M=\begin{pmatrix}53 & 252\\ 12 & 55\end{pmatrix}$$

Repeating the above process, the general solution (x_n, z_n) to (1) is given by

$$\begin{pmatrix} x_n \\ z_n \end{pmatrix} = M^n \begin{pmatrix} x_0 \\ z_0 \end{pmatrix}$$

To find M^n, the eigen values of M are $\alpha = 152, \ \beta = -87$

We know that $M^n = \frac{\alpha^n}{(\alpha - \beta)}(M - \beta I) + \frac{\beta^n}{(\beta - \alpha)}(M - \alpha I)$

Using the above formula, we have

$$M^n = \begin{pmatrix} \frac{54\alpha^n + 56\beta^n}{110} & \frac{252(\alpha^n - \beta^n)}{110} \\ \frac{12\alpha^n - 12\beta^n}{110} & \frac{56\alpha^n + 54\beta^n}{110} \end{pmatrix}$$

Thus the general solution (x_n, y_n, z_n) to (1) is given by

$$x_n = \left(\frac{54\alpha^n + 56\beta^n}{110}\right)x_0 + \frac{252}{110}(\alpha^n - \beta^n)z_0$$
$$y_n = y_0$$
$$z_n = \left(\frac{12(\alpha^n - \beta^n)}{110}\right)x_0 + \left(\frac{56\alpha^n + 54\beta^n}{110}\right)z_0$$

Case (ii)

Let

$$x_1 = 4x_0$$
$$y_1 = h + 4y_0 \qquad , h \neq 0$$
$$z_1 = h - 4z_0$$

Repeating the process as in the case (i) the corresponding general solution (x_n, y_n, z_n) to (1)

is given by

$$x_n = 4^n x_0$$
$$y_n = \left(\frac{\alpha^n(\sqrt{3297} - 21) + \beta^n(21 + \sqrt{3297}}{2\sqrt{3297}}\right)y_0 + \frac{84(\alpha^n - \beta^n)}{2\sqrt{3297}}z_0$$
$$z_n = \frac{34(\alpha^n - \beta^n)}{2\sqrt{3297}}y_0 + \left(\frac{\alpha^n(21 + \sqrt{3297} + \beta^n(-21 + \sqrt{3297}}{2\sqrt{3297}}\right)z_0$$

Case (iii)

Let

$$x_1 = h - 18x_0$$
$$y_1 = h - 18y_0 \ , h \neq 0$$
$$z_1 = 18z_0$$

Repeating the process as in the case (i) the corresponding general solution (x_n, y_n, z_n) to (1) is given by

$$x_n = \left(\frac{2\alpha^n - 2\beta^n}{36}\right)x_0 + \frac{34(\alpha^n - \beta^n)}{36}y_0$$
$$y_n = \frac{2(\alpha^n - \beta^n)}{36}x_0 + \left(\frac{34\alpha^n - \beta^n}{36}\right)y_0$$
$$z_n = 18^n z_0$$

Case (iv)

Let

$$x_1 = 3x_0 + h$$
$$y_1 = 3y_0 + h \ , h \neq 0$$
$$z_1 = h - 3z_0$$

Repeating the process as in the case (i) the corresponding general solution (x_1, y_1, z_1) to (1) is given by

$$x_1 = 5x_0 + 34y_0 + 42z_0$$
$$y_1 = 2x_0 + 37y_0 + 42z_0$$
$$z_1 = 2x_0 + 34y_0 + 39z_0$$

The required matrix is represented below:

$$(x_1, y_1, z_1)^t = \begin{pmatrix} 5 & 34 & 42 \\ 2 & 37 & 42 \\ 2 & 34 & 39 \end{pmatrix}(x_0, y_0, z_0)^t$$

$$M^2 = \begin{bmatrix} 219 & 3570 & 6237 \\ 210 & 3579 & 6237 \\ 198 & 3366 & 5868 \end{bmatrix}$$

$$M^3 = \begin{bmatrix} 20709 & 351594 & 612927 \\ 20682 & 351621 & 612927 \\ 19458 & 330786 & 576612 \end{bmatrix}$$

The general term of the matrix is denoted by

$$M^{n+1} = \begin{bmatrix} \frac{y_n - 3^{n+1}}{18} + 3^{n+1} & \frac{17\left(y_n - 3^{n+1}\right)}{18} & 21x_n \\ \frac{y_n - 3^{n+1}}{18} & \frac{17\left(y_n - 3^{n+1}\right)}{18} & 21x_n \\ x_n & 17x_n & y_n \end{bmatrix}$$

Where
$$x_n = x_0 y_{n-1} + y_0 x_{n-1}$$
$$y_n = y_0 y_{n-1} + 378 x_0 x_{n-1}$$
$$y_{-1} = 1, x_{-1} = 0$$

CONCLUSION

In this paper, we have made an attempt to obtain all integer solutions to (1). As (1) is symmetric in x, y, z it is to be noted that, if (x, y, z) is any positive integer solution to (1), then the triples (-x, y, z), (x,-y, z), (x, y,-z), (x, -y, -z), (-x, y, -z), (-x, -y, z) (-x, -y, -z) also satisfy (1). To conclude, one may search for integer solutions to other choices of homogeneous cones along with suitable properties.

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