Buckling Analysis of Uniformly Supported-Edged Thin Plates Using Euler-Bernoulli Approach with Trigonometric Shape Functions

Ibearugbulem O. M¹, Anyaogu L², Okorie P. O.³, Uzoukwu C. S.⁴

¹,²,³,⁴Civil Engineering Department, Federal University of Technology Owerri, Imo State, Nigeria
DOI: https://doi.org/10.55248/gengpi.234.5.37107

ABSTRACT

The research presents exact buckling analysis of thin rectangular flat plate using strong (Euler) form of plate equilibrium equation. In this work, the total potential energy equation of a plate was derived from first principles. The derived equation (functional) was differentiated with respect to deflection to obtain the strong form of the equilibrium equation. The strong equilibrium equation was integrated to obtain its exact general solution with unknown coefficients of deflection. The boundary conditions (simple support designated with S and clamp support designated with C) of plates were satisfied in the exact general solution to obtain particular solutions that are a product of unknown coefficient and exact shape functions. The plates include SSSS (all edges simply supported) and CCCC (all edges clamped). The exact shape functions were substituted into the strong form of equilibrium to obtain the exact stiffness coefficients of plates of various boundary conditions. With the exact shape functions and their corresponding exact stiffness coefficients, exact buckling loads were determined for plates of various aspect ratios. After this, stiffness coefficients and buckling loads were computed using the Ritz approach. The exactness of results obtained for both Euler approach and Ritz approach were tested by directly substituting the results into the Plate strong equilibrium equation. It is observed herein that results from Ritz Approach are not exact and the average percentage differences in computed buckling loads (between Euler and Ritz methods) recorded for SSSS and CCCC are 0.00%, 3.18%, 2.50%, 4.97%, 10.34% and 6.07% respectively. These percentage differences agree with the safety factors of 1.2 to 1.5 applied to the results obtained from approximate methods to augment for their

Symbols: S -- simple support C -- clamped support

a -- Length of the primary dimension of the plate

b -- Width of the secondary dimension of the plate

t -- Tertiary dimension (thickness) of the plate

D -- the flexural Rigidity

Π= Total Potential energy

Keywords: Buckling, Euler-Bernolli, Energy, Differential Equations

Introduction

The uniformly supported edged plate under consideration are SSSS and CCCC. In both cases, the edge conditions are the same. The edge is either SS or CC support, both in the x and y-axis. In each case, the shape functions were derived. The integral value of their shape functions gave their corresponding stiffnesses coefficient. From the first principle , the Total Potential Energy were derived as detailed below

From first principle the Total Potential energy was derived as

$$\Pi = U + w_{kb}$$

Considering the U as the Strain and the $w_{kb}$ as the work performed. Mathematically expressed as

$$U = \frac{D}{2} \int_{0}^{a} \int_{0}^{b} \left( \frac{\partial^{2}w}{\partial x^{2}} \right)^{2} + 2 \left( \frac{\partial^{2}w}{\partial x \partial y} \right)^{2} + \left( \frac{\partial^{2}w}{\partial y^{2}} \right)^{2} \partial x \partial y$$

and

$$w_{kb} = \frac{1}{2} \int_{0}^{a} \int_{0}^{b} \left( N_{x} \left( \frac{\partial w}{\partial x} \right)^{2} + 2 N_{xy} \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} + N_{y} \left( \frac{\partial w}{\partial y} \right)^{2} \right) \partial x \partial y$$
Bring them together gives
\[ II = \frac{D}{2} \int_0^L \left( \frac{\partial^2 w}{\partial x^2} \right)^2 dx + 2 \left( \frac{\partial^2 w}{\partial x \partial y} \right)^2 \partial x \partial y + \frac{1}{2} \int_0^L \left( Nx \frac{\partial w}{\partial x} \right)^2 + 2 Nxy \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} + Nw \frac{\partial w}{\partial y} \right) dx \partial y \] 4

But further differentiation of the Total Potential energy with respect to the deflection give the Euer-Bernoulli formular for calculating the critical buckling load. That is

\[ Nx = \frac{\tilde{N}_x}{a^2} \] 5

where

\[ \tilde{N} = \frac{k_{xxxx} + \frac{2}{\pi^2} k_{xxyy} + \frac{1}{\pi^4} k_{yyyy}}{-k_{xx}} \] 6

### 1.2 Determination of Exact Deflection Functions for Thin Buckling Analysis of Rectangular Plate

To achieve this, some assumptions that will enable, eliminate some of the unknowns were made, since it’s very difficult to solve directly. The plate governing equation is a 4th order partial differential equation with 4 unknowns. Also the 4th order Partial differential equation was reduced into simpler differential equations for easy derivation of the solution using the exact approach. On decoupling the Governing Equation and expressing them in exponential forms gives

For x-axis
\[ w_x = c_0 + c_1 R + c_2 \cos k R + c_3 \sin k R \] 7
and differential to First, Second, Third and Fourth order gives
\[ c_1 - c_2 k \sin k R + c_2 k^2 \cos k R - c_2 k^2 \sin k R, \quad c_2 k^3 \sin k R - c_2 k^3 \cos k R, \quad c_2 k^4 \cos k R + c_2 k^4 \sin k R \]

respectively. Similarly

For y-axis,
\[ w_y = l_0 + l_1 Q + l_2 \cos m Q + l_3 \sin m Q \] 8
and differential to First, Second, Third and Fourth order gives
\[ l_1 - l_2 m \sin m Q + l_2 m \cos m Q - l_2 m^2 \sin m Q, \quad l_2 m^3 \sin m Q - l_2 m^3 \cos m Q, \quad l_2 m^4 \cos m Q + l_2 m^4 \sin m Q \]

Multiplying Equation 7 and 8 together gives
\[ w = (c_0 + c_1 R + c_2 \cos k R + c_3 \sin k R)(l_0 + l_1 Q + l_2 \cos m Q + l_3 \sin m Q) \] 10

### 1.1 Application of Deflection equation to plates of Selected Boundary conditions.

The derived equation for deflection was introduced into the plate conditions under consideration. That is the The analysis were as detailed below

Particular Deflection equation and shape function for SS (Simple support at opposite ends)

\[ N \rightarrow \text{SS} \rightarrow N \]

**Figure 1:** Cross-section of plate Under Buckling Load (N) along x-direction for SS

Since R varies between 0 and 1, for between $x = 0$, and $x = L$

The boundary conditions for the SS beam are stated mathematically as:

At $R = 0, \quad w_R = w''_R = 0$ 11
At $R = 1, \quad w_R = w''_R = 0$ 12

Where,
\[ \frac{d^2 w}{dR^2} = w'' \] 13

Therefore substituting the boundary conditions into Equation 7, gives:
\[ w_R(0) = c_0 + c_1(0) + c_2 \cos k(0) + c_3 \sin k(0) = 0 \] 14

That is:
\[ w_R(0) = c_0 + c_2 = 0 \] 15
Hence:

c₀ + c₂ = 0

From equation 13

\[ \frac{d^2 w_0}{dR^2} = w_0 = -c_2 k^2 \cos k R - c_3 k^2 \sin k R \]

Considering the boundary conditions of Equation 11 gives:

\[ \frac{d^2 w_0}{dR^2} = w_0 = -c_2 k^2 \cos(0) - c_3 k^2 \sin(0) = 0 \]

That is:

\[ w_0(0) = -c_2 k^2 = 0 \]

Hence:

\[ -c_2 k^2 = 0 \]

Again, considering the boundary conditions of Equation 3.149 for Equation 3.134 gives:

\[ w_0(1) = c_0 + c_1 + c_2 \cos k + c_3 \sin k = 0 \]

Expressing simultaneous equations 14 to 21 in matrix form will give

\[
\begin{bmatrix}
    1 & 0 & 1 & 0 & |c_0| \\
    0 & 0 & -k^2 & 0 & |c_1| \\
    1 & 1 & \cos k & \sin k & |c_2| \\
    0 & 0 & -k^2 \cos k & -k^2 \sin k & |c_3|
\end{bmatrix} = 0
\]

For Equation 22 to be true, the determinant of the square matrix must be zero. That is:

\[ 1 \times \begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & -k^2 & 0 \\ 1 & 1 & \cos k & \sin k \\ 0 & 0 & -k^2 \cos k & -k^2 \sin k \end{vmatrix} = 0 \]

That is:

\[ 0 + k^2 \times \begin{vmatrix} 1 & \sin k \\ 0 & -k^2 \sin k \end{vmatrix} + \{0\} = 0 \]

That is:

\[ k^2 \times -k^2 \sin k = 0. \]

That is:

\[ k^4 \sin k = 0 \]

For Equation 24 to be zero, then \( \sin k \) must be zero. The only condition for this to happen is when \( k \) is equal to the product of a positive integer and \( \pi \).

That is:

\[ \sin k = 0 \text{ (when } k = n\pi \text{ and } n = 0, 1, 2, 3 \text{ etc.)} \]

but

\[ \frac{d w_0}{dR} = c_1 - c_2 k \sin k R + c_3 k \cos k R \]

\[ \frac{d^2 w_0}{dR^2} = -c_2 k^2 \cos k R - c_3 k^2 \sin k R \]

\[ \frac{d^3 w_0}{dR^3} = c_2 k^3 \sin k R - c_3 k^3 \cos k R \]

\[ \frac{d^4 w_0}{dR^4} = c_2 k^4 \cos k R + c_3 k^4 \sin k R \]

Substituting equation 25 into all equations (from 1st to 4th order) and satisfying the boundary conditions from equation 12 gives:

\[ c_0 = c_1 = c_2 = 0 \]
Substituting the $c_0 + c_1 R + c_2 \cos kR + c_3 \sin kR$ for deflection $w_R$, gives:

$$w_R = c_3 \sin(n\pi R)$$

When a similar procedure is done on the y-direction. The outcome will:

$$w_Q = l_3 \sin(n\pi Q)$$

Particular Deflection equation and shape function for CC (Clamp support at opposite ends)

When a similar procedure is done on the y-direction. The outcome will;

$$w_Q = l_3 \sin(n\pi Q)$$

Particular Deflection equation and shape function for CC (Clamp support at opposite ends)

Expressing simultaneous equations 38 to 41 in matrix form will give –

$$\begin{bmatrix}
w_R(0) \\
w_R'(0) \\
w_R(1) \\
w_R'(1)
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & 0 & k \\
1 & 1 & \cos k & \sin k \\
0 & 1 & -\sin k & k\cos k
\end{bmatrix}
\begin{bmatrix}
c_0 \\
c_1 \\
c_2 \\
c_3
\end{bmatrix} = 0$$

For equation 42 to be equal to zero, then:

$$c_0 + c_2 = 0$$
\[
\begin{bmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & 0 & k \\
1 & 1 & \cos k & \sin k \\
0 & 1 & -k\sin k & k\cos k
\end{bmatrix} = 0
\]

That is:
\[
1 \times \begin{bmatrix}
1 & 0 & k \\
1 & \cos k & \sin k \\
1 & -k\sin k & k\cos k \\
0 & 1 & k\cos k
\end{bmatrix} - 0 + 1 \times \begin{bmatrix}
0 & 1 & k \\
1 & \sin k & \cos k \\
0 & 1 & k\sin k \\
1 & k\cos k
\end{bmatrix} - 0 = 0
\]

That is:
\[
1 \times \begin{bmatrix}
\cos k & \sin k \\
-k\sin k & k\cos k \\
-k\sin k & k\cos k \\
0 & 1 & k\cos k
\end{bmatrix} - 0 + 1 \times \begin{bmatrix}
1 & \cos k \\
-k\sin k & k\cos k \\
0 & 1 & k\cos k
\end{bmatrix} + 0 = 0
\]

That is:
\[
k\cos^2 k + k\sin^2 k + k(-k\sin k - \cos k) - k\cos k + k = 0
\]

Rearranging gives:
\[
\cos^2 k + \sin^2 k - k\sin k - 2\cos k + 1 = 0
\]

Equation 47 above represents the characteristics equation for the matrix. Solving for k:
\[
(C\cos^2 k + \sin^2 k) - k\sin k - 2\cos k + 1 = 0
\]

From Trigonometry:
\[
(C\cos^2 k + \sin^2 k) = 0
\]

Therefore;
\[
1 - k\sin k - 2\cos k + 1 = 0
\]
\[
k\sin k + 2\cos k - 2 = 0
\]

The value of k that satisfies equation 51 above is:
\[
k = 2\pi\text{[where} n = 1, 2, 3, \ldots]\]

Substituting the values of k into equations 32 to 41 and satisfying the boundary conditions gives:
\[
c_1 = c_3 = 0; \ c_0 = -c_2
\]

Substituting equation 52 and 52a above into the equation of deflection yields;
\[
w_x = c_0(1 - \cos 2\pi n\pi R)
\]

When a similar procedure is done on the y-direction. We obtain;
\[
w_y = l_y(1 - \cos 2\pi mQ)
\]

### Table 1  Summary of Deflection Equations of the Shape Orientations

<table>
<thead>
<tr>
<th>Combined Support Condition</th>
<th>Equation of Deflection (w)</th>
<th>Shape Function (h)</th>
</tr>
</thead>
<tbody>
<tr>
<td>SS (Pinned at Both ends)</td>
<td>(c_3 \sin n\pi R) [(n = 1, 2, 3, \ldots)]</td>
<td>(\sin n\pi R) [(n = 1, 2, 3, \ldots)]</td>
</tr>
<tr>
<td>CC (Clamped at Both ends)</td>
<td>(c_0(1 - \cos 2\pi mR)) [(n = 1, 2, 3, \ldots)]</td>
<td>(1 - \cos 2\pi mR) [(n = 1, 2, 3, \ldots)]</td>
</tr>
</tbody>
</table>

Deflection Equation of plates of Selected Boundary Equation

SSSS Rectangular Plate

![Figure 3: SSSS Rectangular Plate](image)

Considering the x direction, the appropriate deflection equation is that for SS. See Figure
\[ w_R = c_3 (\sin n\pi R) \]

by virtue of similar support conditions at both sides, the shape function in y direction is the same as that of the x direction.

Hence:

\[ w_Q = l_3 (\sin n\pi Q) \]

Therefore, \( w = A (\sin n\pi R \cdot \sin n\pi Q) \)

where \( A = c_3 \cdot l_3 \)

**Table 2: Deflection Equation for Various Plates of Different Edge Conditions**

<table>
<thead>
<tr>
<th>S/No.</th>
<th>Edge Condition</th>
<th>Deflection Equation</th>
<th>Constants</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>SSSS</td>
<td>( A (\sin n\pi R \cdot \sin n\pi Q) ) [Single Mode, ( n=1 )]</td>
<td>( A = c_3 \cdot l_3 )</td>
</tr>
<tr>
<td></td>
<td>CCCC</td>
<td>( A(1 - \cos 2n\pi R)(1 - \cos 2n\pi Q) ) [Single Mode, ( n=1 )]</td>
<td>( A = c_0 \cdot l_0 )</td>
</tr>
</tbody>
</table>

**1.4 Determination of the Euler-Bernoulli stiffness coefficients of thin rectangular plates**

The stiffness coefficients \( k_{xx}, k_{yy}, k_{xyx}, \) and \( k_{xxy} \) needed to calculate the critical buckling loads of thin rectangular plates considered in this work are determined here. In doing so, the derivatives of the shape functions were obtained and subsequently substituted into the various formulars for the stiffness coefficients. The integrations were done within the domain of effective lengths (spans) in both orthogonal axes.

The Derivatives of the Deflection Equations and Shape Functions

The Derivatives of the SSSS Shape function

\[ w = A (\sin n\pi R \cdot \sin n\pi Q) \]

\[ h = (\sin n\pi R \cdot \sin n\pi Q) \]

The first derivative of the SSSS deflection equation with respect to \( R \) is:

\[ \frac{\partial h}{\partial R} = n(\cos n\pi R) (\sin n\pi Q) \]

While the second derivative with respect to \( R \) is:

\[ \frac{\partial^2 h}{\partial R^2} = -n^2(\sin n\pi R) (\sin n\pi Q) \]

And the third derivative with respect to \( R \) is:

\[ \frac{\partial^3 h}{\partial R^3} = -n^3(\cos n\pi R) (\sin n\pi Q) \]

It follows that the fourth derivative with respect to \( R \) is:

\[ \frac{\partial^4 h}{\partial R^4} = n^4(\sin n\pi R) (\sin n\pi Q) \]

Similarly, the fourth derivative with respect to \( Q \) is:
\[ \frac{\partial^4 h}{\partial Q^4} = \pi^4 (\sin \pi R) (\sin \pi Q) \]

While the Second Partial derivate with respects to R and Q is:

\[ \frac{\partial^4 h}{\partial R^2 \partial Q^2} = \pi^4 (\sin \pi R) (\sin \pi Q) \]

The Derivatives of the CCCC Shape function

\[ w = A (1 - \cos 2\pi R)(1 - \cos 2\pi Q) \]
\[ h = (1 - \cos 2\pi R)(1 - \cos 2\pi Q) \]

The first derivative of the CCCC deflection equation with respect to R is:

\[ \frac{\partial h}{\partial R} = 2\pi (\sin 2\pi R)(1 - \cos 2\pi Q) \]

While the second derivative with respect to R is:

\[ \frac{\partial^2 h}{\partial R^2} = 4\pi^2 (\cos 2\pi R)(1 - \cos 2\pi Q) \]

And the third derivative with respect to R is:

\[ \frac{\partial^3 h}{\partial R^3} = -8\pi^3 (\sin 2\pi R)(1 - \cos 2\pi Q) \]

It follows that the fourth derivative with respect to R is:

\[ \frac{\partial^4 h}{\partial R^4} = -16\pi^4 (\cos 2\pi R)(1 - \cos 2\pi Q) \]

Similarly, the fourth derivative with respect to Q is:

\[ \frac{\partial^4 h}{\partial Q^4} = -16\pi^4 (1 - \cos 2\pi R)(\cos 2\pi Q) \]

While the Second Partial derivate with respects to R and Q is:

\[ \frac{\partial^4 h}{\partial R^2 \partial Q^2} = 16\pi^4 (\cos 2\pi R)(\cos 2\pi Q) \]

Effective Span and Bent Domain of Single Buckled mode of Line Continuum

When line continuum buckles in first deformation mode, a region within the span takes bent configuration. The span of this bent region is largely dependent on the support conditions of the line continuum. The point the continuum starts to bend is designated as \( d_1 \) and the last point when the continuum is bent is designated \( d_2 \).

Let the Effective Span be \( L_e \).

For SS Continuum, \( L_e = L \).

For CC Continuum, \( L_e = 0.5L \).

\[ \text{Figure 5 Effective Lengths for various Support conditions} \]
Calculation of the stiffness coefficients for plates of various boundary conditions

The stiffness coefficients were represented as

\[ k_{xx} = \int_{d_1}^{d_2} \int_{d_1}^{d_2} \frac{\partial^4 w}{\partial R^4} \partial R \partial Q \]

\[ k_{xy} = \int_{d_1}^{d_2} \int_{d_1}^{d_2} \frac{\partial^4 w}{\partial R^2 \partial Q^2} \partial R \partial Q \]

\[ k_{yy} = \int_{d_1}^{d_2} \int_{d_1}^{d_2} \frac{\partial^4 w}{\partial Q^4} \partial R \partial Q \]

\[ k_{xx} = \int_{d_1}^{d_2} \int_{d_1}^{d_2} \frac{\partial^2 w}{\partial R^2} \partial R \partial Q \]

\[ k_T = k_{xxxx} + \frac{2}{\beta^2} k_{xxyy} + \frac{4}{\beta^2} k_{yyyy} \]

We will use these equations to compute stiffness coefficients for the various boundary conditions. The boundary conditions are SSSS, CCCC, CSSS, CSCS and CCCS. The stiffness coefficients to be computed include:

\[ K_{xx}, K_{xxxx}, K_{xxyy}, K_{yyyy}, K_T \]

We will also compute the Ratio of stiffness coefficients:

\[ \tilde{N} = \frac{K_T}{-K_{xx}} \]

As stated in equation 84

The Integrals of the SS Deflection Equation

\[ \text{For SS}, d_1 = 0, \quad d_2 = 1 \]

\[ k_{xx} = -\pi^2 \int_{0}^{1} \int_{0}^{1} (\sin \pi R) (\sin \pi Q) \ dR dQ = -\pi^2 \left( \int_{0}^{1} \frac{1}{\pi} \cos \pi R \right) \left( \int_{0}^{1} \frac{1}{\pi} \cos \pi Q \right) = -\pi^2 \left( \frac{1}{\pi} \cos \pi + \frac{1}{\pi} \cos 0 \right) \left( \frac{1}{\pi} \cos 0 + \frac{1}{\pi} \cos 0 \right) \]

Further resolution gives

\[ k_T = 4\pi^2 + \frac{8}{\beta^2} \pi^2 + \frac{4}{\beta^3} \pi^2 \]

Substituting back gives:

\[ \frac{N_s a^2}{D} = \frac{4\pi^2 + \frac{8}{\beta^2} \pi^2 + \frac{4}{\beta^3} \pi^2}{4} = \pi^2 + \frac{2}{\beta^2} \pi^2 + \frac{1}{\beta^3} \pi^2 = \pi^2 \left( 1 + \frac{2}{\beta^2} + \frac{1}{\beta^3} \right) \]

The Integrals of the CCCC Deflection Equation

\[ \text{For CC}, d_1 = 0.25, \quad d_2 = 0.75 \]
\[ k_{xx} = 4\pi^2 \int_0^{0.75} \int_0^{0.75} (\cos 2\pi R)(1 - \cos 2\pi Q)dRdQ = 4\pi^2 \left( \frac{1}{2\pi} \sin 2\pi R \right)^{0.75}_{0.25} \left( Q - \frac{1}{2\pi} \sin 2\pi Q \right)^{0.75}_{0.25} 90 \]
\[ = 4\pi^2 \left( \frac{1}{2\pi} \sin 2\pi (0.75) - \frac{1}{2\pi} \sin 2\pi (0.25) \right) \left( 0.75 - \frac{1}{2\pi} \sin 2\pi (0.75) - 0.25 + \frac{1}{2\pi} \sin 2\pi (0.25) \right) \]
\[ = 4\pi^2 \left( \frac{1}{2\pi} - \frac{1}{2\pi} \right) (0.75 + \frac{1}{2\pi} - 0.25 + \frac{1}{2\pi}) = 4\pi^2 \left( \frac{1}{\pi} \right) (0.5 + \frac{1}{\pi}) = 4\pi^2 \left( \frac{1}{2\pi} - \frac{1}{\pi^2} \right) \]
\[ = -2\pi - 4 \]

\[ k_{xxxx} = -16\pi^4 \int_0^{0.25} \int_0^{0.25} (\cos 2\pi R)(\cos 2\pi Q)dRdQ = -16\pi^4 \left( \frac{1}{2\pi} \sin 2\pi R \right)^{0.25}_{0.25} \left( Q - \frac{1}{2\pi} \sin 2\pi Q \right)^{0.25}_{0.25} \]
\[ = -16\pi^4 \left( \frac{1}{2\pi} \sin 2\pi (0.75) - \frac{1}{2\pi} \sin 2\pi (0.25) \right) \left( 0.75 - \frac{1}{2\pi} \sin 2\pi (0.75) - 0.25 + \frac{1}{2\pi} \sin 2\pi (0.25) \right) \]
\[ = -16\pi^4 \left( \frac{1}{2\pi} - \frac{1}{2\pi} \right) (0.75 + \frac{1}{2\pi} - 0.25 + \frac{1}{2\pi}) = -16\pi^4 \left( \frac{1}{\pi} \right) (0.5 + \frac{1}{\pi}) = -16\pi^4 \left( \frac{1}{2\pi} - \frac{1}{\pi^2} \right) \]
\[ = 8\pi^3 + 16\pi^2 \]

\[ k_{xyy} = 16\pi^4 \int_0^{1} \int_0^{1} (\cos 2\pi R)(\cos 2\pi Q)dRdQ = 16\pi^4 \left( \frac{1}{2\pi} \sin 2\pi R \right)^{1}_{0.25} \left( \frac{1}{2\pi} \sin 2\pi Q \right)^{1}_{0.25} \]
\[ = 16\pi^4 \left( \frac{1}{2\pi} \sin 2\pi (0.75) - \frac{1}{2\pi} \sin 2\pi (0.25) \right) \left( \frac{1}{2\pi} \sin 2\pi (0.75) - \frac{1}{2\pi} \sin 2\pi (0.25) \right) = 16\pi^4 \left( \frac{1}{2\pi} - \frac{1}{2\pi} \right)^2 \]
\[ = 16\pi^2 \]

\[ k_{yyyy} = -16\pi^4 \int_0^{1} \int_0^{1} (1 - \cos 2\pi R)(\cos 2\pi Q)dRdQ = -16\pi^4 \left( R - \frac{1}{2\pi} \sin 2\pi R \right)^{0.75}_{0.25} \left( \frac{1}{2\pi} \sin 2\pi Q \right)^{0.75}_{0.25} \]
\[ = -16\pi^4 \left( 0.75 - \frac{1}{2\pi} \sin 2\pi (0.75) - 0.25 + \frac{1}{2\pi} \sin 2\pi (0.25) \right) \left( \frac{1}{2\pi} \sin 2\pi (0.75) - \frac{1}{2\pi} \sin 2\pi (0.25) \right) \]
\[ = -16\pi^4 \left( 0.75 + \frac{1}{2\pi} - 0.25 + \frac{1}{2\pi} \right) (\frac{1}{2\pi} - \frac{1}{2\pi}) = -16\pi^4 \left( \frac{1}{\pi} \right) (0.5 + \frac{1}{\pi}) = -16\pi^4 \left( \frac{1}{2\pi} - \frac{1}{\pi^2} \right) \]
\[ = 8\pi^3 + 16\pi^2 \]

Substituting Equations 91 to 91b into

\[ k_T = k_{xxxx} + 2\frac{2}{\beta^2} k_{xyy} + \frac{1}{\beta^4} k_{yyyy} \]

\[ = 8\pi^3 + 16\pi^2 + \frac{2}{\beta^2} 16\pi^2 + \frac{1}{\beta^4} (8\pi^3 + 16\pi^2) \]

Substituting Equations 90a and 92 into Equation 91c gives:

\[ \frac{N_{ax}}{D} = \frac{8\pi^3 + 16\pi^2 + \frac{2}{\beta^2} 16\pi^2 + \frac{1}{\beta^4} (8\pi^3 + 16\pi^2)}{2\pi + 4} \]

Table 1: Summary of Stiffness coefficients in terms of \( \pi \)

<table>
<thead>
<tr>
<th>INTEGRAL</th>
<th>SSSS</th>
<th>CCCC</th>
</tr>
</thead>
<tbody>
<tr>
<td>( k_{xx} )</td>
<td>-4</td>
<td>-2\pi - 4</td>
</tr>
<tr>
<td>( k_{xxxx} )</td>
<td>( 4\pi^2 )</td>
<td>8\pi^3 + 16\pi^2</td>
</tr>
<tr>
<td>( k_{xyy} )</td>
<td>( 4\pi^2 )</td>
<td>16\pi^2</td>
</tr>
<tr>
<td>( k_{yyyy} )</td>
<td>( 4\pi^2 )</td>
<td>8\pi^3 + 16\pi^2</td>
</tr>
</tbody>
</table>
Table 2: Summary of Stiffness Coefficients for all plate types

<table>
<thead>
<tr>
<th></th>
<th>SSSS</th>
<th>CCC</th>
</tr>
</thead>
<tbody>
<tr>
<td>(k_{xx})</td>
<td>-4.00000</td>
<td>-10.28319</td>
</tr>
<tr>
<td>(k_{xxxx})</td>
<td>39.478418</td>
<td>405.96388</td>
</tr>
<tr>
<td>(k_{xyy})</td>
<td>39.478418</td>
<td>157.91367</td>
</tr>
<tr>
<td>(k_{yyyy})</td>
<td>39.478418</td>
<td>405.96388</td>
</tr>
</tbody>
</table>

Table 3: Stiffness Coefficients for different Aspect ratios for SSSS Plates

| b/a | \(k_{xxxx}\) | \(k_{xyy}\) | \(k_{yyyy}\) | \(k_{xx}\) | \(k_{T}\) | \(\bar{N}\)  
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1</td>
<td>39.47842</td>
<td>39.47842</td>
<td>39.47842</td>
<td>-4.00000</td>
<td>131.69629</td>
<td>32.92407</td>
</tr>
<tr>
<td>1.3</td>
<td>39.47842</td>
<td>39.47842</td>
<td>39.47842</td>
<td>-4.00000</td>
<td>100.02093</td>
<td>25.00523</td>
</tr>
<tr>
<td>1.4</td>
<td>39.47842</td>
<td>39.47842</td>
<td>39.47842</td>
<td>-4.00000</td>
<td>90.03907</td>
<td>22.50977</td>
</tr>
<tr>
<td>1.5</td>
<td>39.47842</td>
<td>39.47842</td>
<td>39.47842</td>
<td>-4.00000</td>
<td>82.36855</td>
<td>20.59214</td>
</tr>
<tr>
<td>1.6</td>
<td>39.47842</td>
<td>39.47842</td>
<td>39.47842</td>
<td>-4.00000</td>
<td>76.34486</td>
<td>19.08621</td>
</tr>
<tr>
<td>1.7</td>
<td>39.47842</td>
<td>39.47842</td>
<td>39.47842</td>
<td>-4.00000</td>
<td>71.52589</td>
<td>17.88147</td>
</tr>
<tr>
<td>1.9</td>
<td>39.47842</td>
<td>39.47842</td>
<td>39.47842</td>
<td>-4.00000</td>
<td>64.37944</td>
<td>16.09486</td>
</tr>
</tbody>
</table>

Table 4: Stiffness Coefficients for different Aspect ratios for CCC Plates

| b/a | \(k_{xxxx}\) | \(k_{xyy}\) | \(k_{yyyy}\) | \(k_{xx}\) | \(k_{T}\) | \(\bar{N}\)  
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>405.96388</td>
<td>157.91367</td>
<td>405.96388</td>
<td>-10.28319</td>
<td>1127.75511</td>
<td>109.66982</td>
</tr>
<tr>
<td>1.1</td>
<td>405.96388</td>
<td>157.91367</td>
<td>405.96388</td>
<td>-10.28319</td>
<td>944.25701</td>
<td>91.82534</td>
</tr>
<tr>
<td>1.2</td>
<td>405.96388</td>
<td>157.91367</td>
<td>405.96388</td>
<td>-10.28319</td>
<td>821.06576</td>
<td>79.84547</td>
</tr>
<tr>
<td>1.3</td>
<td>405.96388</td>
<td>157.91367</td>
<td>405.96388</td>
<td>-10.28319</td>
<td>734.98321</td>
<td>71.47427</td>
</tr>
<tr>
<td>1.4</td>
<td>405.96388</td>
<td>157.91367</td>
<td>405.96388</td>
<td>-10.28319</td>
<td>672.77601</td>
<td>65.42487</td>
</tr>
<tr>
<td>1.5</td>
<td>405.96388</td>
<td>157.91367</td>
<td>405.96388</td>
<td>-10.28319</td>
<td>626.52199</td>
<td>60.92684</td>
</tr>
<tr>
<td>1.6</td>
<td>405.96388</td>
<td>157.91367</td>
<td>405.96388</td>
<td>-10.28319</td>
<td>591.27911</td>
<td>57.49961</td>
</tr>
<tr>
<td>1.7</td>
<td>405.96388</td>
<td>157.91367</td>
<td>405.96388</td>
<td>-10.28319</td>
<td>563.85291</td>
<td>54.83251</td>
</tr>
<tr>
<td>1.8</td>
<td>405.96388</td>
<td>157.91367</td>
<td>405.96388</td>
<td>-10.28319</td>
<td>542.11352</td>
<td>52.71844</td>
</tr>
<tr>
<td>1.9</td>
<td>405.96388</td>
<td>157.91367</td>
<td>405.96388</td>
<td>-10.28319</td>
<td>524.60175</td>
<td>51.01549</td>
</tr>
<tr>
<td>2</td>
<td>405.96388</td>
<td>157.91367</td>
<td>405.96388</td>
<td>-10.28319</td>
<td>510.29346</td>
<td>49.62407</td>
</tr>
</tbody>
</table>

INTEGRAL
SSSS
CCCC

\(k_{xx}\)
\(-4.00000\)
\(-10.28319\)

\(k_{xxxx}\)
\(39.478418\)
\(405.96388\)

\(k_{xyy}\)
\(39.478418\)
\(157.91367\)

\(k_{yyyy}\)
\(39.478418\)
\(405.96388\)
1.6 Determination of Ritz Stiffness Co-efficients and Buckling loads

In this section, we determine critical buckling loads and stiffness co-efficients as used in the Ritz approach. Recall Ritz formulation for calculating the Critical Buckling load of a thin Rectangular Plate from Equation 2.7:

\[
N_x = D \left[ \int_0^a \int_0^b \left( \frac{\partial^2 H}{\partial R^2} \right)^2 \partial R \partial Q + 2 \int_0^a \int_0^b \left( \frac{\partial^2 H}{\partial R \partial Q} \right)^2 \partial R \partial Q \right. \\
\left. + \int_0^a \int_0^b \left( \frac{\partial^2 H}{\partial Q^2} \right)^2 \partial R \partial Q \right] \\
\frac{1}{a^2} \int_0^a \int_0^b \left( \frac{\partial H}{\partial R} \right)^2 \partial R \partial Q
\]

For Simplicity, Let:

\[
k_{xx} = \int_0^a \int_0^b \left( \frac{\partial^2 H}{\partial R^2} \right)^2 \partial R \partial Q
\]

\[
k_{xy} = \int_0^a \int_0^b \left( \frac{\partial^2 H}{\partial R \partial Q} \right)^2 \partial R \partial Q
\]

\[
k_{yy} = \int_0^a \int_0^b \left( \frac{\partial^2 H}{\partial Q^2} \right)^2 \partial R \partial Q
\]

\[
k_N = \int_0^a \int_0^b \left( \frac{\partial H}{\partial R} \right)^2 \partial R \partial Q
\]

Hence, the critical buckling load becomes:

\[
N_x = \frac{D \left[ k_{xx} + 2 \frac{1}{a^2} k_{xy} \right. \left. + \frac{1}{a^2} k_{yy} \right]}{k_N}
\]

We will then proceed to compute these stiffness coefficients \((k_{xx}, k_{xy}, k_{yy}, \text{and} k_N)\) needed to calculate the Ritz critical buckling loads of thin rectangular plates. In doing so, the derivatives of the shape functions obtained will be determined as required by Ritz Formulation.

The integrations were done within the domain of 0 to 1 in both orthogonal axes, as required by Ritz Formulation.

The Ritz Derivatives of the Deflection Equations

The derivatives of the shape functions were determined. These derivatives will be Computed as required by Ritz Formulation

The Ritz Derivative for the SSSS Shape Functions

\[
h = (\sin \pi R) (\sin \pi Q)
\]

The first derivative of the SSSS deflection equation with respect to R is:

\[
\frac{\partial h}{\partial R} = \pi (\cos \pi R) (\sin \pi Q)
\]

The Square of the first derivative with respect to R is:

\[
\left[ \frac{\partial h}{\partial R} \right]^2 = \pi^2 (\cos^2 \pi R) (\sin^2 \pi Q)
\]

While the second derivative with respect to R is:

\[
\frac{\partial^2 h}{\partial R^2} = -\pi^2 (\sin \pi R) (\sin \pi Q)
\]

And the The Square of the Second derivative with respect to R is:

\[
\left[ \frac{\partial^2 h}{\partial R^2} \right]^2 = \pi^4 (\sin^2 \pi R) (\sin^2 \pi Q)
\]

It follows that the Second derivative with respect to Q is:

\[
\frac{\partial^2 h}{\partial Q^2} = -\pi^2 (\sin \pi R) (\sin \pi Q)
\]

Similarly, the square of the Second derivative with respect to Q is:

\[
\left[ \frac{\partial^2 h}{\partial Q^2} \right]^2 = \pi^4 (\sin^2 \pi R) (\cos^2 \pi Q)
\]
While the First Partial derivative with respects to R and Q is:
\[ \frac{\partial^2 H}{\partial R \partial Q} = \pi^2 (\cos \pi R) (\cos \pi Q) \]

While the Square of the First Partial derivative with respects to R and Q is:
\[ \left( \frac{\partial^2 h}{\partial R \partial Q} \right)^2 = 16\pi^4 (\sin^2 2\pi R)(\sin^2 2\pi Q) \]

The Ritz Derivative for the CCCC Shape Functions
\[ h = (1 - \cos 2\pi R)(1 - \cos 2\pi Q) \]

The first derivative of the CCCC deflection equation with respect to R is:
\[ \frac{\partial h}{\partial R} = 2\pi (\sin 2\pi R)(1 - \cos 2\pi Q) \]

The Square of the first derivative with respect to R is
\[ \left( \frac{\partial h}{\partial R} \right)^2 = 4\pi^2 (\sin^2 2\pi R)(1 - \cos 2\pi Q)^2 \]

While the second derivative with respect to R is:
\[ \frac{\partial^2 h}{\partial R^2} = 4\pi^2 (\cos 2\pi R)(1 - \cos 2\pi Q) \]

And the Square of the Second derivative with respect to R is:
\[ \left( \frac{\partial^2 h}{\partial R^2} \right)^2 = 16\pi^4 (\cos^2 2\pi R)(\cos^2 2\pi Q) \]

It follows that the Second derivative with respect to Q is:
\[ \frac{\partial^2 h}{\partial Q^2} = 4\pi^2 (1 - \cos 2\pi R)(\cos Q) \]

Similarly, the square of the Second derivative with respect to Q is:
\[ \left( \frac{\partial^2 h}{\partial Q^2} \right)^2 = 16\pi^4 (1 - \cos 2\pi R)^2(\cos^2 2\pi Q) \]

While the First Partial derivative with respects to R and Q is:
\[ \frac{\partial^2 h}{\partial R \partial Q} = 4\pi^2 (\sin 2\pi R)(\sin 2\pi Q) \]

While the Square of the First Partial derivative with respects to R and Q is:
\[ \left( \frac{\partial^2 h}{\partial R \partial Q} \right)^2 = 16\pi^4 (\sin^2 2\pi R)(\sin^2 2\pi Q) \]

1.7 RESULTS AND DISCUSSIONS

Presentation of Results

In this section, the formulations obtained in this work are presented. Among these are the general expression of Total Potential Energy for plate under buckling load, the Euler-Bernoulli functional for Buckling analysis of thin rectangular plates, exact deflection functions for thin rectangular plates and the Euler-Bernoulli Stiffness coefficients for thin rectangular plates. Furthermore, the comparative results of tests for exact buckling loads and approximate buckling loads in the strong form of the governing equation for plates of selected boundary conditions are presented.

The results are presented for the two types of support conditions of thin rectangular plates: SSSS and CCCC The results for the non-dimensional buckling load are evaluated for different aspect ratios, β (b/a: where, 1 ≤ β ≤ 2).

The Total Potential Energy Functional for Thin Rectangular Plate under Buckling Load
The general expression for the total potential energy for thin rectangular plate under buckling load (biaxial and in-plane shear load) as obtained in this work is presented in Equation 4.1

\[ \Pi = \frac{D}{2} \int_{0}^{a} \int_{0}^{b} \left( \frac{\partial^2 w}{\partial x^2} \right)^2 + \left( \frac{\partial^2 w}{\partial y^2} \right)^2 + \frac{\partial^2 w}{\partial x \partial y} \right) dx \, dy + \frac{1}{2} \int_{0}^{a} \int_{0}^{b} \left[ N_x \frac{\partial w}{\partial x} \right]^2 + 2N_{xy} \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} + N_y \left( \frac{\partial w}{\partial y} \right)^2 \right) dx \, dy \]

4.1
The Euler-Bernoulli Formula for Buckling Analysis of Thin Rectangular Plate

The Euler-Bernoulli Formula for Buckling analysis of thin Rectangular plate as obtained in this work is presented in Equation 4.2

\[
N_f = \frac{k_T}{-k_{ss}a^2}
\]

Where:

\[
k_T = k_{xxx} + \frac{2}{\beta}k_{xyy} + \frac{1}{\beta}k_{yyy}
\]

\[
k_{ss} = \int_{d_1}^{d_2} \int_{d_1}^{d_2} \left( \frac{\partial^2w}{\partial R^2} \right) \partial R \partial Q, k_{xxx} = \int_{d_1}^{d_2} \int_{d_1}^{d_2} \left( \frac{\partial^3w}{\partial R^3} \right) \partial R \partial Q \partial R, k_{xyy} = \int_{d_1}^{d_2} \int_{d_1}^{d_2} \left( \frac{\partial^3w}{\partial R^2 \partial Q^2} \right) \partial R \partial Q \partial R \partial R, k_{yyy} = \int_{d_1}^{d_2} \int_{d_1}^{d_2} \left( \frac{\partial^4w}{\partial R^4} \right) \partial R \partial Q
\]

Results for the exact deflection functions for buckling analysis of thin rectangular plate

Equations presented in For SSSS, \( w = A (\sin \pi R) (\sin \pi Q) \) [Single Mode, \( n=1 \)], with Constant as \( A = c_3 \cdot I_3 \)

Also for CCCC, \( w = A (1 - \cos 2\pi R)(1 - \cos 2\pi Q) \) [Single Mode, \( n=1 \)] with Constant as \( A = c_0 \cdot I_0 \) are the exact deflection equations of Kirchhoff’s Plates for the two selected boundary conditions. The Equations are in trigonometric form.

For SSSS, \( w = A (\sin \pi R) (\sin \pi Q) \) [Single Mode, \( n=1 \)], with Constant as \( A = c_3 \cdot I_3 \)

Also for CCCC, \( w = A (1 - \cos 2\pi R)(1 - \cos 2\pi Q) \) [Single Mode, \( n=1 \)] with Constant as \( A = c_0 \cdot I_0 \)

1.8 Results for the Euler-Bernoulli Stiffness coefficients for Buckling Analysis of Thin Rectangular Plates

Results for Euler-Bernoulli Stiffness coefficients as computed are presented here, from Table to Error! Reference source not found.. First of all, the summary of primary stiffness coefficients are presented. Then followed by stiffness coefficients generated by considering different aspect ratios.

Table 7: Summary of Euler Stiffness Coefficients for Plates with different Support Conditions

<table>
<thead>
<tr>
<th>INTEGRAL</th>
<th>SSSS</th>
<th>CCCC</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1</td>
<td>39.47842</td>
<td>405.96388</td>
</tr>
<tr>
<td>1.2</td>
<td>39.47842</td>
<td>405.96388</td>
</tr>
<tr>
<td>1.3</td>
<td>39.47842</td>
<td>405.96388</td>
</tr>
<tr>
<td>1.4</td>
<td>39.47842</td>
<td>405.96388</td>
</tr>
<tr>
<td>1.5</td>
<td>39.47842</td>
<td>405.96388</td>
</tr>
<tr>
<td>1.6</td>
<td>39.47842</td>
<td>405.96388</td>
</tr>
<tr>
<td>1.7</td>
<td>39.47842</td>
<td>405.96388</td>
</tr>
<tr>
<td>1.8</td>
<td>39.47842</td>
<td>405.96388</td>
</tr>
<tr>
<td>1.9</td>
<td>39.47842</td>
<td>405.96388</td>
</tr>
<tr>
<td>2</td>
<td>39.47842</td>
<td>405.96388</td>
</tr>
</tbody>
</table>

Table 8: Euler Stiffness Coefficients for different Aspect ratios for SSSS Plates

<table>
<thead>
<tr>
<th>b/a</th>
<th>k_{xxx}</th>
<th>k_{xyy}</th>
<th>k_{yyy}</th>
<th>k_{ss}</th>
<th>k_T</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1</td>
<td>39.47842</td>
<td>39.47842</td>
<td>39.47842</td>
<td>-4.00000</td>
<td>131.69629</td>
</tr>
<tr>
<td>1.3</td>
<td>39.47842</td>
<td>39.47842</td>
<td>39.47842</td>
<td>-4.00000</td>
<td>100.02093</td>
</tr>
<tr>
<td>1.4</td>
<td>39.47842</td>
<td>39.47842</td>
<td>39.47842</td>
<td>-4.00000</td>
<td>90.03907</td>
</tr>
<tr>
<td>1.5</td>
<td>39.47842</td>
<td>39.47842</td>
<td>39.47842</td>
<td>-4.00000</td>
<td>82.36855</td>
</tr>
<tr>
<td>1.6</td>
<td>39.47842</td>
<td>39.47842</td>
<td>39.47842</td>
<td>-4.00000</td>
<td>76.34486</td>
</tr>
<tr>
<td>1.7</td>
<td>39.47842</td>
<td>39.47842</td>
<td>39.47842</td>
<td>-4.00000</td>
<td>71.52589</td>
</tr>
<tr>
<td>1.9</td>
<td>39.47842</td>
<td>39.47842</td>
<td>39.47842</td>
<td>-4.00000</td>
<td>64.37944</td>
</tr>
</tbody>
</table>

Table 9: Euler Stiffness Coefficients for different Aspect ratios for CCCC Plates

<table>
<thead>
<tr>
<th>b/a</th>
<th>k_{xxx}</th>
<th>k_{xyy}</th>
<th>k_{yyy}</th>
<th>k_{ss}</th>
<th>k_T</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>405.96388</td>
<td>157.91367</td>
<td>405.96388</td>
<td>-10.28319</td>
<td>1127.75511</td>
</tr>
<tr>
<td>1.1</td>
<td>405.96388</td>
<td>157.91367</td>
<td>405.96388</td>
<td>-10.28319</td>
<td>944.25701</td>
</tr>
</tbody>
</table>
Results for the Ritz Stiffness coefficients for Buckling Analysis of Thin Rectangular Plates

Results for Ritz Stiffness coefficients as computed, are presented here, from Table 10 and Error! Reference source not found.. First of all, the summary of primary stiffness coefficients are presented, then followed by stiffness coefficients generated by considering different aspect ratios.

Table 10: Summary of Ritz Stiffness Coefficients for Plates with different Support Conditions

<table>
<thead>
<tr>
<th>k_{xx}</th>
<th>2.46740</th>
<th>29.60881</th>
</tr>
</thead>
<tbody>
<tr>
<td>k_{xxxx}</td>
<td>24.35227</td>
<td>1168.90909</td>
</tr>
<tr>
<td>k_{xyy}</td>
<td>24.35227</td>
<td>389.63636</td>
</tr>
<tr>
<td>k_{yyyy}</td>
<td>24.35227</td>
<td>1168.90909</td>
</tr>
</tbody>
</table>

Table 11: Ritz Stiffness Coefficients for different Aspect ratios for SSSS Plates

<table>
<thead>
<tr>
<th>b/a</th>
<th>k_{xxxx}</th>
<th>k_{xyy}</th>
<th>k_{yyyy}</th>
<th>k_{xx}</th>
<th>k_{T}</th>
<th>N</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>24.35227</td>
<td>24.35227</td>
<td>24.35227</td>
<td>2.46740</td>
<td>97.40909</td>
<td>39.47842</td>
</tr>
<tr>
<td>1.1</td>
<td>24.35227</td>
<td>24.35227</td>
<td>24.35227</td>
<td>2.46740</td>
<td>81.23689</td>
<td>32.92407</td>
</tr>
<tr>
<td>1.2</td>
<td>24.35227</td>
<td>24.35227</td>
<td>24.35227</td>
<td>2.46740</td>
<td>69.91883</td>
<td>28.33704</td>
</tr>
<tr>
<td>1.3</td>
<td>24.35227</td>
<td>24.35227</td>
<td>24.35227</td>
<td>2.46740</td>
<td>61.69794</td>
<td>25.00523</td>
</tr>
<tr>
<td>1.4</td>
<td>24.35227</td>
<td>24.35227</td>
<td>24.35227</td>
<td>2.46740</td>
<td>55.54063</td>
<td>22.50977</td>
</tr>
<tr>
<td>1.5</td>
<td>24.35227</td>
<td>24.35227</td>
<td>24.35227</td>
<td>2.46740</td>
<td>50.80906</td>
<td>20.59214</td>
</tr>
<tr>
<td>1.6</td>
<td>24.35227</td>
<td>24.35227</td>
<td>24.35227</td>
<td>2.46740</td>
<td>47.09335</td>
<td>19.08262</td>
</tr>
<tr>
<td>1.7</td>
<td>24.35227</td>
<td>24.35227</td>
<td>24.35227</td>
<td>2.46740</td>
<td>44.12076</td>
<td>17.88147</td>
</tr>
<tr>
<td>1.8</td>
<td>24.35227</td>
<td>24.35227</td>
<td>24.35227</td>
<td>2.46740</td>
<td>41.70433</td>
<td>16.90213</td>
</tr>
<tr>
<td>1.9</td>
<td>24.35227</td>
<td>24.35227</td>
<td>24.35227</td>
<td>2.46740</td>
<td>39.71247</td>
<td>16.09486</td>
</tr>
<tr>
<td>2</td>
<td>24.35227</td>
<td>24.35227</td>
<td>24.35227</td>
<td>2.46740</td>
<td>38.05043</td>
<td>15.42126</td>
</tr>
</tbody>
</table>

Table 12: Ritz Stiffness Coefficients for different Aspect ratios for CCCC Plates

<table>
<thead>
<tr>
<th>b/a</th>
<th>k_{xxxx}</th>
<th>k_{xyy}</th>
<th>k_{yyyy}</th>
<th>k_{xx}</th>
<th>k_{T}</th>
<th>N</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1168.90909</td>
<td>389.63636</td>
<td>1168.90909</td>
<td>29.60881</td>
<td>3117.09091</td>
<td>105.27578</td>
</tr>
<tr>
<td>1.1</td>
<td>1168.90909</td>
<td>389.63636</td>
<td>1168.90909</td>
<td>29.60881</td>
<td>2611.31678</td>
<td>88.19390</td>
</tr>
<tr>
<td>1.2</td>
<td>1168.90909</td>
<td>389.63636</td>
<td>1168.90909</td>
<td>29.60881</td>
<td>2273.78073</td>
<td>76.79405</td>
</tr>
<tr>
<td>1.3</td>
<td>1168.90909</td>
<td>389.63636</td>
<td>1168.90909</td>
<td>29.60881</td>
<td>2039.28473</td>
<td>68.87425</td>
</tr>
<tr>
<td>1.4</td>
<td>1168.90909</td>
<td>389.63636</td>
<td>1168.90909</td>
<td>29.60881</td>
<td>1870.77385</td>
<td>63.18301</td>
</tr>
<tr>
<td>1.5</td>
<td>1168.90909</td>
<td>389.63636</td>
<td>1168.90909</td>
<td>29.60881</td>
<td>1746.14815</td>
<td>58.97393</td>
</tr>
<tr>
<td>1.6</td>
<td>1168.90909</td>
<td>389.63636</td>
<td>1168.90909</td>
<td>29.60881</td>
<td>1651.67387</td>
<td>55.78318</td>
</tr>
<tr>
<td>1.7</td>
<td>1168.90909</td>
<td>389.63636</td>
<td>1168.90909</td>
<td>29.60881</td>
<td>1578.50755</td>
<td>53.31208</td>
</tr>
</tbody>
</table>
Table 13: Summary of Approximate Critical Buckling Loads from Ritz Method

<table>
<thead>
<tr>
<th>Aspect Ratio</th>
<th>SSSS</th>
<th>CCCC</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.8</td>
<td>1168.90909</td>
<td>389.63636</td>
</tr>
<tr>
<td>1.9</td>
<td>1168.90909</td>
<td>389.63636</td>
</tr>
<tr>
<td>2</td>
<td>1168.90909</td>
<td>389.63636</td>
</tr>
</tbody>
</table>

Table 1: Comparative Study of the Present with That of Previous

1.9 COMPARISON OF THE PRESENT WITH THAT OF PREVIOUS

Results from Testing the Exact Critical Buckling Loads and Values from Ritz Method in The Strong Form of The Governing Equation for Plates of Selected Boundary Conditions.

In this section results obtained by substituting the exact and approximate non-dimensional buckling loads into the Strong Form of the equilibrium equation are presented in Table 1:

<table>
<thead>
<tr>
<th>Aspect Ratio</th>
<th>SSSS</th>
<th>CCCC</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>39.48</td>
<td>105.28</td>
</tr>
<tr>
<td>1.1</td>
<td>32.92</td>
<td>88.19</td>
</tr>
<tr>
<td>1.2</td>
<td>28.34</td>
<td>76.79</td>
</tr>
<tr>
<td>1.3</td>
<td>25.01</td>
<td>68.87</td>
</tr>
<tr>
<td>1.4</td>
<td>22.51</td>
<td>63.18</td>
</tr>
<tr>
<td>1.5</td>
<td>20.59</td>
<td>58.97</td>
</tr>
<tr>
<td>1.6</td>
<td>19.09</td>
<td>55.78</td>
</tr>
<tr>
<td>1.7</td>
<td>17.88</td>
<td>53.31</td>
</tr>
<tr>
<td>1.8</td>
<td>16.90</td>
<td>51.36</td>
</tr>
<tr>
<td>1.9</td>
<td>16.09</td>
<td>49.80</td>
</tr>
<tr>
<td>2</td>
<td>15.42</td>
<td>48.53</td>
</tr>
</tbody>
</table>

Table 15: Test Results for Approximate Critical Buckling Loads from Ritz Method

<table>
<thead>
<tr>
<th>Aspect Ratio</th>
<th>SSSS</th>
<th>CCCC</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>194.82</td>
<td>6234.18</td>
</tr>
<tr>
<td>1.1</td>
<td>162.47</td>
<td>5222.63</td>
</tr>
<tr>
<td>1.2</td>
<td>139.84</td>
<td>4547.56</td>
</tr>
<tr>
<td>1.3</td>
<td>123.40</td>
<td>4078.57</td>
</tr>
<tr>
<td>1.4</td>
<td>111.08</td>
<td>3741.55</td>
</tr>
<tr>
<td>1.5</td>
<td>101.62</td>
<td>3492.30</td>
</tr>
<tr>
<td>1.6</td>
<td>94.19</td>
<td>3303.35</td>
</tr>
<tr>
<td>1.7</td>
<td>88.24</td>
<td>3157.02</td>
</tr>
<tr>
<td>1.8</td>
<td>83.41</td>
<td>3041.55</td>
</tr>
</tbody>
</table>
Table 16: Comparison of Critical Buckling Loads for SSSS Plate

<table>
<thead>
<tr>
<th>ASPECT RATIO</th>
<th>( \bar{N} ) from present study</th>
<th>( \bar{N} ) from Ritz</th>
<th>Percentage Difference (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>39.48</td>
<td>39.48</td>
<td>0.00</td>
</tr>
<tr>
<td>1.1</td>
<td>32.92</td>
<td>32.92</td>
<td>0.00</td>
</tr>
<tr>
<td>1.2</td>
<td>28.34</td>
<td>28.34</td>
<td>0.00</td>
</tr>
<tr>
<td>1.3</td>
<td>25.01</td>
<td>25.01</td>
<td>0.00</td>
</tr>
<tr>
<td>1.4</td>
<td>22.51</td>
<td>22.51</td>
<td>0.00</td>
</tr>
<tr>
<td>1.5</td>
<td>20.59</td>
<td>20.59</td>
<td>0.00</td>
</tr>
<tr>
<td>1.6</td>
<td>19.09</td>
<td>19.09</td>
<td>0.00</td>
</tr>
<tr>
<td>1.7</td>
<td>17.88</td>
<td>17.88</td>
<td>0.00</td>
</tr>
<tr>
<td>1.8</td>
<td>16.90</td>
<td>16.90</td>
<td>0.00</td>
</tr>
<tr>
<td>1.9</td>
<td>16.09</td>
<td>16.09</td>
<td>0.00</td>
</tr>
<tr>
<td>2</td>
<td>15.42</td>
<td>15.42</td>
<td>0.00</td>
</tr>
</tbody>
</table>

Table 17: Comparison of Critical Buckling Loads for CCCC Plate

<table>
<thead>
<tr>
<th>ASPECT RATIO</th>
<th>( N_x ) from present study</th>
<th>( N_x ) from Ritz</th>
<th>Percentage Difference (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.00</td>
<td>109.67</td>
<td>105.28</td>
<td>4.01</td>
</tr>
<tr>
<td>1.10</td>
<td>91.83</td>
<td>88.19</td>
<td>3.95</td>
</tr>
<tr>
<td>1.20</td>
<td>79.85</td>
<td>76.79</td>
<td>3.82</td>
</tr>
<tr>
<td>1.30</td>
<td>71.47</td>
<td>68.87</td>
<td>3.64</td>
</tr>
<tr>
<td>1.40</td>
<td>65.42</td>
<td>63.18</td>
<td>3.43</td>
</tr>
<tr>
<td>1.50</td>
<td>60.93</td>
<td>58.97</td>
<td>3.21</td>
</tr>
<tr>
<td>1.60</td>
<td>57.50</td>
<td>55.78</td>
<td>2.99</td>
</tr>
<tr>
<td>1.70</td>
<td>54.83</td>
<td>53.31</td>
<td>2.77</td>
</tr>
<tr>
<td>1.80</td>
<td>52.72</td>
<td>51.36</td>
<td>2.57</td>
</tr>
<tr>
<td>1.90</td>
<td>51.02</td>
<td>49.80</td>
<td>2.39</td>
</tr>
<tr>
<td>2.00</td>
<td>49.62</td>
<td>48.53</td>
<td>2.21</td>
</tr>
</tbody>
</table>

1.10 CONCLUSIONS

Based on the results of this research, it can be said that:

Since only the basic governing equations of the plates are used and there are no predetermined functions, the present method overcomes the deficiency of the conventional semi-inverse methods thus serves as a completely rational model in solving plate buckling problems.

The strong form of equilibrium of forces expression for plate derived in this study is satisfactory in determining the deformed shape of thin rectangular plates of various boundary conditions.

REFERENCES


