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Observations on the Positive Pell Equation $x^2 = 42y^2 + 28$

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ABSTRACT:

This paper deals with the problem of obtaining non-zero distinct Integer solutions to the non-homogeneous binary quadratic equations with two unknowns $x^2 = 42y^2 + 28$. A few interesting relations among the solutions are given. Employing the linear combination among the solutions of the given equation, integer solutions for other choices of hyperbola & parabola and a few relations among special polygonal numbers are illustrated.

Keywords: Binary quadratic, non-homogeneous quadratic, Pell equation, Positive Pell equation, hyperbola
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Notations:

- $t_{m,n}$: *Polygonal number of rank n with m sides* $= n\left[1 + \frac{(n-1)(m-2)}{2}\right]$
- S_n : *Star number of rank* $n = 6n(n-1)+1$

Introduction:

A binary quadratic equation of the form $y^2 = Dx^2 + 1$, where D is non-square positive integer has been studied by various mathematicians for its non-trivial integral solutions when D takes different integral values [1-2]. For an extensive review of various problems, one may refer [3-22]. In this communication, yet another interesting hyperbola given by $x^2 = 42y^2 + 28$ is considered and infinitely many integer solutions are obtained. A few interesting relations among the solutions are given. The construction of second order Ramanujan Numbers with base numbers as real integers and Gaussian integers is illustrated.

Further, employing the solutions of the above hyperbola, we have obtained solutions of other choices of hyperbola and parabola.

METHOD OF ANALYSIS:

The Positive Pell equation representing hyperbola under consideration is

$$x^2 = 42y^2 + 28 \qquad (1)$$

whose initial solution is

$$x_0 = 14, \quad y_0 = 2$$

To obtain the other solutions of (1), consider the Pell equation

$$x^2 = 42y^2 + 1 \qquad (2)$$

whose general solution is given by

$$\tilde{x}_n = \frac{1}{2} f_n \, , \quad \tilde{y}_n = \frac{1}{2\sqrt{42}} g_n$$

where

$$f_n = (13+2\sqrt{42})^{n+1} + (13-2\sqrt{42})^{n+1}$$
$$g_n = (13+2\sqrt{42})^{n+1} - (13-2\sqrt{42})^{n+1}$$

Applying Brahmagupta lemma between (x_0, y_0) & $(\tilde{x}_n, \tilde{y}_n)$ the other integer solutions of (1) are given by

$$x_{n+1} = 7 f_n + \sqrt{42} g_n$$
$$6 y_{n+1} = 6 f_n + \sqrt{42} g_n$$

Replacing n by n+1, n+2 in turn in the above two equations, we have

$$x_{n+1} = 7 f_n + \sqrt{42} g_n \quad (3)$$

$$x_{n+2} = 175 f_n + 27\sqrt{42} g_n \quad (4)$$

$$x_{n+3} = 4543 f_n + 701\sqrt{42} g_n \quad (5)$$

$$6 y_{n+1} = 6 f_n + \sqrt{42} g_n \quad (6)$$

$$6 y_{n+2} = 162 f_n + 25\sqrt{42} g_n \quad (7)$$

$$6 y_{n+3} = 4206 f_n + 649\sqrt{42} g_n \quad (8)$$

Eliminating f_n & g_n among (3)-(5) and (6)-(8), the recurrence relation for x & y are given by

$$x_{n+1} - 26 x_{n+2} + x_{n+3} = 0$$

$$y_{n+1} - 26 y_{n+2} + y_{n+3} = 0$$

A few numerical examples are given in the following Table: 1

Table: 1 Numerical Examples

n	x_{n+1}	y_{n+1}
-1	$x_0 = 14$	$y_0 = 2$
0	$x_1 = 350$	$y_1 = 54$
1	$x_2 = 9086$	$y_2 = 1402$
2	$x_3 = 235886$	$y_3 = 36398$

3	$x_4 = 6123950$	$y_4 = 944946$
4	$x_5 = 158986814$	$y_5 = 24532198$

From the above table, the results observed are presented below:

- x_{n+1} *and* y_{n+1} *are even.*
- $x_{n+1} \equiv 0 (\bmod 14), n = -1, 0, 1, 2...$
- $y_{n+1} \equiv 0 (\bmod 2), \ n = -1, 0, 1, 2...$

1. Relations between solutions

- $x_{n+3} = 26x_{n+2} - x_{n+1}$
- $84y_{n+1} = x_{n+2} - 13x_{n+1}$
- $84y_{n+2} = 13x_{n+2} - x_{n+1}$
- $84y_{n+3} = 337x_{n+2} - 13x_{n+1}$
- $26x_{n+2} = x_{n+1} + x_{n+3}$
- $2184y_{n+1} = x_{n+3} - 337x_{n+1}$
- $168y_{n+2} = x_{n+3} - x_{n+1}$
- $2184y_{n+3} = 337x_{n+3} - x_{n+1}$
- $x_{n+2} = 13x_{n+1} + 84y_{n+1}$
- $x_{n+3} = 337x_{n+1} + 2184y_{n+1}$
- $y_{n+2} = 2x_{n+1} + 13y_{n+1}$
- $y_{n+3} = 52x_{n+1} + 337y_{n+1}$
- $13x_{n+2} = x_{n+1} + 84y_{n+2}$
- $x_{n+3} = x_{n+1} + 168y_{n+2}$
- $13y_{n+1} = y_{n+2} - 2x_{n+1}$
- $13y_{n+3} = 2x_{n+1} + 337y_{n+2}$
- $337x_{n+2} = 13x_{n+1} + 84y_{n+3}$
- $337x_{n+3} = x_{n+1} + 2184y_{n+3}$
- $337y_{n+1} = y_{n+3} - 52x_{n+1}$

- $337y_{n+2} = 13y_{n+3} - 2x_{n+1}$
- $x_{n+1} = 26x_{n+2} - x_{n+3}$
- $84y_{n+1} = 13x_{n+3} - 337x_{n+2}$
- $84y_{n+2} = x_{n+3} - 13x_{n+2}$
- $84y_{n+3} = 13x_{n+3} - x_{n+2}$
- $13x_{n+1} = x_{n+2} - 84y_{n+1}$
- $13x_{n+3} = 337x_{n+2} + 84y_{n+1}$
- $13y_{n+2} = 2x_{n+2} + y_{n+1}$
- $y_{n+3} = 4x_{n+2} + y_{n+1}$
- $x_{n+1} = 13x_{n+2} - 84y_{n+2}$
- $x_{n+3} = 13x_{n+2} + 84y_{n+2}$
- $y_{n+1} = 13y_{n+2} - 2x_{n+2}$
- $y_{n+3} = 2x_{n+2} + 13y_{n+2}$
- $13x_{n+1} = 337x_{n+2} - 84y_{n+3}$
- $13x_{n+3} = x_{n+2} + 84y_{n+3}$
- $y_{n+1} = y_{n+3} - 4x_{n+2}$
- $13y_{n+2} = y_{n+3} - 2x_{n+2}$
- $337x_{n+1} = x_{n+3} - 2184y_{n+1}$
- $337x_{n+2} = 13x_{n+3} - 84y_{n+1}$
- $337y_{n+2} = 2x_{n+3} + 13y_{n+1}$
- $337y_{n+3} = 52x_{n+3} + y_{n+1}$
- $x_{n+1} = x_{n+3} - 168y_{n+2}$
- $13x_{n+2} = x_{n+3} - 84y_{n+2}$
- $13y_{n+1} = 337y_{n+2} - 2x_{n+3}$
- $13y_{n+3} = 2x_{n+3} + y_{n+2}$
- $x_{n+1} = 337x_{n+3} - 2184y_{n+3}$

- $x_{n+2} = 13x_{n+3} - 84y_{n+3}$
- $y_{n+1} = 337y_{n+3} - 52x_{n+3}$
- $y_{n+2} = 13y_{n+3} - 2x_{n+3}$

  $2x_{n+1} = y_{n+2} - 13y_{n+1}$
- $2x_{n+2} = 13y_{n+2} - y_{n+1}$
- $2x_{n+3} = 337y_{n+2} - 13y_{n+1}$
- $y_{n+3} = 26y_{n+2} - y_{n+1}$
- $52x_{n+1} = y_{n+3} - 337y_{n+1}$
- $4x_{n+2} = y_{n+3} - y_{n+1}$
- $52x_{n+3} = 337y_{n+3} - y_{n+1}$
- $26y_{n+2} = y_{n+3} + y_{n+1}$
- $2x_{n+1} = 13y_{n+3} - 337y_{n+2}$
- $2x_{n+2} = y_{n+3} - 13y_{n+2}$
- $2x_{n+3} = 13y_{n+3} - y_{n+2}$
- $y_{n+1} = 26y_{n+2} - y_{n+3}$

2. Each of the following expressions represents a Cubical Integer

- $\frac{1}{14}\left[27x_{3n+3} - x_{3n+4} + 81x_{n+1} - 3x_{n+2}\right]$
- $\frac{1}{364}\left[701x_{3n+3} - x_{3n+5} + 2103x_{n+1} - 3x_{n+3}\right]$
- $x_{3n+3} - 6y_{3n+3} + 3x_{n+1} - 18y_{n+1}$
- $\frac{1}{13}\left[25x_{3n+3} - 6y_{3n+4} + 75x_{n+1} - 18y_{n+2}\right]$
- $\frac{1}{337}\left[649x_{3n+3} - 6y_{3n+5} + 1947x_{n+1} - 18y_{n+3}\right]$
- $\frac{1}{14}\left[701x_{3n+4} - 27x_{3n+5} + 2108x_{n+2} - 81x_{n+3}\right]$
- $\frac{1}{13}\left[x_{3n+4} - 162y_{3n+3} + 3x_{n+2} - 486y_{n+1}\right]$

- $25x_{3n+4}-162y_{3n+4}+75x_{n+2}-486y_{n+2}$
- $\frac{1}{13}\left[649x_{3n+4}-162y_{3n+5}+1947x_{n+2}-486y_{n+3}\right]$
- $\frac{1}{337}\left[x_{3n+5}-4206y_{3n+3}+3x_{n+3}-12618y_{n+1}\right]$
- $\frac{1}{13}\left[25x_{3n+5}-4206y_{3n+4}+75x_{n+3}-12618y_{n+2}\right]$
- $649x_{3n+5}-4206y_{3n+5}+1947x_{n+3}-12618y_{n+3}$
- $\frac{1}{2}\left[y_{3n+4}-25y_{3n+3}+3y_{n+2}-75y_{n+1}\right]$
- $\frac{1}{52}\left[y_{3n+5}-649y_{3n+3}+3y_{n+3}-1947y_{n+1}\right]$
- $\frac{1}{2}\left[25y_{3n+5}-649y_{3n+4}+75y_{n+3}-1947y_{n+2}\right]$

3. Each of the following expression represents a Bi-Quadratic Integer

- $\frac{1}{14}\left[27x_{4n+4}-x_{4n+5}+108x_{2n+2}-4x_{2n+3}+84\right]$
- $\frac{1}{364}\left[701x_{4n+4}-x_{4n+6}+2804x_{2n+2}-4x_{2n+4}+2184\right]$
- $x_{4n+4}-6y_{4n+4}+4x_{2n+2}-24y_{2n+2}+6$
- $\frac{1}{13}\left[25y_{4n+4}-6y_{4n+5}+100x_{2n+2}-24y_{2n+3}+78\right]$
- $\frac{1}{337}\left[649x_{4n+4}-6y_{4n+6}+2596x_{2n+2}-24y_{2n+4}+2022\right]$
- $\frac{1}{14}\left[701x_{4n+5}-27x_{4n+6}+2804x_{2n+3}-108x_{2n+4}+84\right]$

- $\frac{1}{13}\left[x_{4n+5}-162y_{4n+4}+4x_{2n+3}-648y_{2n+2}+78\right]$
- $25x_{4n+5}-162y_{4n+5}+100x_{2n+3}-648y_{2n+3}+6$
- $\frac{1}{13}\left[649x_{4n+5}-162y_{4n+6}+2596x_{2n+3}-648y_{2n+4}+78\right]$
- $\frac{1}{337}\left[x_{4n+6}-4206y_{4n+4}+4x_{2n+4}-16824y_{2n+2}+2022\right]$
- $\frac{1}{13}\left[25x_{4n+6}-4206y_{4n+5}+100x_{2n+4}-16824y_{2n+3}+78\right]$
- $649x_{4n+6}-4206y_{4n+6}+2596x_{2n+4}-16824y_{2n+4}+6$
- $\frac{1}{2}\left[y_{4n+5}-25y_{4n+4}+4y_{2n+3}-100y_{2n+2}+12\right]$
- $\frac{1}{52}\left[y_{4n+6}-649y_{4n+4}+4y_{2n+4}-2596y_{2n+2}+312\right]$
- $\frac{1}{2}\left[25y_{4n+6}-649y_{4n+5}+100y_{2n+4}-2596y_{2n+3}+12\right]$

4. Each of the following expressions represents a Quintic Integer

- $\frac{1}{14}\left[27x_{5n+5}-x_{5n+6}+135x_{3n+3}-5x_{3n+4}+270x_{n+1}-20x_{n+2}\right]$
- $\frac{1}{364}\left[701x_{5n+5}-x_{5n+7}+3505x_{3n+3}-5x_{3n+5}+102010x_{n+1}-10x_{n+3}\right]$
- $x_{5n+5}-6y_{5n+5}+5x_{3n+3}-30y_{3n+3}+10x_{n+1}-84y_{n+1}$

- $\frac{1}{13}\left[25x_{5n+5}-6y_{5n+6}+125x_{3n+3}-30y_{3n+4}+250x_{n+1}-60y_{n+2}\right]$
- $\frac{1}{337}\left[649x_{5n+5}-6y_{5n+7}+3245x_{3n+3}-30y_{3n+5}+6490x_{n+1}-60y_{n+3}\right]$
- $\frac{1}{14}\left[701x_{5n+6}-27x_{5n+7}+3505x_{3n+4}-135x_{3n+5}+7010x_{n+2}-270x_{n+3}\right]$
- $\frac{1}{13}\left[x_{5n+6}-162y_{5n+5}+5x_{3n+4}-810y_{3n+3}+10x_{n+2}-1620y_{n+1}\right]$
- $25x_{5n+6}-162y_{5n+6}+125x_{3n+4}-810y_{3n+4}+270x_{n+2}-1620y_{n+2}$
- $\frac{1}{13}\left[649x_{5n+6}-162y_{5n+7}+3245x_{3n+4}-810y_{3n+5}+6490x_{n+2}-1620y_{n+3}\right]$
- $\frac{1}{337}\left[x_{5n+7}-4206y_{5n+5}+5x_{3n+5}-21030y_{3n+3}+10x_{n+3}-42060y_{n+1}\right]$
- $\frac{1}{13}\left[25x_{5n+7}-4206y_{5n+6}+125x_{3n+5}-21030y_{3n+4}+20x_{n+3}-42060y_{n+2}\right]$
- $649x_{5n+7}-4206y_{5n+7}+3245x_{3n+5}-21030y_{3n+5}+6490x_{n+3}-42060y_{n+3}$
- $\frac{1}{2}\left[y_{5n+6}-25y_{5n+5}+5y_{3n+4}-125y_{3n+3}+10y_{n+2}-250y_{n+1}\right]$
- $\frac{1}{52}\left[y_{5n+7}-649y_{5n+5}+5y_{3n+5}-3245y_{3n+3}+10y_{n+3}-6490y_{n+1}\right]$
- $\frac{1}{2}\left[25y_{5n+7}-649y_{5n+6}+125y_{3n+5}-3245y_{3n+4}+250y_{n+3}-6490y_{n+2}\right]$

Remarkable Observations:

1. Employing linear combinations among the solutions of (1),one may generate integer solutions for other choices of hyperbola which are presented in the Table: 2 below

Table: 2 Hyperbola

S.No	Hyperbola	(P,Q)
1	$6P^2-7Q^2=4704$	$\left(27x_{n+1}-x_{n+2}\ ,\ x_{n+2}-25x_{n+1}\right)$
2	$6P^2-7Q^2=3179904$	$(701x_{n+1}-x_{n+3}\ ,\ x_{n+3}-649x_{n+1})$
3	$42P^2-Q^2=168$	$(x_{n+1}-6y_{n+1}\ ,\ 42y_{n+1}-6x_{n+1})$
4	$42P^2-Q^2=28392$	$(25x_{n+1}-6y_{n+2}\ ,\ 42y_{n+2}-162x_{n+1})$
5	$42P^2-Q^2=19079592$	$(649x_{n+1}-6y_{n+3}\ ,\ 42y_{n+3}-4206x_{n+1})$
6	$42P^2-Q^2=32928$	$(701x_{n+2}-27x_{n+3}\ ,175x_{n+3}-4543x_{n+2})$
7	$42P^2-Q^2=28392$	$(x_{n+2}-162y_{n+1}\ ,1050y_{n+1}-6x_{n+2})$
8	$42P^2-Q^2=168$	$(25x_{n+2}-162y_{n+2}\ ,1050y_{n+2}-162x_{n+2})$
9	$42P^2-Q^2=28392$	$(649x_{n+2}-162y_{n+3}\ ,1050y_{n+3}-4206x_{n+2})$
10	$42P^2-Q^2=19079592$	$(x_{n+3}-4206y_{n+1}\ ,\ 27258y_{n+1}-6x_{n+3})$
11	$42P^2-Q^2=28392$	$\left(25x_{n+3}-4206y_{n+2}\ ,\ 27258y_{n+2}-162x_{n+3}\right)$
12	$42P^2-Q^2=168$	$(649x_{n+3}-4206y_{n+3}\ ,\ 27258y_{n+3}-4206x_{n+3})$
13	$42P^2-Q^2=24192$	$(6y_{n+2}-150y_{n+1}\ ,\ 972y_{n+1}-36y_{n+2})$
14	$42P^2-Q^2=16353792$	$(6y_{n+3}-3894y_{n+1}\ ,\ 25236y_{n+1}-36y_{n+3})$
15	$42P^2-Q^2=24192$	$(150y_{n+3}-3894y_{n+2}\ ,\ 25236y_{n+2}-972y_{n+3})$

2. Employing linear combinations among the solutions of (1),one may generate integer solutions for other choices of parabola which are presented in the Table: 3

Table: 3 Parabola

S. No	Parabola	(R,Q)
1	$12R-Q^2=672$	$\left(27x_{2n+2}-x_{2n+3}+28\ ,\ x_{n+2}-25x_{n+1}\right)$
2	$312R-Q^2=454272$	$(701x_{2n+2}-x_{2n+4}+728\ ,\ x_{n+3}-649x_{n+1})$

3	$42R-Q^2=168$	$(x_{2n+2}-6y_{2n+2}+2\,,\,42y_{n+1}-6x_{n+1})$
4	$546R-Q^2=28392$	$(25x_{2n+2}-6y_{2n+3}+26\,,\,42y_{n+2}-162x_{n+1})$
5	$14154R-Q^2=19079592$	$(649x_{2n+2}-6y_{2n+4}+674\,,\,42y_{n+3}-4206x_{n+1})$
6	$588R-Q^2=32928$	$(701x_{2n+3}-27x_{2n+4}+28\,,\,175x_{n+3}-4543x_{n+2})$
7	$546R-Q^2=28392$	$(x_{2n+3}-162y_{2n+2}+26\,,\,1050y_{n+1}-6x_{n+2})$
8	$42R-Q^2=168$	$(25x_{2n+3}-162y_{2n+3}+2\,,\,1050y_{n+2}-162x_{n+2})$
9	$546R-Q^2=28392$	$(649x_{2n+3}-162y_{2n+4}+26\,,\,1050y_{n+3}-4206x_{n+2})$
10	$14154R-Q^2=19079592$	$(x_{2n+4}-4206y_{2n+2}+674\,,\,27258y_{n+1}-6x_{n+3})$
11	$546R-Q^2=28392$	$(25x_{2n+4}-4206y_{2n+3}+26\,,\,27258y_{n+2}-162x_{n+3})$
12	$42R-Q^2=168$	$(649x_{2n+4}-4206y_{2n+4}+2\,,\,27258y_{n+3}-4206x_{n+3})$
13	$504R-Q^2=24192$	$(6y_{2n+3}-150y_{2n+2}+24\,,\,972y_{n+1}-36y_{n+2})$
14	$13104R-Q^2=16353792$	$(6y_{2n+4}-3894y_{2n+2}+624\,,\,25236y_{n+1}-36y_{n+3})$
15	$504R-Q^2=24192$	$(150y_{2n+4}-3894y_{2n+3}+24\,,\,25236y_{n+2}-972y_{n+3})$

3. From the suitable values of $x_{n+1}\,,\,y_{n+1}$, one may generate second order Ramanujan numbers with base numbers as real integers as well as gaussian integers.

Illustration 1

Consider

$$x_0=14=1*14=2*7 \tag{9}$$

$$\Rightarrow(14+1)^2+(7-2)^2=(14-1)^2+(7+2)^2$$

$$\Rightarrow 15^2+\ 5^2=13^2+\ 9^2=250$$

Thus, 250 is second order Ramanujan numbers with base numbers as real integers.

Further, from (9), one may write

$$(14+i)^2+(7-2i)^2=(14-i)^2+(7+2i)^2=240$$

Here, 240 is second order Ramanujan numbers with base numbers as Gaussian integers.

Illustration 2

Consider

$$x_1 = 350 = 1*350 = 2*175 = 5*70 = 7*50 = 10*35 = 14*25 \tag{10}$$

Now,

$$(i) \quad 1*350 = 2*175$$

$$\Rightarrow (350+1)^2 + (175-2)^2 = (350-1)^2 + (175+2)^2$$

$$\Rightarrow 351^2 + 173^2 = 349^2 + 177^2 = 153130$$

Thus, 15130 is second order Ramanujan numbers with base numbers as real integers.

Further, from (10), one may write

$$(350+i)^2 + (175-2i)^2 = (350-i)^2 + (175+2i)^2 = 153120$$

Here, 153120 is second order Ramanujan numbers with base numbers as Gaussian integers.

Now,

$$(ii) \quad 1*350 = 5*70$$

$$\Rightarrow (350+1)^2 + (70-5)^2 = (350-1)^2 + (70+5)^2$$

$$\Rightarrow 351^2 + 65^2 = 349^2 + 75^2 = 127426$$

Thus, 127426 is second order Ramanujan numbers with base numbers as real integers.

Further, from (10), one may write

$$(350+i)^2 + (70-5i)^2 = (350-i)^2 + (70+5i)^2 = 127374$$

Here, 127374 is second order Ramanujan numbers with base numbers as Gaussian integers.

Now,

$$(iii) \quad 1*350 = 7*50$$

$$\Rightarrow (350+1)^2 + (50-7)^2 = (350-1)^2 + (50+7)^2$$

$$\Rightarrow 351^2 + 43^2 = 349^2 + 57^2 = 125050$$

Thus, 125050 is second order Ramanujan numbers with base numbers as real integers.

Further, from (10), one may write

$$(350+i)^2 + (50-7i)^2 = (350-i)^2 + (50+7i)^2 = 124950$$

Here, 124950 is second order Ramanujan numbers with base numbers as Gaussian integers.

Now,

$$(iv) \quad 1*350 = 10*35$$

$$\Rightarrow (350+1)^2 + (35-10)^2 = (350-1)^2 + (35+10)^2$$

$$\Rightarrow 351^2 + 25^2 = 349^2 + 45^2 = 123826$$

Thus, 123826 is second order Ramanujan numbers with base numbers as real integers.

Further, from (10), one may write

$$(350+i)^2+(35-10i)^2=(350-i)^2+(35+10i)^2=123624$$

Here, 123624 is second order Ramanujan numbers with base numbers as Gaussian integers.

Now,

$$(v)\quad 1*350=14*25$$

$$\Rightarrow (350+1)^2+(25-14)^2=(350-1)^2+(25+14)^2$$

$$\Rightarrow 351^2+\ 11^2=349^2+\ 39^2=123322$$

Thus, 123322 is second order Ramanujan numbers with base numbers as real integers.

Further, from (10), one may write

$$(350+i)^2+(25-14i)^2=(350-i)^2+(25+14i)^2=122928$$

Here, 122928 is second order Ramanujan numbers with base numbers as Gaussianintegers.

Now,

$$(vi)\quad 2*175=5*70$$

$$\Rightarrow (175+2)^2+(70-5)^2=(175-2)^2+(70+5)^2$$

$$\Rightarrow 177^2+\ 65^2=173^2+\ 75^2=35554$$

Thus, 35554 is second order Ramanujan numbers with base numbers as real integers.

Further, from (10), one may write

$$(175+2i)^2+(70-5i)^2=(175-2i)^2+(70+5i)^2=35496$$

Here, 35496 is second order Ramanujan numbers with base numbers as Gaussian integers.

Now,

$$(vii)\quad 2*175=7*50$$

$$\Rightarrow (175+2)^2+(7-50)^2=(175-2)^2+(7+50)^2$$

$$\Rightarrow 177^2+\ 43^2=173^2+\ 57^2=33178$$

Thus, 33178 is second order Ramanujan numbers with base numbers as real integers.

Further, from (10), one may write

$$(175+2i)^2+(50-7i)^2=(175-2i)^2+(50+7i)^2=33072$$

Here, 33072 is second order Ramanujan numbers with base numbers as Gaussian integers.

Now,

$$(viii)\quad 2*175=10*35$$

$$\Rightarrow (175+2)^2+(35-10)^2=(175-2)^2+(35+10)^2$$

$$\Rightarrow 177^2+\ 25^2=173^2+\ 45^2=31954$$

Thus, 31954 is second order Ramanujan numbers with base numbers as real integers.

Further, from (10), one may write

$$(175+2i)^2+(35-10i)^2=(175-2i)^2+(35+10i)^2=31746$$

Here, 31746 is second order Ramanujan numbers with base numbers as Gaussian integers.

Now,

$$(ix) \quad 2*175 = 14*25$$

$$\Rightarrow (175+2)^2 + (25-14)^2 = (175-2)^2 + (25+14)^2$$

$$\Rightarrow 177^2 + 11^2 = 173^2 + 39^2 = 31450$$

Thus, 31450 is second order Ramanujan numbers with base numbers as real integers.

Further, from (10), one may write

$$(175+2i)^2 + (25-14i)^2 = (175-2i)^2 + (25+14i)^2 = 29800$$

Here, 29800 is second order Ramanujan numbers with base numbers as Gaussian integers.

Now,

$$(x) \quad 5*70 = 7*50$$

$$\Rightarrow (70+5)^2 + (50-7)^2 = (70-5)^2 + (50+7)^2$$

$$\Rightarrow 75^2 + 43^2 = 65^2 + 57^2 = 7474$$

Thus, 7474 is second order Ramanujan numbers with base numbers as real integers.

Further, from (10), one may write

$$(70+5i)^2 + (50-7i)^2 = (70-5i)^2 + (50+7i)^2 = 7326$$

Here, 7326 is second order Ramanujan numbers with base numbers as Gaussian integers.

Now,

$$(xi) \quad 5*70 = 10*35$$

$$\Rightarrow (70+5)^2 + (35-10)^2 = (70-5)^2 + (35+10)^2$$

$$\Rightarrow 75^2 + 25^2 = 65^2 + 45^2 = 6250$$

Thus, 6250 is second order Ramanujan numbers with base numbers as real integers.

Further, from (10), one may write

$$(70+5i)^2 + (35-10i)^2 = (70-5i)^2 + (35+10i)^2 = 6000$$

Here, 6000 is second order Ramanujan numbers with base numbers as Gaussian integers.

Now,

$$(xii) \quad 5*70 = 14*25$$

$$\Rightarrow (70+5)^2 + (25-14)^2 = (70-5)^2 + (25+14)^2$$

$$\Rightarrow 75^2 + 11^2 = 65^2 + 39^2 = 5746$$

Thus, 5746 is second order Ramanujan numbers with base numbers as real integers.

Further, from (10), one may write

$$(70+5i)^2 + (25-14i)^2 = (70-5i)^2 + (25+14i)^2 = 5304$$

Here, 5304 is second order Ramanujan numbers with base numbers as Gaussian integers.

Now,

$$(xiii) \quad 7*50 = 10*35$$

$$\Rightarrow (50+7)^2 + (35-10)^2 = (50-7)^2 + (35+10)^2$$

$$\Rightarrow 57^2 + 25^2 = 43^2 + 45^2 = 3874$$

Thus, 3874 is second order Ramanujan numbers with base numbers as real integers.

Further, from (10), one may write

$$(50+7i)^2 + (35-10i)^2 = (50-7i)^2 + (35+10i)^2 = 3576$$

Here, 3576 is second order Ramanujan numbers with base numbers as Gaussian integers.

Now,

$$(xiv) \quad 7*50 = 14*25$$

$$\Rightarrow (50+7)^2 + (25-14)^2 = (50-7)^2 + (25+14)^2$$

$$\Rightarrow 57^2 + 11^2 = 43^2 + 39^2 = 3370$$

Thus, 3370 is second order Ramanujan numbers with base numbers as real integers.

Further, from (10), one may write

$$(50+7i)^2 + (25-14i)^2 = (50-7i)^2 + (25+14i)^2 = 2880$$

Here, 2880 is second order Ramanujan numbers with base numbers as Gaussian integers.

Now,

$$(xv) \quad 10*35 = 14*25$$

$$\Rightarrow (35+10)^2 + (25-14)^2 = (35-10)^2 + (25+14)^2$$

$$\Rightarrow 45^2 + 11^2 = 25^2 + 39^2 = 2146$$

Thus, 2146 is second order Ramanujan numbers with base numbers as real integers.

Further, from (10), one may write

$$(35+10i)^2 + (25-14i)^2 = (35-10i)^2 + (25+14i)^2 = 1554$$

Here, 1554 is second order Ramanujan numbers with base numbers as Gaussian integers.

Illustration 3

Consider

$$y_1 = 54 = 1*54 = 2*27 = 3*18 = 6*9 \tag{11}$$

Now,

$$(i) \quad 1*54 = 2*27$$

$$\Rightarrow (54+1)^2 + (27-2)^2 = (54-1)^2 + (27+2)^2$$

$$\Rightarrow 55^2 + 25^2 = 53^2 + 29^2 = 3650$$

Thus, 3650 is second order Ramanujan numbers with base numbers as real integers.

Further, from (11), one may write

$$(54+i)^2 + (27-2i)^2 = (54-i)^2 + (27+2i)^2 = 3640$$

Here, 3640 is second order Ramanujan numbers with base numbers as Gaussian integers.

Now,

$(ii)\quad 1*54 = 3*18$

$$\Rightarrow (54+1)^2 + (18-3)^2 = (54-1)^2 + (18+3)^2$$
$$\Rightarrow 55^2 + 15^2 = 53^2 + 21^2 = 3250$$

Thus, 3250 is second order Ramanujan numbers with base numbers as real integers.

Further, from (11), one may write

$$(54+i)^2 + (18-3i)^2 = (54-i)^2 + (18+3i)^2 = 3230$$

Here, 3230 is second order Ramanujan numbers with base numbers as Gaussian integers.

Now,

$(iii)\quad 1*54 = 6*9$

$$\Rightarrow (54+1)^2 + (9-6)^2 = (54-1)^2 + (9+6)^2$$
$$\Rightarrow 55^2 + 3^2 = 53^2 + 15^2 = 3034$$

Thus, 3034 is second order Ramanujan numbers with base numbers as real integers.

Further, from (11), one may write

$$(54+i)^2 + (9-6i)^2 = (54-i)^2 + (9+6i)^2 = 2960$$

Here, 2960 is second order Ramanujan numbers with base numbers as Gaussian integers.

Now,

$(iv)\quad 2*27 = 3*18$

$$\Rightarrow (27+2)^2 + (18-3)^2 = (27-2)^2 + (18+3)^2$$
$$\Rightarrow 29^2 + 15^2 = 25^2 + 21^2 = 1066$$

Thus, 1066 is second order Ramanujan numbers with base numbers as real integers.

Further, from (11), one may write

$$(27+2i)^2 + (18-3i)^2 = (27-2i)^2 + (18+3i)^2 = 1040$$

Here, 1040 is second order Ramanujan numbers with base numbers as Gaussian integers.

Now,

$(v)\quad 2*27 = 6*9$

$$\Rightarrow (27+2)^2 + (9-6)^2 = (27-2)^2 + (9+6)^2$$
$$\Rightarrow 29^2 + 3^2 = 25^2 + 15^2 = 850$$

Thus, 850 is second order Ramanujan numbers with base numbers as real integers.

Further, from (11), one may write

$$(27+2i)^2 + (9-6i)^2 = (27-2i)^2 + (9+6i)^2 = 770$$

Here, 770 is second order Ramanujan numbers with base numbers as Gaussian integers.

Now,

$(vi)\quad 3*18 = 6*9$

$$\Rightarrow (18+3)^2+(9-6)^2=(18-3)^2+(9+6)^2$$
$$\Rightarrow 21^2+\ 3^2=15^2+\ 15^2=450$$

Thus, 450 is second order Ramanujan numbers with base numbers as real integers.

Further, from (11), one may write

$$(18+3i)^2+(9-6i)^2=(18-3i)^2+(9+6i)^2=360$$

Here, 360 is second order Ramanujan numbers with base numbers as Gaussian integers.

Illustration 4

Consider

$$y_2=1402=1*1402=2*701 \qquad (12)$$
$$\Rightarrow (1402+1)^2+(701-2)^2=(1402-1)^2+(701+2)^2$$
$$\Rightarrow 1403^2+\ 699^2=1401^2+\ 703^2=2457010$$

Thus, 2457010 is second order Ramanujan numbers with base numbers as real integers.

Further, from (12), one may write

$$(1402+i)^2+(701-2i)^2=(1402-i)^2+(701+2i)^2=2457000$$

Here, 2457000 is second order Ramanujan numbers with base numbers as Gaussian integers.

4. Let $\{a_{n+1}\}$ and $\{b_{n+1}\}$ be two sequences of positive integers, where $a_{n+1}=\frac{x_{n+1}}{2}, b_{n+1}=\frac{y_{n+1}}{2}$

Then, the following relations are observed:

- $7t_{4,a_{n+1}}-49$ is a nasty number.
- $t_{4,a_{n+1}}=42t_{4,b_{n+1}}+7$

5. Let $\{a_{n+1}\}$ and $\{b_{n+1}\}$ be two sequences of positive integers, where $a_{n+1}=\frac{x_{n+1}+2}{2}, b_{n+1}=\frac{y_{n+1}+2}{2}$

- $t_{10,a_{n+1}}-168t_{4,b_{n+1}}-5a_{n+1}=24$
- $14t_{3,a_{n+1}-1}-7a_{n+1}-42$ is a nasty number.
- $S_{a_{n+1}}-t_{506,b_{n+1}}-6a_{n+1}+253b_{n+1}=43$

6. Let $\{a_{n+1}\}$ and $\{b_{n+1}\}$ be two sequences of positive integers, where $a_{n+1}=\frac{x_{n+1}-2}{2}, b_{n+1}=\frac{y_{n+1}-2}{2}$

- $2t_{3,a_{n+1}}-84t_{3,b_{n+1}}+a_{n+1}-42b_{n+1}=48$
- $t_{4,a_{n+1}}-42t_{4,b_{n+1}}+2a_{n+1}-84b_{n+1}=48$

Conclusion:

In this paper, we have presented infinitely many integer solutions for the hyperbola represented by the Positive Pell equation $x^2 = 42y^2 + 28$. As the binary quadratic Diophantine equations are rich in variety, one may search for the other choices of positive Pell equations and determine their integer solutions along with suitable properties.

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