WWW.IJRPR.COM

International Journal of Research Publication and Reviews

Journal homepage: www.ijrpr.com ISSN 2582-7421

Observations on the Hyperbola $x^2 = 20y^2 + 45$

J. Shanthi[1], M. Parkavi[2]

[1]Assistant Professor, Department of Mathematics, Shrimati Indira Gandhi College, Affiliated to Bharathidasan University, Trichy-620 002, Tamil Nadu, India. Email: shanthivishvaa@gmail.com

[2]PG Scholar, Department of Mathematics, Shrimati Indira Gandhi College, Affiliated to Bharathidasan University, Trichy-620 002, Tamil Nadu, India. Email: parkavimathiyalagan@gmail.com

ABSTRACT:

The binary quadratic equation represented by the positive pellian $x^2 = 20y^2 + 45$ is analysed for its distinct Integer solutions. A few interesting relation among the solutions are given. Further, employing the solutions of the above hyperbola, we have obtained solutions of other choices of hyperbolas & parabolas and a few relations among special polygonal numbers are obtained.

Keywords: Binary quadratic, non-homogeneous quadratic, Pell equation, Positive Pell equation, hyperbola, Integral solutions. 2010 Mathematics Subject Classification: 11D09

Notations:

- $t_{m,n}$: Polygonal number of rank n with side $m = n\left[1 + \frac{(n-1)(m-2)}{2}\right]$
- S_n : Star number of rank $n = 6n(n-1) + 1$

Introduction:

The binary quadratic equation of the form $y^2 = Dx^2 + 1$, where D is non-square positive integer has been studied by various mathematicians for its non-trivial integral solutions when D takes different integral values [1-2]. For an extensive review of various problems, one may refer [3-22]. In this communication, yet another interesting hyperbola given by $x^2 = 20y^2 + 45$ is considered and infinitely many integer solutions are obtained. A few interesting properties among the solutions are obtained. Further, employing the solutions of the above hyperbola, we have obtained solutions of other choices of hyperbola, parabola.

METHOD OF ANALYSIS:

The Positive Pell equation representing hyperbola under consideration is

$$x^2 = 20y^2 + 45 \qquad (1)$$

whose initial solution is

$$x_0 = 15, \quad y_0 = 3$$

To obtain the other solutions of (1), consider the Pell equation

$$x^2 = 20y^2 + 1$$

whose general solution is given by

$$\tilde{x}_n = \frac{1}{2} f_n, \quad \tilde{y}_n = \frac{1}{2\sqrt{20}} g_n$$

where

$$f_n = (9 + 2\sqrt{20})^{n+1} + (9 - 2\sqrt{20})^{n+1}$$
$$g_n = (9 + 2\sqrt{20})^{n+1} - (9 - 2\sqrt{20})^{n+1}$$

Applying Brahmagupta lemma between (x_0, y_0) & $(\tilde{x}_n, \tilde{y}_n)$ the other integer solution of (1) are given by

$$2x_{n+1} = 15 f_n + 3\sqrt{20} g_n$$
$$8y_{n+1} = 12 f_n + 3\sqrt{20} g_n$$

Replacing n by n+1, n+2 in turn in the above two equations, we have

$$2x_{n+1} = 15 f_n + 3\sqrt{20} g_n \quad (2)$$

$$2x_{n+2} = 255 f_n + 57\sqrt{20} g_n \quad (3)$$

$$2x_{n+3} = 4575 f_n + 1023\sqrt{20} g_n \quad (4)$$

$$8y_{n+1} = 12 f_n + 3\sqrt{20} g_n \quad (5)$$

$$8y_{n+2} = 228 f_n + 51\sqrt{20} g_n \quad (6)$$

$$8y_{n+3} = 4092 f_n + 915\sqrt{20} g_n \quad (7)$$

Eliminating f_n & g_n among (2)-(4) and (5)-(7), the recurrence relation for x &y are given by

$$x_{n+1} - 18x_{n+2} + x_{n+3} = 0$$

$$y_{n+1} - 18y_{n+2} + y_{n+3} = 0$$

A few numerical examples are given in the following Table: 1

Table: 1 Numerical examples

n	x_{n+1}	y_{n+1}
-1	$x_0 = 15$	$y_0 = 3$
0	$x_1 = 255$	$y_1 = 57$
1	$x_2 = 4575$	$y_2 = 1023$
2	$x_3 = 82095$	$y_3 = 18357$
3	$x_4 = 1473135$	$y_4 = 329403$

From the above table, the result observed are presented below:

- x_{n+1} & y_{n+1} are odd
- $x_{n+1} \equiv 0 (mod\ 15)$, n = -1,0,1,2,……
- $y_{n+1} \equiv 0 (mod\ 3)$, n = -1,0,1,2,…….

1. **Relations between solutions**

- $x_{n+3} = 18x_{n+2} - x_{n+1}$
- $40y_{n+1} = x_{n+2} - 9x_{n+1}$
- $40y_{n+2} = 9x_{n+2} - x_{n+1}$
- $40y_{n+3} = 161x_{n+2} - 9x_{n+1}$

- $18x_{n+2} = x_{n+1} + x_{n+3}$
- $720y_{n+1} = x_{n+3} - 161x_{n+1}$
- $80y_{n+2} = x_{n+3} - x_{n+1}$
- $720y_{n+3} = 161x_{n+3} - x_{n+1}$
- $x_{n+2} = 9x_{n+1} + 40y_{n+1}$
- $x_{n+3} = 161x_{n+1} + 720y_{n+1}$
- $y_{n+2} = 2x_{n+1} + 9y_{n+1}$
- $y_{n+3} = 36x_{n+1} + 161y_{n+1}$
- $9x_{n+2} = x_{n+1} + 40y_{n+2}$
- $x_{n+3} = x_{n+1} + 80y_{n+2}$
- $9y_{n+1} = y_{n+2} - 2x_{n+1}$
- $9y_{n+3} = 2x_{n+1} + 161y_{n+2}$
- $161x_{n+2} = 9x_{n+1} + 40y_{n+3}$
- $161x_{n+3} = x_{n+1} + 720y_{n+3}$
- $161y_{n+1} = y_{n+3} - 36x_{n+1}$
- $161y_{n+2} = 9y_{n+3} - 2x_{n+1}$
- $x_{n+1} = 18x_{n+2} - x_{n+3}$
- $40y_{n+1} = 9x_{n+3} - 161x_{n+2}$
- $40y_{n+2} = x_{n+3} - 9x_{n+2}$
- $40y_{n+3} = 9x_{n+3} - x_{n+2}$
- $9x_{n+1} = x_{n+2} - 40y_{n+1}$
- $9x_{n+3} = 161x_{n+2} + 40y_{n+1}$
- $9y_{n+2} = 2x_{n+2} + y_{n+1}$
- $y_{n+3} = 4x_{n+2} + y_{n+1}$
- $x_{n+1} = 9x_{n+2} - 40y_{n+2}$
- $x_{n+3} = 9x_{n+2} + 40y_{n+2}$
- $y_{n+1} = 9y_{n+2} - 2x_{n+2}$
- $y_{n+3} = 2x_{n+2} + 9y_{n+2}$
- $9x_{n+1} = 161x_{n+2} - 40y_{n+3}$

- $9x_{n+3} = x_{n+2} - 40y_{n+3}$
- $y_{n+1} = y_{n+3} - 4x_{n+2}$
- $9y_{n+2} = y_{n+3} - 2x_{n+2}$
- $161x_{n+1} = x_{n+3} - 720y_{n+1}$
- $161x_{n+2} = 9x_{n+3} - 40y_{n+1}$
- $161y_{n+2} = 2x_{n+3} + 9y_{n+1}$
- $161y_{n+3} = 36x_{n+3} + y_{n+1}$
- $x_{n+1} = x_{n+3} - 80y_{n+2}$
- $9x_{n+2} = x_{n+3} - 40y_{n+2}$
- $9y_{n+1} = 161y_{n+2} - 2x_{n+3}$
- $9y_{n+3} = 2x_{n+3} + y_{n+2}$
- $x_{n+1} = 161x_{n+3} - 720y_{n+3}$
- $x_{n+2} = 9x_{n+3} - 40y_{n+3}$
- $y_{n+1} = 161y_{n+3} - 36x_{n+3}$
- $y_{n+2} = 9y_{n+3} - 2x_{n+3}$
- $2x_{n+1} = y_{n+2} - 9y_{n+1}$
- $2x_{n+2} = 9y_{n+2} - y_{n+1}$
- $2x_{n+3} = 161y_{n+2} - 9y_{n+1}$
- $y_{n+3} = 18y_{n+2} - y_{n+1}$
- $36x_{n+1} = y_{n+3} - 161y_{n+1}$
- $4x_{n+2} = y_{n+3} - y_{n+1}$
- $36x_{n+3} = 161y_{n+3} - y_{n+1}$
- $18y_{n+2} = y_{n+3} + y_{n+1}$
- $2x_{n+1} = 9y_{n+3} - 161y_{n+2}$
- $2x_{n+2} = y_{n+3} - 9y_{n+2}$
- $2x_{n+3} = 9y_{n+3} - y_{n+2}$
- $y_{n+1} = 18y_{n+2} - y_{n+3}$

2. Each of the following expressions represents a cubical integer

- $\frac{1}{15}\left[19x_{3n+3} - x_{3n+4} + 57x_{n+1} - 3x_{n+2}\right]$
- $\frac{1}{270}\left[341x_{3n+3} - x_{3n+5} + 1023x_{n+1} - 3x_{n+3}\right]$
- $\frac{1}{3}\left[2x_{3n+3} - 8y_{3n+3} + 6x_{n+1} - 24y_{n+1}\right]$
- $\frac{1}{27}\left[34x_{3n+3} - 8y_{3n+4} + 102x_{n+1} - 24y_{n+2}\right]$
- $\frac{1}{483}\left[610y_{3n+3} - 8y_{3n+5} + 1830x_{n+1} - 24y_{n+3}\right]$

- $\frac{1}{30}\left[682x_{3n+4}-38x_{3n+5}+2046x_{n+2}-114x_{n+3}\right]$
- $\frac{1}{27}\left[2x_{3n+4}-152y_{n+3}+6x_{n+2}-456y_{n+1}\right]$
- $\frac{1}{9}\left[102x_{3n+4}-456y_{3n+4}+306x_{n+2}-1368y_{n+2}\right]$
- $\frac{1}{81}\left[1830x_{3n+4}-456y_{3n+5}+5490x_{n+2}-1368y_{n+3}\right]$
- $\frac{1}{483}\left[2x_{3n+5}-2728y_{3n+3}+6x_{n+3}-8184y_{n+1}\right]$
- $\frac{1}{81}\left[102x_{3n+5}-8184y_{3n+4}+306x_{n+3}-24552y_{n+2}\right]$
- $\frac{1}{9}\left[1830x_{3n+5}-8184y_{3n+5}+5490x_{n+3}-24552y_{n+3}\right]$
- $\frac{1}{3}\left[y_{3n+4}-17y_{3n+3}+3y_{n+2}-51y_{n+1}\right]$
- $\frac{1}{54}\left[y_{3n+5}-305y_{3n+3}+3y_{n+3}-915y_{n+1}\right]$
- $\frac{1}{3}\left[17y_{3n+5}-305y_{3n+4}+51y_{n+3}-915y_{n+2}\right]$

3. Each of the following expression represents a bi-quadratic integer

- $\frac{1}{15}\left[19x_{4n+4}-x_{4n+5}+76x_{2n+2}-4x_{2n+3}+90\right]$
- $\frac{1}{270}\left[341x_{4n+4}-x_{4n+6}+1364x_{2n+2}-4x_{2n+4}+1620\right]$
- $\frac{1}{3}\left[2x_{4n+4}-8y_{4n+4}+8x_{2n+2}-32y_{2n+2}+24\right]$
- $\frac{1}{27}\left[34x_{4n+4}-8x_{4n+5}+136x_{2n+2}-32y_{2n+3}+162\right]$
- $\frac{1}{483}\left[610x_{4n+4}-8x_{4n+6}+2440x_{2n+2}-32y_{2n+4}+2898\right]$
- $\frac{1}{30}\left[682x_{4n+5}-38x_{4n+6}+2728x_{2n+3}-152x_{2n+4}+180\right]$
- $\frac{1}{27}\left[2y_{4n+5}-152y_{4n+4}+8x_{2n+3}-608y_{2n+2}+162\right]$
- $\frac{1}{9}\left[102x_{4n+5}-456y_{4n+5}+408x_{2n+3}-1824y_{2n+3}+54\right]$
- $\frac{1}{81}\left[1830x_{4n+5}-456y_{4n+6}+7320x_{2n+3}-1824y_{2n+4}+486\right]$
- $\frac{1}{483}\left[2x_{4n+6}-2728y_{4n+4}+8x_{2n+4}-10912y_{2n+2}+24\right]$
- $\frac{1}{81}\left[102x_{4n+6}-8184y_{4n+5}+408x_{2n+4}-32736y_{2n+3}+24\right]$

- $\frac{1}{9}\left[1830x_{4n+6}-8184y_{4n+6}+7320x_{2n+4}-32736y_{2n+4}-36\right]$
- $\frac{1}{3}\left[y_{4n+5}-17y_{4n+4}+4y_{2n+3}-68y_{2n+2}+18\right]$
- $\frac{1}{54}\left[y_{4n+6}-305y_{4n+4}+4y_{2n+4}-1220y_{2n+2}+324\right]$
- $\frac{1}{3}\left[17y_{4n+6}-305y_{4n+5}+68y_{2n+4}-1220y_{2n+3}+18\right]$

4. Each of the following expressions represents a quintic integer

- $\frac{1}{15}\left[19x_{5n+5}-x_{5n+6}+95x_{3n+2}-5x_{3n+4}+190x_{n+1}-10x_{n+2}\right]$
- $\frac{1}{270}\left[341x_{5n+5}-x_{5n+7}+1705x_{3n+3}-5x_{3n+5}+3410x_{n+1}-10x_{n+3}\right]$
- $\frac{1}{3}\left[2x_{5n+5}-8y_{5n+5}+10x_{3n+3}-40y_{3n+3}+20x_{n+1}-80y_{n+1}\right]$
- $\frac{1}{27}\left[34x_{5n+5}-8x_{5n+6}+170x_{3n+3}-40y_{3n+4}+340x_{n+1}-80y_{n+2}\right]$
- $\frac{1}{483}\left[610x_{5n+5}-8y_{5n+7}+3050x_{3n+3}-40y_{3n+5}+6100x_{n+1}-80y_{n+3}\right]$
- $\frac{1}{90}\left[2046x_{5n+6}-114x_{5n+7}+10230x_{3n+4}-570x_{3n+5}+20460x_{n+2}-1140x_{n+3}\right]$
- $\frac{1}{27}\left[2x_{5n+6}-152y_{5n+5}+10x_{3n+4}-760y_{n+3}+20x_{n+2}-1520y_{n+1}\right]$
- $\frac{1}{9}\left[102x_{5n+6}-456y_{5n+6}+510x_{3n+4}-2280y_{3n+4}+1020x_{n+2}-4560y_{n+2}\right]$
- $\frac{1}{81}\left[1830x_{5n+6}-456y_{5n+7}+9150x_{3n+4}-2280y_{3n+5}+18300x_{n+2}-4560y_{n+3}\right]$
- $\frac{1}{483}\left[2x_{5n+7}-2728y_{5n+5}+10x_{3n+5}-13640y_{3n+3}+20x_{n+3}-27280y_{n+1}\right]$
- $\frac{1}{81}\left[102x_{5n+7}-8184y_{5n+6}+510x_{3n+5}-40920y_{3n+4}+1020x_{n+3}-81840y_{n+2}\right]$
- $\frac{1}{9}\left[1830x_{5n+7}-8184y_{5n+7}+9150x_{3n+5}-40920y_{3n+5}+18300x_{n+3}-81840y_{n+3}\right]$
- $\frac{1}{3}\left[y_{5n+6}-17y_{5n+5}+5y_{3n+4}-85y_{3n+3}+10y_{n+2}-170y_{n+1}\right]$
- $\frac{1}{54}\left[y_{5n+7}-305y_{5n+5}+5y_{3n+5}-1525y_{3n+3}+10y_{n+3}-3050y_{n+1}\right]$
- $\frac{1}{3}\left[17y_{5n+7}-305y_{5n+6}+85y_{3n+5}-1525y_{3n+4}+170y_{n+3}-3050y_{n+2}\right]$

REMARKABLE OBSERVATIONS:

1. Employing linear combinations among the solutions of (1),one may generate integer solutions for other choices of hyperbola which are presented in the Table: 2 below

Table: 2 Hyperbola

S.No	Hyperbola	(P,Q)
1	$4P^2-5Q^2=14400$	$\left(38x_{n+1}-2x_{n+2},2x_{n+2}-34x_{n+1}\right)$
2	$4P^2-5Q^2=1166400$	$(341x_{n+1}-x_{n+3},x_{n+3}-305x_{n+1})$
3	$180P^2-Q^2=6480$	$(2x_{n+1}-8y_{n+1},120y_{n+1}-24x_{n+1})$
4	$180P^2-Q^2=524880$	$(34x_{n+1}-8y_{n+2},120y_{n+2}-456x_{n+1})$
5	$180P^2-Q=167968080$	$(610x_{n+1}-8y_{n+3},120y_{n+3}-8184x_{n+1})$
6	$20P^2-Q^2=648000$	$(2046x_{n+2}-114x_{n+3},510x_{n+3}-9150x_{n+2})$
7	$20P^2-Q^2=58320$	$(2x_{n+2}-152y_{n+1},680y_{n+1}-8x_{n+2})$
8	$20P^2-Q^2=6480$	$(102x_{n+2}-456y_{n+2},2040y_{n+2}-456x_{n+2})$
9	$20P^2-Q^2=524880$	$(1830x_{n+2}-456y_{n+3},2040y_{n+3}-8184x_{n+2})$
10	$20P^2-Q^2=18663120$	$(2x_{n+3}-2728y_{n+1},12200y_{n+1}-8x_{n+1})$
11	$20P^2-Q^2=524880$	$(102x_{n+3}-8184y_{n+2},36600y_{n+2}-456x_{n+3})$
12	$20P^2-Q^2=6480$	$(1830x_{n+3}-8184y_{n+3},36600y_{n+3}-8184x_{n+3})$
13	$20P^2-Q^2=720$	$(y_{n+2}-17y_{n+1},76y_{n+1}-4y_{n+2})$
14	$20P^2-Q^2=233280$	$(y_{n+3}-305y_{n+1},1364y_{n+1}-4y_{n+3})$
15	$20P^2-Q^2=720$	$(17y_{n+3}-305y_{n+2},1364y_{n+2}-76y_{n+3})$

2. Employing linear combinations among the solutions of (1),one may generate integer solutions for other choices of parabola which are presented in the Table: 3

Table: 3 Parabola

S. No	Parabola	(R,Q)
1	$24R-Q^2=2880$	$(38x_{2n+2}-2x_{2n+3}+60,2x_{n+2}-34x_{n+1})$
2	$216R-Q^2=233280$	$(341x_{2n+2}-x_{2n+4}+540,x_{n+3}-305x_{n+1})$
3	$540R-Q^2=6480$	$(2x_{2n+2}-8y_{2n+2}+6,120y_{n+1}-24x_{n+1})$
4	$4860R-Q^2=524880$	$(34x_{2n+2}-8y_{2n+3}+54,120y_{n+2}-456x_{n+1})$
5	$86940R-Q^2=167968080$	$(610x_{2n+2}-8y_{2n+4}+966,120y_{n+3}-8184x_{n+1})$
6	$1800R-Q^2=648000$	$(2046x_{2n+3}-114x_{2n+4}+180,510x_{n+3}-9150x_{n+2})$

7	$540R-Q^2=58320$	$(2x_{2n+3}-152y_{2n+2}+54{,}680y_{n+1}-8x_{n+2})$
8	$180R-Q^2=6480$	$(102x_{2n+3}-456y_{2n+3}+18{,}2040y_{n+2}-456x_{n+2})$
9	$1620R-Q^2=524880$	$(1830x_{2n+3}-456y_{2n+4}+162{,}2040y_{n+3}-8184x_{n+2})$
10	$9660R-Q^2=18663120$	$(2x_{n+4}-2728y_{2n+2}+966{,}12200y_{n+1}-8x_{n+3})$
11	$1620R-Q^2=52480$	$(102x_{2n+4}-8184y_{2n+3}+162{,}36600y_{n+2}-456x_{n+3})$
12	$180R-Q^2=6480$	$(1830x_{2n+4}-8184y_{2n+4}+18{,}36600y_{n+3}-8184x_{n+3})$
13	$60R-Q^2=720$	$(y_{2n+3}-17y_{2n+2}+6{,}76y_{n+1}-4y_{n+2})$
14	$1080R-Q^2=233280$	$(y_{2n+4}-305y_{2n+2}+108{,}1364y_{n+1}-4y_{n+3})$
15	$60R-Q^2=720$	$(17y_{2n+4}-305y_{2n+3}+6{,}1364y_{n+2}-76y_{n+3})$

3. From the suitable values of x_{n+1}, y_{n+1}, one may generate second order Ramanujan numbers with base numbers as real integers as well as Gaussian integers.

Illustration 1

Consider

$$x_0 = 15 = 1*15 = 3*4 \qquad (8)$$

$$\Rightarrow 8^2 - 7^2 = 4^2 - 1^2$$

$$\Rightarrow 8^2 + 1^2 = 7^2 + 4^2 = 65$$

Also, from (8), we have

$$\Rightarrow (15+1)^2 + (5-3)^2 = (15-1)^2 + (5+3)^2$$

$$\Rightarrow 16^2 + 2^2 = 14^2 + 8^2 = 260$$

Thus, 65 and 260 are second order Ramanujan numbers with base numbers as real integers. Further, from (8), one may write

$$\Rightarrow (15+i)^2 + (5-3i)^2 = (15-i)^2 + (5+3i)^2 = 240$$

Here, 240 is second order Ramanujan numbers with base numbers as Gaussian integers.

Illustration 2

Consider

$$x_1 = 255 = 1*255 = 5*51 = 3*85 = 15*17 \qquad (9)$$

Now

(i) $1*255 = 5*51$

$$\Rightarrow 128^2 - 127^2 = 28^2 - 23^2$$

$$\Rightarrow 28^2 + 127^2 = 128^2 + 23^2 = 16913$$

Also, from (9), we have

$$\Rightarrow (51+5)^2 + (1-255)^2 = (51-5)^2 + (1+255)^2$$

$$\Rightarrow 56^2 + 254^2 = 46^2 + 256^2 = 67652$$

Thus, 16913 and 67652 are second order Ramanujan numbers with base numbers as real integers. Further, from (9), one may write

$$\Rightarrow (51+5i)^2 + (255-i)^2 = (51-5i)^2 + (255+i)^2 = 67600$$

Here, 67600is second order Ramanujan numbers with base numbers as Gaussian integers.

(ii) $1*255 = 3*85$

$$\Rightarrow 128^2 - 127^2 = 44^2 - 41^2$$

$$\Rightarrow 128^2 + 41^2 = 127^2 + 44^2 = 18065$$

Also, from (9), we have

$$\Rightarrow (1+255)^2 + (85-3)^2 = (1-255)^2 + (85+3)^2$$

$$\Rightarrow 256^2 + 82^2 = 254^2 + 88^2 = 72260$$

Thus, 18065 and 72260 are second order Ramanujan numbers with base numbers as real integers.

Further, from (9), one may write

$$\Rightarrow (255+i)^2 + (85-3i)^2 = (255-i)^2 + (85+3i)^2 = 72240$$

Here, 72240 is second order Ramanujan numbers with base numbers as Gaussian integers.

(iii) $1*255 = 51*17$

$$\Rightarrow 128^2 - 127^2 = 16^2 - 1^2$$

$$\Rightarrow 128^2 + 1^2 = 127^2 + 16^2 = 16385$$

Also, from (9), we have

$$\Rightarrow (1+255)^2 + (15-17)^2 = (1-255)^2 + (15+17)^2$$

$$\Rightarrow 256^2 + 2^2 = 254^2 + 32^2 = 65540$$

Thus, 16385 and 65540 are second order Ramanujan numbers with base numbers as real integers.

Further, from (9), one may write

$$\Rightarrow (255+i)^2 + (15-17i)^2 = (255-i)^2 + (15+17i)^2 = 64990$$

Here, 64990 is second order Ramanujan numbers with base numbers as Gaussian integers.

(iv) $5*51 = 3*85$

$$\Rightarrow 28^2 - 23^2 = 44^2 - 41^2$$

$$\Rightarrow 28^2 + 41^2 = 44^2 + 23^2 = 2465$$

Also, from (9), we have

$$\Rightarrow (51+5)^2 + (85-3)^2 = (51-5)^2 + (85+3)^2$$

$$\Rightarrow 56^2 + 82^2 = 46^2 + 88^2 = 9860$$

Thus, 2465 and 9860 are second order Ramanujan numbers with base numbers as real integers.

Further, from (9), one may write

$$\Rightarrow (51+5i)^2 + (85-3i)^2 = (51-5i)^2 + (85+3i)^2 = 9792$$

Here, 9792 is second order Ramanujan numbers with base numbers as Gaussian integers.

(v) $5*51=15*17$

$$\Rightarrow 28^2 - 23^2 = 16^2 - 1^2$$

$$\Rightarrow 28^2 + 1^2 = 16^2 + 23^2 = 785$$

Also, from (9), we have

$$\Rightarrow (15+17)^2 + (51-5)^2 = (15-17)^2 + (51+5)^2$$

$$\Rightarrow 32^2 + 46^2 = 2^2 + 56^2 = 3140$$

Thus, 785 and 3140 are second order Ramanujan numbers with base numbers as real integers.

Further, from (9), one may write

$$\Rightarrow (15+17i)^2 + (51-5i)^2 = (15-17i)^2 + (51+5i)^2 = 2512$$

Here, 2512 is second order Ramanujan numbers with base numbers as Gaussian integers.

(vi) $3*85=15*17$

$$\Rightarrow 44^2 - 41^2 = 16^2 - 1^2$$

$$\Rightarrow 44^2 + 1^2 = 16^2 + 41^2 = 1937$$

Also, from (9), we have

$$\Rightarrow (3+85)^2 + (15-17)^2 = (3-85)^2 + (15+17)^2$$

$$\Rightarrow 88^2 + 2^2 = 82^2 + 32^2 = 7748$$

Thus, 1937 and 7748 are second order Ramanujan numbers with base numbers as real integers.

Further, from (9), one may write

$$\Rightarrow (85+3i)^2 + (15-17i)^2 = (85-3i)^2 + (15+17i)^2 = 7152$$

Here, 7152 is second order Ramanujan numbers with base numbers as Gaussian integers.

1) Illustration 3

Consider

$$y_1 = 57 = 1*57 = 3*19 \qquad (10)$$

$$\Rightarrow 29^2 - 28^2 = 11^2 - 8^2$$

$$\Rightarrow 29^2 + 8^2 = 11^2 + 28^2 = 905$$

Also, from (10), we have

$$\Rightarrow (1+57)^2 + (3-19)^2 = (1-57)^2 + (3+19)^2$$

$$\Rightarrow 58^2 + 16^2 = 56^2 + 22^2 = 3620$$

Thus, 905 and 3620 are second order Ramanujan numbers with base numbers as real integers.

Further, from (10), one may write

$$\Rightarrow (1+57i)^2 + (3-19i)^2 = (1-57i)^2 + (3+19i)^2 = 3600$$

Here, 3600 is second order Ramanujan numbers with base numbers as Gaussian integers.

2) Illustration 4

Consider

$$x_0 = 1023 = 1*1023 = 31*33 = 11*93 = 3*341 \tag{11}$$

Now

(i) $1*1023 = 31*33$

$$\Rightarrow 512^2 - 54^2 = 32^2 - 1^2$$

$$\Rightarrow 512^2 + 1^2 = 32^2 + 511^2 = 262145$$

Also, from (11), we have

$$\Rightarrow (1+1023)^2 + (33-31)^2 = (1-1023)^2 + (33+31)^2$$

$$\Rightarrow 1024^2 + 2^2 = 1022^2 + 64^2 = 1048580$$

Thus, 262145 and 1048580 are second order Ramanujan numbers with base numbers as real integers. Further, from (11), one may write

$$\Rightarrow (1+1023i)^2 + (33-31i)^2 = (1-1023i)^2 + (33+31i)^2 = 1046400$$

Here, 1046400 is second order Ramanujan numbers with base numbers as Gaussian integers.

(ii) $1*1023 = 11*93$

$$\Rightarrow 512^2 - 511^2 = 52^2 - 41^2$$

$$\Rightarrow 512^2 + 41^2 = 511^2 + 52^2 = 263825$$

Also, from (11), we have

$$\Rightarrow (1+1023)^2 + (93-11)^2 = (1-1023)^2 + (93+11)^2$$

$$\Rightarrow 1024^2 + 82^2 = 1022^2 + 104^2 = 1055300$$

Thus, 263825 and 1055300 are second order Ramanujan numbers with base numbers as real integers.

Further, from (11), one may write

$$\Rightarrow (1+1023i)^2 + (93-11i)^2 = (1-1023i)^2 + (93+11i)^2 = 1038000$$

Here, 1038000 is second order Ramanujan numbers with base numbers as Gaussian integers.

(iii) $1*1023 = 3*341$

$$\Rightarrow 512^2 - 511^2 = 172^2 - 169^2$$

$$\Rightarrow 512^2 + 169^2 = 511^2 + 172^2 = 290705$$

Also, from (11), we have

$$\Rightarrow (1+1023)^2 + (341-3)^2 = (1-1023)^2 + (341+3)^2$$

$$\Rightarrow 1024^2 + 338^2 = 1022^2 + 344^2 = 1162820$$

Thus, 290705 and 1162820 are second order Ramanujan numbers with base numbers as real integers.

Further, from (11), one may write

$$\Rightarrow (1+1023i)^2 + (341-3i)^2 = (1-1023i)^2 + (341+3i)^2 = 930256$$

Here, 930256 is second order Ramanujan numbers with base numbers as Gaussian integers.

(iv) $3*341 = 11*93$

$$\Rightarrow 172^2 - 169^2 = 52^2 - 41^2$$

$$\Rightarrow 172^2 + 41^2 = 52^2 + 169^2 = 31265$$

Also, from (11), we have

$$\Rightarrow (341+3)^2 + (93-11)^2 = (341-3)^2 + (93+11)^2$$

$$\Rightarrow 344^2 + 82^2 = 338^2 + 104^2 = 125060$$

Thus, 31265 and 125060 are second order Ramanujan numbers with base numbers as real integers.

Further, from (11), one may write

$$\Rightarrow (341+3i)^2 + (93-11i)^2 = (341-3i)^2 + (93+11i)^2 = 124800$$

Here, 124800 is second order Ramanujan numbers with base numbers as Gaussian integers.

(v) $3*341 = 31*33$

$$\Rightarrow 172^2 - 169^2 = 32^2 - 1^2$$

$$\Rightarrow 172^2 + 1^2 = 32^2 + 169^2 = 29585$$

Also, from (11), we have

$$\Rightarrow (341+3)^2 + (33-31)^2 = (341-3)^2 + (33+31)^2$$

$$\Rightarrow 344^2 + 2^2 = 338^2 + 64^2 = 118340$$

Thus, 29585 and 118340 are second order Ramanujan numbers with base numbers as real integers.

Further, from (11), one may write

$$\Rightarrow (341+3i)^2 + (33-31i)^2 = (341-3i)^2 + (33+31i)^2 = 116400$$

Here, 116400 is second order Ramanujan numbers with base numbers as Gaussian integers.

(vi) $9*11 = 33*31$

$$\Rightarrow 52^2 - 41^2 = 32^2 - 1^2$$

$$\Rightarrow 52^2 + 1^2 = 32^2 + 41^2 = 2705$$

Also, from (11), we have

$$\Rightarrow (93+11)^2 + (33-31)^2 = (93-11)^2 + (33+31)^2$$

$$\Rightarrow 104^2 + 2^2 = 82^2 + 64^2 = 10820$$

Thus, 2705 and 10820 are second order Ramanujan numbers with base numbers as real integers.

Further, from (11), one may write

$$\Rightarrow (93+11i)^2 + (33-31i)^2 = (93-11i)^2 + (33+31i)^2 = 8656$$

Here, 8656 is second order Ramanujan numbers with base numbers as Gaussian integers.

4. Let $\{a_{n+1}\}$ and $\{b_{n+1}\}$ be two sequences of positive integers, where $a_{n+1} = \frac{x_{n+1}+1}{2}, b_{n+1} = \frac{y_{n+1}+1}{2}$

Then, the following relations are observed:

1. $t_{3,a_{n+1}-1} = 20t_{3,b_{n+1}-1} + 8$
2. $160t_{3,b_{n+1}-1} - t_{8,a_{n+1}} + 65$ is a square integer.
3. $S_{a_{n+1}} - 20S_{b_{n+1}} = 77$

5. Let $\{a_{n+1}\}$ and $\{b_{n+1}\}$ be two sequences of positive integers, where $a_{n+1} = \frac{x_{n+1}-3}{2}, b_{n+1} = \frac{y_{n+1}+3}{2}$

Then, the following relation is observed:

- $t_{3,a_{n+1}} + a_{n+1} = 20t_{3,b_{n+1}-1} + 20b_{n+1} + 27$

CONCLUSION:

In this paper, we have presented infinitely many integer solutions for the hyperbola represented by the positive Pell equation $x^2 = 20y^2 + 45$. As the binary quadratic Diophantine equations are rich in variety, one may search for the other choices of positive Pell equations and determine their Integer solutions along with suitable properties.

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