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Buckling Analysis of Uniformly Supported-Edged Thin Plates Using Euler-Bernoulli Approach with Trigonometric Shape Functions

Ibearugbulem O. M[1], Anyaogu L[2], Okorie P. O.[3], Uzoukwu C. S.[4]

[1,2,3,4]Civil Engineering Department, Federal University of Technology Owerri, Imo State, Nigeria

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ABSTRACT

The research presents exact buckling analysis of thin rectangular flat plate using strong (Euler) form of plate equilibrium equation. In this work, the total potential energy equation of a plate was derived from first principles. The derived equation (functional) was differentiated with respect to deflection to obtain the strong form of the equilibrium equation. The strong equilibrium equation was integrated to obtain its exact general solution with unknown coefficients of deflection. The boundary conditions (simple support designated with S and clamp support designated with C) of plates of were satisfied in the exact general solution to obtained particular solutions that are a product of unknown coefficient and exact shape functions. The plates include SSSS (all edges simply supported) and CCCC (all edges clamped). The exact shape functions were substituted into the strong form of equilibrium to obtain the exact stiffness coefficients of plates of various boundary conditions. With the exact shape functions and their corresponding exact stiffness coefficients, exact buckling loads were determined for plates of various aspect ratios. After this, stiffness coefficients and buckling loads were computed using the Ritz approach. The exactness of results obtained for both Euler approach and Ritz approach were tested by directly substituting the results into the Plate strong equilibrium equation. It is observed herein that results from Ritz Approach are not exact and the average percentage differences in computed buckling loads (between Euler and Ritz methods) recorded for SSSS and CCCC are 0.00%, 3.18%, 2.50%, 4.97%, 10.34% and 6.07% respectively. These percentage differences agree with the safety factors of 1.2 to 1.5 applied to the results obtained from approximate methods to augment for their

Symbols: S -- simple support C -- clamped support

a -- Length of the primary dimension of the plate

b -- Width of the secondary dimension of the plate

t -- Tertiary dimension (thickness) of the plate

D -- the flexural Rigidity

Π= Total Potential energy

Keywords: Buckling, Euler-Bernolli, Energy, Differential Equations

Introduction

The uniformly supported edged plate under consideration are SSSS and CCCC. In both cases, the edge conditions are the same. The edge is either SS or CC support, both in the x and y-axis. In each case, the shape functions were derived. The integral value of their shape functions gave their corresponding stiffnesses coefficient. From the first principle , the Total Potential Energy were derived as detailed below

From first principle the Total Potential energy was derived as

$$\Pi = U + w_{kb} \quad 1$$

Considering the U as the Strain and the w_{kb} as the work performed. Mathematically expressed as

$$\mathrm{U} = \frac{\mathrm{D}}{2}\int_0^{\mathrm{a}}\int_0^{\mathrm{b}}\left(\left[\frac{\partial^2 \mathrm{w}}{\partial \mathrm{x}^2}\right]^2 + 2\left[\frac{\partial^2 \mathrm{w}}{\partial x \partial y}\right]^2 + \left[\frac{\partial^2 \mathrm{w}}{\partial \mathrm{y}^2}\right]^2\right)\partial \mathrm{x}\,\partial \mathrm{y} \quad 2$$

and

$$w_{kb} = \frac{1}{2}\int_0^a\int_0^a\left[Nx\left(\frac{\partial w}{\partial x}\right)^2 + 2Nxy\frac{\partial w}{\partial x}\cdot\frac{\partial w}{\partial y} + Ny\left(\frac{\partial w}{\partial y}\right)^2\right]\partial x \partial y \quad 3$$

Bring them together gives

$$\Pi = \frac{D}{2}\int_0^a\int_0^b\left(\left[\frac{\partial^2 w}{\partial x^2}\right]^2 + 2\left[\frac{\partial^2 w}{\partial x\partial y}\right]^2 + \left[\frac{\partial^2 w}{\partial y^2}\right]^2\right)\partial x\,\partial y + \frac{1}{2}\int_0^a\int_0^a\left[Nx\left(\frac{\partial w}{\partial x}\right)^2 + 2Nxy\frac{\partial w}{\partial x}.\frac{\partial w}{\partial y} + Ny\left(\frac{\partial w}{\partial y}\right)^2\right]\partial x\partial y \quad 4$$

But further differentiation of the Total Potential energy with respect to the deflection give the

Euer-Bernoulli formular for calculating the critical buckling load. That is

$$Nx = \bar{N}\left[\frac{D}{a^2}\right] \quad 5$$

where

$$\bar{N} = \frac{k_{xxxx} + \frac{2}{\beta^2}k_{xxyy} + \frac{1}{\beta^4}k_{yyyyy}}{-k_{xx}} \quad 6$$

1.2 Determination of Exact Deflection Functions for Thin Buckling Analysis of Rectangular Plate

To achieve this, some assumptions that will enable, eliminate some of the unknowns were made, since it's very difficult to solve directly. The plate governing equation is a 4th order partial differential equation with 4 unknowns. Also the 4th order Partial differential equation was reduced into simpler differential equations for easy derivation of the solution using the exact approach. On decoupling the Governing Equation and expressing them in exponential forms gives

For x-axis

$$w_R = c_0 + c_1 R + c_2 CoskR + c_3 SinkR \quad 7$$

and differential to First, Second, Third and Fourth order gives

$c_1 - c_2 ksinkR + c_3 kcoskR, \quad -c_2 k^2 coskR - c_3 k^2 sinkR, \quad c_2 k^3 sinkR - c_3 k^3 coskR$ and $c_2 k^4 coskR + c_3 k^4 sinkR$

respectively. Similarly

For y-axis,

$$w_Q = l_0 + l_1 Q + l_2 cosmQ + l_3 sinmQ \quad 8$$

and differential to First, Second, Third and Fourth order gives

$l_1 - l_2 msinmQ + l_3 mcosmQ, \quad -l_2 m^2 cosmQ - l_3 m^2 sinmQ, \quad l_2 m^3 sinmQ - l_3 m^3 cosmQ,$ and $l_2 m^4 cosmQ + l_3 m^4 sinmQ$

Multiplying Equation 7 and 8 together gives

$$w = (c_0 + c_1 R + c_2 coskR + c_3 sinkR)(l_0 + l_1 Q + l_2 cosmQ + l_3 sinmQ) \quad 10$$

1.1 Application of Deflection equation to plates of Selected Boundary conditions.

The derived equation for deflection was introduced into the plate conditions under consideration. That is the The analysis were as detailed below

Particular Deflection equation and shape function for SS (Simple support at opposite ends)

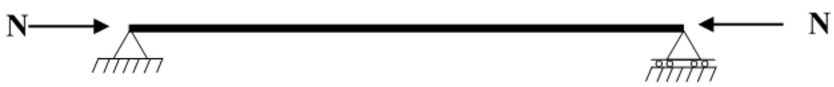

Figure 1: Cross-section of plate Under Buckling Load (N) along x-direction for SS

$Since\ R\ varies\ between\ 0\ and\ 1, for\ between\ x = 0, and\ x = L$

The boundary conditions for the SS beam are stated mathematically as:

$$At\ R = 0, \quad w_R = w_R'' = 0 \quad 11$$

$$At\ R = 1, \quad w_R = w_R'' = 0 \quad 12$$

Where,

$$\frac{d^2 w_R}{dR^2} = w_R'' \quad 13$$

Therefore substituting the boundary conditions into Equation 7, gives:

$$w_R(0) = c_0 + c_1(0) + c_2 cosk(0) + c_3 sink(0) = 0 \quad 14$$

That is:

$$w_R(0) = c_0 + c_2 = 0 \quad 15$$

Hence:

$$c_0 + c_2 = 0 \quad 16$$

From equation 13

$$\frac{d^2 w_R}{dR^2} = w_R'' = -c_2 k^2 coskR - c_3 k^2 sinkR \quad 17$$

Considering the boundary conditions of Equation 11 gives:

$$\frac{d^2 w_R}{dR^2} = w_R'' = -c_2 k^2 cosk(0) - c_3 k^2 sink(0) \quad = 0 \quad 18$$

That is:

$$w_R''(0) = -c_2 k^2 = 0 \quad 19$$

Hence:

$$-c_2 k^2 = 0 \quad 20$$

Again, considering the boundary conditions of Equation 3.149 for Equation 3.134gives:

$$w_R(1) = c_0 + c_1 + c_2 cosk + c_3 sink = 0 \quad 3.156$$

Also, considering the boundary conditions of Equation 11 for Equation 17 gives:

$$w_R''(1) = -c_2 k^2 cosk - c_3 k^2 sink = 0 \quad 21$$

Expressing simultaneous equations 14 to 21 in matrix form will give –

$$\begin{vmatrix} w_R(0) \\ w_R''(0) \\ w_R(1) \\ w_R''(1) \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & -k^2 & 0 \\ 1 & 1 & cosk & sink \\ 0 & 0 & -k^2 cosk & -k^2 sink \end{vmatrix} \begin{vmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{vmatrix} = 0 \quad 22$$

For Equation 322 to be true, the determinant of the square matrix must be zero. That is:

$$\begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & -k^2 & 0 \\ 1 & 1 & cosk & sink \\ 0 & 0 & -k^2 cosk & -k^2 sink \end{vmatrix} = 0 \quad 23$$

That is:

$$1 \times \begin{vmatrix} 0 & -k^2 & 0 \\ 1 & cosk & sink \\ 0 & -k^2 cosk & -k^2 sink \end{vmatrix} - 0 + 1 \times \begin{vmatrix} 0 & 0 & 0 \\ 1 & 1 & sink \\ 0 & 0 & -k^2 sink \end{vmatrix} - 0 = 0$$

That is:

$$\left\{0 + k^2 \times \begin{vmatrix} 1 & sink \\ 0 & -k^2 sink \end{vmatrix} + 0\right\} + \{0\} = 0$$

That is:

$$k^2 \times -k^2 sink = 0.$$

That is:

$$k^4 sink = 0 \quad 24$$

For Equation 24 to be zero, then sin k must be zero. The only condition for this to happen is when k is equal to the product of a positive integer and pi. That is:

$$sink = 0 \; (whenk = n\pi andn = 0, 1, 2, 3 \; etc) \quad 25$$

but

$$\frac{dw_R}{dR} = c_1 - c_2 ksinkR + c_3 kcoskR \quad 26$$

$$\frac{d^2 w_R}{dR^2} = -c_2 k^2 coskR - c_3 k^2 sinkR \quad 27$$

$$\frac{d^3 w_R}{dR^3} = c_2 k^3 sinkR - c_3 k^3 coskR \quad 28$$

$$\frac{d^4 w_R}{dR^4} = c_2 k^4 coskR + c_3 k^4 sinkR \quad 28a$$

Substituting equation 25 into all equations (from Ist to 4^{th} order) and satisfying the boundary conditions from equation 12 gives:

$$c_0 = c_1 = c_2 = 0 \quad 29$$

Substituting the $c_0 + c_1R + c_2coskR + c_3sinkR$ for deflection w_R, gives;

$$w_R = c_3 \sin(n\pi R) \quad 30$$

When a similar procedure is done on the y-direction. The outcome will;

$$w_Q = l_3 \sin(n\pi Q) \quad 31$$

Particular Deflection equation and shape function for CC (Clamp support at opposite ends)

Figure 2: Cross-section of plate Under Buckling Load (N) along x-direction for CC

$$AtR = 0, \quad w_R = w_R{}' = 0 \quad 32$$

$$AtR = 1, \quad w_R = w_R{}' = 0 \quad 33$$

Where:

$$\frac{dw_R}{dR} = w_R{}' \quad 34$$

Substituting Equation 32 boundary conditions into this

$$w_R = c_0 + c_1R + c_2coskR + c_3sinkR \quad 34a$$

gives:

$$w_R(0) = c_0 + c_1(0) + c_2cosk(0) + c_3sink(0) = 0 \quad 35$$

That is:

$$w_R(0) = c_0 + c_2 = 0 \quad 36$$

Therefore:

$$c_0 + c_2 = 0 \quad 37$$

Substituting the boundary conditions of Equation 32 into

$$\frac{dw_R}{dR} = c_1 - c_2ksinkR + c_3kcoskR \quad 37a$$

gives:

$$w_R'(0) = c_1 - c_2kSinkR + c_3kCoskR = 0 \quad 38$$

That is:

$$w_R'(0) = c_1 + c_3k = 0 \quad 39$$

Again considering the boundary conditions of Equation 33 as done above give:

$$w_R(1) = c_0 + c_1 + c_2Cosk + c_3Sink = 0 \quad 40$$

Also:

$$w_R'(1) = c_1 - c_2kSink + c_3kCosk = 0 \quad 41$$

Expressing simultaneous equations 38 to 41 in matrix form will give –

$$\begin{vmatrix} w_R(0) \\ w_R'(0) \\ w_R(1) \\ w_R'(1) \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & k \\ 1 & 1 & Cosk & Sink \\ 0 & 1 & -kSink & kCosk \end{vmatrix} \begin{vmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{vmatrix} = 0 \quad 42$$

For equation 42 to be equal to zero, then :

$$\begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & k \\ 1 & 1 & Cosk & Sink \\ 0 & 1 & -kSink & kCosk \end{vmatrix} = 0 \quad 43$$

That is:

$$1 \times \begin{vmatrix} 1 & 0 & k \\ 1 & Cosk & Sink \\ 1 & -kSink & kCosk \end{vmatrix} - 0 + 1 \times \begin{vmatrix} 0 & 1 & k \\ 1 & 1 & Sink \\ 0 & 1 & kCosk \end{vmatrix} - 0 = 0 \quad 44$$

That is:

$$1 \times \begin{vmatrix} Cosk & Sink \\ -kSink & kCosk \end{vmatrix} - 0 + k \times \begin{vmatrix} 1 & Cosk \\ 1 & -kSink \end{vmatrix} + 0 - 1 \times \begin{vmatrix} 1 & k \\ 1 & kCosk \end{vmatrix} + 0 = 0 \quad 45$$

That is:

$$kCos^2k + \ kSin^2k + k(-kSink - Cosk) - kCosk + k = 0 \quad 46$$

Rearranging gives:

$$Cos^2k + Sin^2k - kSink - 2Cosk + 1 = 0 \quad 47$$

Equation 47 above represents the characteristics equation for the matrix. Solving for k:

$$(Cos^2k + Sin^2k) - kSink - 2Cosk + 1 = 0 \quad 48$$

From Trigonometry:

$$(Cos^2k + Sin^2k) = 0 \quad 49$$

Therefore;

$$1 - kSink - 2Cosk + 1 = 0 \quad 50$$

$$kSink + 2Cosk - 2 = 0 \quad 51$$

The value of k that satisfies equation 51 above is:

$$k \ = \ 2n\pi[where n = 1,2,3, \ldots] \quad 52$$

Substituting the values of k into equations 32 to 41 and satisfying the boundary conditions gives:

$$c_1 = c_3 = 0; \ c_0 = -c_2 \quad 52a$$

Substituting equation 52 and 52a above into the equation of deflection yields;

$$w_R = c_0(1 - Cos2n\pi R) \quad 53$$

When a similar procedure is done on the y-direction. We obtain;

$$w_Q = \ l_0\,(1 - \mathrm{Cos}2\,n\pi Q) \quad 54$$

Table 1 Summary of Deflection Equations of the Shape Orientations

Combined Support Condition	**Equation of Deflection (w)**	**Shape Function (h)**
SS (Pinned at Both ends)	$c_3\,(\sin n\pi R\,)[n = 1,2,3, \ldots]$	$\sin n\pi R\ \ [n = 1,2,3, \ldots]$
CC (Clamped at Both ends)	$c_0(1 - Cos2n\pi R)[n = 1,2,3, \ldots]$	$1 - Cos2n\pi R[n = 1,2,3, \ldots]$

Deflection Equation of plates of Selected Boundary Equation

SSSS Rectangular Plate

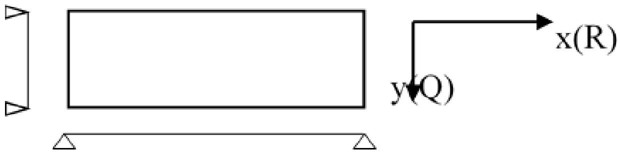

Figure3 : SSSS Rectangular Plate

Considering the x direction, the appropriate deflection equation is that for SS. See Figure

$$w_R = c_3 (\sin n\pi R) \quad 55$$

by virtue of similar support conditions at both sides, the shape function in y direction is the same as that of the x direction.

Hence:

$$w_Q = l_3 (\sin n\pi Q) \quad 56$$

$$Therefore, w = A (\sin n\pi R)(\sin n\pi Q) \quad 57$$

$$where, A = c_3 * l_3 \quad 58$$

CCCC Rectangular Plate

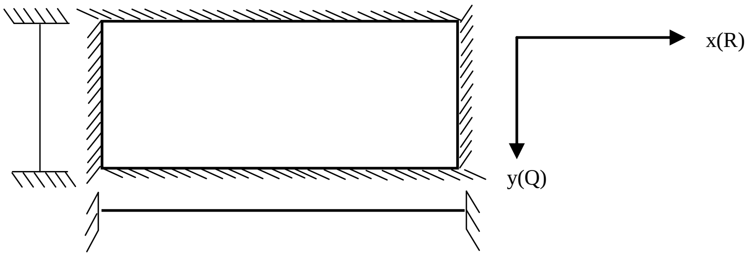

Figure 4: CCCC Rectangular Plate

Similarly:

$$w_R = c_0(1 - Cos2n\pi R) \quad 59$$

And:

$$w_Q = l_0(1 - Cos2n\pi Q) \quad 60$$

Therefore:

$$w = A(1 - Cos2n\pi R)(1 - Cos2n\pi Q) \quad 61$$

Where:

$$A = c_0 * l_0 \quad 62$$

Table 2: Deflection Equation for Various Plates of Different Edge Conditions

S/No.	Edge Condition	Deflection Equation	Constants
	SSSS	$A(\sin\pi R)(\sin\pi Q)$ [Single Mode, n=1]	$A = c_3 * l_3$
	CCCC	$A(1 - Cos2\pi R)(1 - Cos2\pi Q)$[Single Mode, n=1]	$A = c_0 * l_0$

1.4 Determination of the Euler-Bernoulli stiffness coefficients of thin rectangular plates

The stiffness coefficients (k_{xxxx}, k_{xxyy}, k_{yyyy} and k_{xx}) needed to calculate the critical buckling loads of thin rectangular plates considered in this work are determined here. In doing so, the derivatives of the shape functions were obtained and subsequently substituted into the various formulars for the stiffness coefficients. The integrations were done within the domain of effective lengths (spans) in both orthogonal axes.

The Derivatives of the Deflection Equations and Shape Functions

The Derivatives of the SSSS Shape function

$$\mathrm{w} = A(\mathrm{Sin}\,\pi R)(\mathrm{Sin}\,\pi Q) \quad 63$$

$$h = (\mathrm{Sin}\,\pi R)(\mathrm{Sin}\,\pi Q) \quad 64$$

The first derivative of the SSSS deflection equation with respect to R is:

$$\frac{\partial \mathrm{h}}{\partial \mathrm{R}} = \pi(\mathrm{Cos}\pi R)(\mathrm{Sin}\,\pi Q) \quad 65$$

While the second derivative with respect to R is:

$$\frac{\partial^2 \mathrm{h}}{\partial \mathrm{R}^2} = -\pi^2(\mathrm{Sin}\pi R)(\mathrm{Sin}\,\pi Q) \quad 66$$

And the third derivative with respect to R is:

$$\frac{\partial^3 \mathrm{h}}{\partial \mathrm{R}^3} = -\pi^3(\mathrm{Cos}\pi R)(\mathrm{Sin}\,\pi Q) \quad 67$$

It follows that the fourth derivative with respect to R is:

$$\frac{\partial^4 \mathrm{h}}{\partial \mathrm{R}^4} = \pi^4(\mathrm{Sin}\pi R)(\mathrm{Sin}\,\pi Q) \quad 68$$

Similarly, the fourth derivative with respect to Q is:

$$\frac{\partial^4 \mathrm{h}}{\partial \mathrm{Q}^4} = \pi^4 (\mathrm{Sin}\pi R)\,(\mathrm{Sin}\,\pi Q) \quad 69$$

While the Second Partial derivate with respects to R and Q is:

$$\frac{\partial^4 \mathrm{h}}{\partial \mathrm{R}^2\,\partial \mathrm{Q}^2} = \pi^4 (\mathrm{Sin}\pi R)\,(\mathrm{Sin}\,\pi Q) \quad 70$$

The Derivatives of the CCCC Shape function

$$w = A(1 - Cos2\pi R)(1 - Cos2\pi Q) \quad 71$$

$$h = (1 - Cos2\pi R)(1 - Cos2\pi Q) \quad 72$$

The first derivative of the CCCC deflection equation with respect to R is:

$$\frac{\partial h}{\partial R} = 2\pi(Sin2\pi R)(1 - Cos2\pi Q) \quad 73$$

While the second derivative with respect to R is:

$$\frac{\partial^2 h}{\partial R^2} = 4\pi^2(Cos2\pi R)(1 - Cos2\pi Q) \quad 74$$

And the third derivative with respect to R is:

$$\frac{\partial^3 h}{\partial R^3} = -8\pi^3(Sin2\pi R)(1 - Cos2\pi Q) \quad 75$$

It follows that the fourth derivative with respect to R is:

$$\frac{\partial^4 h}{\partial R^4} = -16\pi^4(Cos2\pi R)(1 - Cos2\pi Q) \quad 76$$

Similarly, the fourth derivative with respect to Q is:

$$\frac{\partial^4 h}{\partial Q^4} = -16\pi^4(1 - Cos2\pi R)(Cos2\pi Q) \quad 77$$

While the Second Partial derivate with respects to R and Q is:

$$\frac{\partial^4 h}{\partial R^2 \partial Q^2} = 16\pi^4(Cos2\pi R)(Cos2\pi Q) \quad 78$$

Effective Span and Bent Domain of Single Buckled mode of Line Continuum

When line continuum buckles in first deformation mode, a region within the span takes bent configuration. The span of this bent region is largely dependent on the support conditions of the line continuum. The point the continuum starts to bend is designated as d_1 and the last point when the continuum is bent is designated d_2

Let the Effective Span be L_e

For SS Continuum, $L_e = L$

For CC Continuum, $L_e = 0.5L$

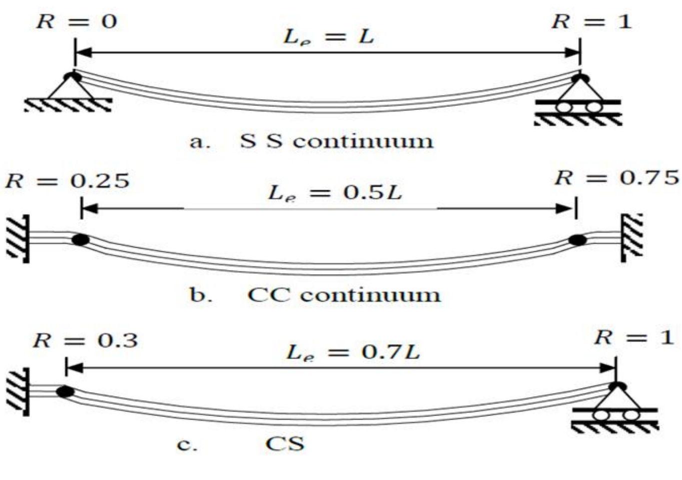

Figure 5 Effective Lengths for various Support conditions

Calculation of the stiffness coefficients for plates of various boundary conditions

The stiffness coefficients were represented as

$$k_{xxxx} = \int_{d_1}^{d_2}\int_{d_1}^{d_2} \frac{\partial^4 \mathrm{w}}{\partial \mathrm{R}^4}\partial R \partial Q \qquad 79$$

$$k_{xxyy} = \int_{d_1}^{d_2}\int_{d_1}^{d_2} \frac{\partial^4 \mathrm{w}}{\partial R^2 \partial Q^2}\partial R \partial Q \qquad 80$$

$$k_{yyyy} = \int_{d_1}^{d_2}\int_{d_1}^{d_2} \frac{\partial^4 \mathrm{w}}{\partial \mathrm{Q}^4}\partial R \partial Q \qquad 81$$

$$k_{xx} = \int_{d_1}^{d_2}\int_{d_1}^{d_2} \frac{\partial^2 w}{\partial R^2}\partial R \partial Q \qquad 82$$

$$k_T = k_{xxxxx} + \frac{2}{\beta^2}k_{xxyy} + \frac{1}{\beta^4}k_{yyyy} \qquad 83$$

We will use these equations to compute stiffness coefficients for the various boundary conditions. The boundary conditions are SSSS, CCCC, CCSS, CSSS, CSCS and CCCS. The stiffness coefficients to be computed include:

$K_{xx}, K_{xxxx}, K_{xxyy}, K_{yyyy}, K_T$

We will also compute the Ratio of stiffness coefficients:

$$\bar{N} = \frac{K_T}{-K_{xx}} \qquad 84$$

As stated in equation 84

The Integrals of the SSSS Deflection Equation

$$For SS, d_1 = 0\,, \qquad d_2 = 1 \qquad 85$$

$$k_{xx} = -\pi^2 \int_0^1\int_0^1 (\mathrm{Sin}\pi R)(\mathrm{Sin}\,\pi Q) \quad dRdQ = -\pi^2\left(-\frac{1}{\pi}Cos\pi R\Big]_0^1\right)\left(-\frac{1}{\pi}Cos\pi Q\Big]_0^1\right) = -\pi^2\left(-\frac{1}{\pi}Cos\pi + \frac{1}{\pi}Cos0\right)\left(-\frac{1}{\pi}Cos\pi + \frac{1}{\pi}Cos0\right)$$

$$= -\pi^2\left(\frac{1}{\pi}+\frac{1}{\pi}\right)\left(\frac{1}{\pi}+\frac{1}{\pi}\right) = -\pi^2\left(\frac{2}{\pi}\right)\left(\frac{2}{\pi}\right) = -4 \qquad 85b \;\; 3.284$$

$$k_{xxxx} = \pi^4 \int_0^1\int_0^1 (\mathrm{Sin}\pi R)(\mathrm{Sin}\,\pi Q) \quad dRdQ = \pi^4\left(\frac{2}{\pi}\right)\left(\frac{2}{\pi}\right) = 4\pi^2 \qquad 86$$

$$k_{yyyy} = \pi^4 \int_0^1\int_0^1 (\mathrm{Sin}\pi R)(\mathrm{Sin}\,\pi Q) \quad dRdQ = \pi^4\left(\frac{2}{\pi}\right)\left(\frac{2}{\pi}\right) = 4\pi^2 \qquad 87$$

$$k_{xxyy} = \pi^4 \int_0^1\int_0^1 (Sin\pi R)(Sin\,\pi Q)\, dRdQ = \pi^4\left(\frac{2}{\pi}\right)\left(\frac{2}{\pi}\right) = 4\pi^2 \qquad 88$$

Further resolution gives

$$k_T = 4\pi^2 + \frac{8}{\beta^2}\pi^2 + \frac{4}{\beta^4}\pi^2 \qquad 88a$$

Substituting back gives:

$$\frac{\mathrm{N}_x a^2}{D} = \frac{4\pi^2 + \frac{8}{\beta^2}\pi^2 + \frac{4}{\beta^4}\pi^2}{4} = \pi^2 + \frac{2}{\beta^2}\pi^2 + \frac{1}{\beta^4}\pi^2 = \pi^2\left(1 + \frac{2}{\beta^2} + \frac{1}{\beta^4}\right) \qquad 89$$

The Integrals of the CCCC Deflection Equation

$$For CC, d_1 = 0.25\,, \qquad d_2 = 0.75 \qquad 90$$

$$k_{xx} = 4\pi^2 \int_{0.25}^{0.75}\int_{0.25}^{0.75} (Cos2\pi R)(1 - Cos2\pi Q)dRdQ = 4\pi^2 \left(\frac{1}{2\pi}Sin2\pi R\right]_{0.25}^{0.75}\left) \left(Q - \frac{1}{2\pi}Sin2\pi Q\right]_{0.25}^{0.75}\right) \quad 90$$

$$= 4\pi^2 \left(\frac{1}{2\pi}Sin2\pi(0.75) - \frac{1}{2\pi}Sin2\pi(0.25)\right)\left(0.75 - \frac{1}{2\pi}Sin2\pi 0.75 - 0.25 + \frac{1}{2\pi}Sin2\pi 0.25\right)$$

$$= 4\pi^2 \left(\frac{-1}{2\pi} - \frac{1}{2\pi}\right)\left(0.75 + \frac{1}{2\pi} - 0.25 + \frac{1}{2\pi}\right) = 4\pi^2\left(\frac{-1}{\pi}\right)\left(0.5 + \frac{1}{\pi}\right) = 4\pi^2\left(\frac{-1}{2\pi} - \frac{1}{\pi^2}\right)$$

$$= -2\pi - 4 \qquad 90a\ 3.291$$

$$k_{xxxx} = -16\pi^4 \int_{0.25}^{0.75}\int_{0.25}^{0.75} (Cos2\pi R)(1 - Cos2\pi Q)dRdQ == -16\pi^4 \left(\frac{1}{2\pi}Sin2\pi R\right]_{0.25}^{0.75}\left) \left(Q - \frac{1}{2\pi}Sin2\pi Q\right]_{0.25}^{0.75}\right)$$

$$= -16\pi^4 \left(\frac{1}{2\pi}Sin2\pi(0.75) - \frac{1}{2\pi}Sin2\pi(0.25)\right)\left(0.75 - \frac{1}{2\pi}Sin2\pi 0.75 - 0.25 + \frac{1}{2\pi}Sin2\pi 0.25\right)$$

$$= -16\pi^4 \left(\frac{-1}{2\pi} - \frac{1}{2\pi}\right)\left(0.75 + \frac{1}{2\pi} - 0.25 + \frac{1}{2\pi}\right) = -16\pi^4\left(\frac{-1}{\pi}\right)\left(0.5 + \frac{1}{\pi}\right) = -16\pi^4\left(\frac{-1}{2\pi} - \frac{1}{\pi^2}\right)$$

$$= 8\pi^3 + 16\pi^2 \qquad 91$$

$$k_{xxyy} = 16\pi^4 \int_0^1\int_0^1 (Cos2\pi R)(Cos2\pi Q)dRdQ = 16\pi^4 \left(\frac{1}{2\pi}Sin2\pi R\right]_{0.25}^{0.75}\left)\left(\frac{1}{2\pi}Sin2\pi Q\right]_{0.25}^{0.75}\right)$$

$$= 16\pi^4 \left(\frac{1}{2\pi}Sin2\pi(0.75) - \frac{1}{2\pi}Sin2\pi(0.25)\right)\left(\frac{1}{2\pi}Sin2\pi(0.75) - \frac{1}{2\pi}Sin2\pi(0.25)\right) = 16\pi^4\left(\frac{-1}{2\pi} - \frac{1}{2\pi}\right)\left(\frac{-1}{2\pi} - \frac{1}{2\pi}\right)$$

$$= 16\pi^4\left(\frac{-1}{\pi}\right)\left(\frac{-1}{\pi}\right) = 16\pi^2 \qquad 91a$$

$$k_{yyyy} = -16\pi^4 \int_0^1\int_0^1 (1 - Cos2\pi R)(Cos2\pi Q)dRdQ = -16\pi^4 \left(R - \frac{1}{2\pi}Sin2\pi R\right]_{0.25}^{0.75}\left)\left(\frac{1}{2\pi}Sin2\pi Q\right]_{0.25}^{0.75}\right) =$$

$$= -16\pi^4 \left(0.75 - \frac{1}{2\pi}Sin2\pi 0.75 - 0.25 + \frac{1}{2\pi}Sin2\pi 0.25\right)\left(\frac{1}{2\pi}Sin2\pi(0.75) - \frac{1}{2\pi}Sin2\pi(0.25)\right)$$

$$= -16\pi^4 \left(0.75 + \frac{1}{2\pi} - 0.25 + \frac{1}{2\pi}\right)\left(\frac{-1}{2\pi} - \frac{1}{2\pi}\right) = -16\pi^4\left(0.5 + \frac{1}{\pi}\right)\left(\frac{-1}{\pi}\right) = -16\pi^4\left(-\frac{1}{\pi^2} - \frac{1}{2\pi}\right)$$

$$= 8\pi^3 + 16\pi^2 \qquad 91b$$

Substituting Equations 91 to 91b into

$$k_T = k_{xxxxx} + \frac{2}{\beta^2}k_{xxyy} + \frac{1}{\beta^4}k_{yyyy} \qquad 91c$$

gives:

$$k_T = 8\pi^3 + 16\pi^2 + \frac{2}{\beta^2}16\pi^2 + \frac{1}{\beta^4}(8\pi^3 + 16\pi^2) \qquad 92$$

Substituting Equations 90a and 92 into Equation 91c gives:

$$\frac{N_x a^2}{D} = \frac{8\pi^3 + 16\pi^2 + \frac{2}{\beta^2}16\pi^2 + \frac{1}{\beta^4}(8\pi^3 + 16\pi^2)}{2\pi + 4} \qquad 93$$

Table 1: Summary of Stiffness coefficients in terms of π

INTEGRAL	**SSSS**	**CCCC**
k_{xx}	-4	$-2\pi - 4$
k_{xxxx}	$4\pi^2$	$8\pi^3 + 16\pi^2$
k_{xxyy}	$4\pi^2$	$16\pi^2$
k_{yyyy}	$4\pi^2$	$8\pi^3 + 16\pi^2$

Table 2: Summary of Stiffness Coefficients for all plate types

INTEGRAL	**SSSS**	**CCCC**
k_{xx}	-4.00000	-10.28319
k_{xxxx}	39.478418	405.96388
k_{xxyy}	39.478418	157.91367
k_{yyyy}	39.478418	405.96388

Table 3: Stiffness Coefficients for different Aspect ratios for SSSS Plates

b/a	k_{xxxx}	k_{xxyy}	k_{yyyy}	k_{xx}	k_T	$\bar{N}$
1	39.47842	39.47842	39.47842	-4.00000	157.91367	39.47842
1.1	39.47842	39.47842	39.47842	-4.00000	131.69629	32.92407
1.2	39.47842	39.47842	39.47842	-4.00000	113.34814	28.33704
1.3	39.47842	39.47842	39.47842	-4.00000	100.02093	25.00523
1.4	39.47842	39.47842	39.47842	-4.00000	90.03907	22.50977
1.5	39.47842	39.47842	39.47842	-4.00000	82.36855	20.59214
1.6	39.47842	39.47842	39.47842	-4.00000	76.34486	19.08621
1.7	39.47842	39.47842	39.47842	-4.00000	71.52589	17.88147
1.8	39.47842	39.47842	39.47842	-4.00000	67.60852	16.90213
1.9	39.47842	39.47842	39.47842	-4.00000	64.37944	16.09486
2	39.47842	39.47842	39.47842	-4.00000	61.68503	15.42126

Table 4: Stiffness Coefficients for different Aspect ratios for CCCC Plates

b/a	k_{xxxx}	k_{xxyy}	k_{yyyy}	k_{xx}	k_T	$\bar{N}$
1	405.96388	157.91367	405.96388	-10.28319	1127.75511	109.66982
1.1	405.96388	157.91367	405.96388	-10.28319	944.25701	91.82534
1.2	405.96388	157.91367	405.96388	-10.28319	821.06576	79.84547
1.3	405.96388	157.91367	405.96388	-10.28319	734.98321	71.47427
1.4	405.96388	157.91367	405.96388	-10.28319	672.77601	65.42487
1.5	405.96388	157.91367	405.96388	-10.28319	626.52199	60.92684
1.6	405.96388	157.91367	405.96388	-10.28319	591.27911	57.49961
1.7	405.96388	157.91367	405.96388	-10.28319	563.85291	54.83251
1.8	405.96388	157.91367	405.96388	-10.28319	542.11352	52.71844
1.9	405.96388	157.91367	405.96388	-10.28319	524.60175	51.01549
2	405.96388	157.91367	405.96388	-10.28319	510.29346	49.62407

1.6 Determination of Ritz Stiffness Co-efficients and Buckling loads

In this section, we determine critical buckling loads and stiffness co-efficients as used in the Ritz approach. Recall Ritz formulation for calculating the Critical Buckling load of a thin Rectangular Plate from Equation 2.7:

$$Nx = \frac{D\left[\int_0^a\int_0^b\left[\frac{\partial^2 H}{\partial R^2}\right]^2 \partial R\,\partial Q + 2\frac{1}{\beta^2}\int_0^a\int_0^b\left[\frac{\partial^2 H}{\partial R\,\partial Q}\right]^2 \partial R\,\partial Q \;+\frac{1}{\beta^4}\int_0^a\int_0^b\left[\frac{\partial^2 H}{\partial Q^2}\right]^2 \partial R\,\partial Q\right]}{a^2\int_0^a\int_0^b\left[\frac{\partial H}{\partial R}\right]^2 \partial R\partial Q} \qquad 94$$

For Simplicity, Let;

$$k_{xx} = \int_0^a\int_0^b\left[\frac{\partial^2 H}{\partial R^2}\right]^2 \partial R\,\partial Q \qquad 95$$

$$k_{xy} = \int_0^a\int_0^b\left[\frac{\partial^2 H}{\partial R\,\partial Q}\right]^2 \partial R\,\partial Q \qquad 97$$

$$k_{yy} = \int_0^a\int_0^b\left[\frac{\partial^2 H}{\partial Q^2}\right]^2 \partial R\,\partial Q \qquad 98$$

$$k_N = \int_0^a\int_0^b\left[\frac{\partial H}{\partial R}\right]^2 \partial R\partial Q \qquad 99$$

Hence, the critical buckling load becomes:

$$Nx = \frac{D\left[k_{xx} + 2\frac{1}{\beta^2}k_{xy} \;+\frac{1}{\beta^4}k_{yy}\right]}{a^2.k_N} \qquad 100$$

We will then proceed to compute these stiffness coefficients (k_{xx}, k_{xy}, k_{yy} and k_N) needed to calculate the Ritz critical buckling loads of thin rectangular plates. In doing so, the derivatives of the shape functions obtained will be determined as required by Ritz Formulation.

The integrations were done within the domain of 0 to 1 in both orthogonal axes, as required by Ritz Formulation.

The Ritz Derivatives of the Deflection Equations

The derivatives of the shape functions were determined. These derivatives will be Computed as required by Ritz Formulation

The Ritz Derivative for the SSSS Shape Functions

$$h = (\mathrm{Sin}\,\pi R\,)\;(\mathrm{Sin}\,\pi Q\,) \qquad 101$$

The first derivative of the SSSS deflection equation with respect to R is:

$$\frac{\partial h}{\partial R} = \pi(\mathrm{Cos}\pi R)\,(\mathrm{Sin}\,\pi Q) \qquad 102$$

The Square of the first derivative with respect to R is

$$\left[\frac{\partial h}{\partial R}\right]^2 = \pi^2(cos^2\pi R)(sin^2\pi Q) \qquad 103$$

While the second derivative with respect to R is:

$$\frac{\partial^2 h}{\partial R^2} = -\pi^2(\mathrm{Sin}\pi R)\,(\mathrm{Sin}\,\pi Q) \qquad 104$$

And the The Square of the Second derivative with respect to R is:

$$\left[\frac{\partial^2 h}{\partial R^2}\right]^2 = \pi^4(sin^2\pi R)(sin^2\pi Q) \qquad 105$$

It follows that the Second derivative with respect to Q is:

$$\frac{\partial^2 h}{\partial Q^2} = -\pi^2(sin\pi R)(sin\pi Q) \qquad 106$$

Similarly, the square of the Second derivative with respect to Q is:

$$\left[\frac{\partial^2 H}{\partial Q^2}\right]^2 = \pi^4(sin^2\pi R)(cos^2\pi Q) \qquad 107$$

While the First Partial derivate with respects to R and Q is:

$$\frac{\partial^2 H}{\partial R \partial Q} = \pi^2 (cos\pi R)(cos\pi Q) \qquad 108$$

While the Square of the First Partial derivate with respects to R and Q is:

$$\left[\frac{\partial^2 h}{\partial R \partial Q}\right]^2 = 16\pi^4 (sin^2 2\pi R)(sin^2 2\pi Q) \qquad 109$$

The Ritz Derivative for the CCCC Shape Functions

$$h = (1 - cos2\pi R)(1 - cos2\pi Q) \qquad 110$$

The first derivative of the CCCC deflection equation with respect to R is:

$$\frac{\partial \mathrm{h}}{\partial \mathrm{R}} = 2\pi (sin2\pi R)(1 - cos2\pi Q) \qquad 111$$

The Square of the first derivative with respect to R is

$$\left[\frac{\partial h}{\partial R}\right]^2 = 4\pi^2 (sin^2 2\pi R)(1 - cos2\pi Q)^2 \qquad 112$$

While the second derivative with respect to R is:

$$\frac{\partial^2 h}{\partial R^2} = 4\pi^2 (cos2\pi R)(1 - cos2\pi Q) \qquad 113$$

And the Square of the Second derivative with respect to R is:

$$\left[\frac{\partial^2 h}{\partial R^2}\right]^2 = 16\pi^4 (cos^2 2\pi R)(1 - cos2\pi Q)^2 \qquad 114$$

It follows that the Second derivative with respect to Q is:

$$\frac{\partial^2 h}{\partial Q^2} = 4\pi^2 (1 - cos2\pi R)(cos\pi Q) \qquad 115$$

Similarly, the square of the Second derivative with respect to Q is:

$$\left[\frac{\partial^2 h}{\partial Q^2}\right]^2 = 16\pi^4 (1 - cos2\pi R)^2 (cos^2 2\pi Q) \qquad 116$$

While the First Partial derivate with respects to R and Q is:

$$\frac{\partial^2 h}{\partial R \partial Q} = 4\pi^2 (sin2\pi R)(sin2\pi Q) \qquad 117$$

While the Square of the First Partial derivate with respects to R and Q is:

$$\left[\frac{\partial^2 h}{\partial R \partial Q}\right]^2 = 16\pi^4 (sin^2 2\pi R)(sin^2 2\pi Q) \qquad 118$$

1.7 RESULTS AND DISCUSSIONS

Presentation of Results

In this section, the formulations obtained in this work are presented. Among these are the general expression of Total Potential Energy for plate under buckling load, the Euler-Bernoulli functional for Buckling analysis of thin rectangular plates, exact deflection functions for thin rectangular plates and the Euler-Bernoulli Stiffness coefficients for thin rectangular plates. Furthermore, the comparative results of tests for exact buckling loads and approximate buckling loads in the strong form of the governing equation for plates of selected boundary conditions are presented.

The results are presented for the two types of support conditions of thin rectangular plates: SSSS and CCCC The results for the non-dimensional buckling load are evaluated for different aspect ratios, β (b/a: where, $1 \leq \beta \leq 2$).

The Total Potential Energy Functional for Thin Rectangular Plate under Buckling Load

The general expression for the total potential energy for thin rectangular plate under buckling load (biaxial and in-plane shear load) as obtained in this work is presented in Equation 4.1

$$\Pi = \frac{\mathrm{D}}{2}\int_0^{\mathrm{a}}\int_0^{\mathrm{b}}\left(\left[\frac{\partial^2 \mathrm{w}}{\partial \mathrm{x}^2}\right]^2 + 2\left[\frac{\partial^2 \mathrm{w}}{\partial x \partial y}\right]^2 + \left[\frac{\partial^2 \mathrm{w}}{\partial \mathrm{y}^2}\right]^2\right)\partial \mathrm{x}\,\partial \mathrm{y} + \frac{1}{2}\int_0^{a}\int_0^{a}\left[Nx\left(\frac{\partial w}{\partial x}\right)^2 + 2Nxy\frac{\partial w}{\partial x}.\frac{\partial w}{\partial y} + Ny\left(\frac{\partial w}{\partial y}\right)^2\right]\partial x \partial y \qquad 4.1$$

The Euler-Bernoulli Formula for Buckling Analysis of Thin Rectangular Plate

The Euler-Bernoulli Formula for Buckling analysis of thin Rectangular plate as obtained in this work is presented in Equation 4.2

$$N_x = \frac{k_T}{-k_{xx}}\left[\frac{D}{\mathrm{a}^2}\right] \qquad 4.2$$

Where:

$$k_T = k_{xxxx} + \frac{2}{\beta^2}k_{xxyy} + \frac{1}{\beta^4}k_{yyyy}$$

$$k_{xx} = \int_{d_1}^{1}\int_{d_1}^{1}\left(\frac{\partial^2 w}{\partial R^2}\right)\partial R\partial Q, k_{xxxx} = \int_{d_1}^{d_2}\int_{d_1}^{d_2}\left(\frac{\partial^4 \mathrm{w}}{\partial \mathrm{R}^4}\right)\partial \mathrm{R}\,\partial \mathrm{Q}, k_{xxyy} = \int_{d_1}^{d_2}\int_{d_1}^{d_2}\left(\frac{\partial^4 \mathrm{w}}{\partial R^2\partial Q^2}\right)\partial \mathrm{R}\,\partial \mathrm{Q} \text{ and } k_{yyyy} = \int_{d_1}^{d_2}\int_{d_1}^{d_2}\left(\frac{\partial^4 \mathrm{w}}{\partial \mathrm{Q}^4}\right)\partial \mathrm{R}\,\partial \mathrm{Q}$$

Results for the exact deflection functions for buckling analysis of thin rectangular plate

Equations presented in For SSSS, $w = A(\sin \pi R)(\sin \pi Q)$ [Single Mode, n=1], with Constant as $A = c_3 * l_3$

Also for CCCC, $w = A(1 - Cos2\pi R)(1 - Cos2\pi Q)$[Single Mode, n=1] with Constant as $A = c_0 * l_0$ are the exact deflection equations of Kirchhoff's Plates for the two selected boundary conditions. The Equations are in trigonometric form.

For SSSS, $w = A(\sin \pi R)(\sin \pi Q)$ [Single Mode, n=1], with Constant as $A = c_3 * l_3$

Also for CCCC, $w = A(1 - Cos2\pi R)(1 - Cos2\pi Q)$[Single Mode, n=1] with Constant as $A = c_0 * l_0$

1.8 Results for the Euler-Bernoulli Stiffness coefficients for Buckling Analysis of Thin Rectangular Plates

Results for Euler-Bernoulli Stiffness coefficients as computed are presented here, from Table to **Error! Reference source not found.**. First of all, the summary of primary stiffness coefficients are presented. Then followed by stiffness coefficients generated by considering different aspect ratios.

Table 7: Summary of Euler Stiffness Coefficients for Plates with different Support Conditions

INTEGRAL	**SSSS**	**CCCC**
k_{xx}	-4.00000	-10.28319
k_{xxxx}	39.478418	405.96388
k_{xxyy}	39.478418	157.91367
k_{yyyy}	39.478418	405.96388

Table 8: Euler Stiffness Coefficients for different Aspect ratios for SSSS Plates

b/a	k_{xxxx}	k_{xxyy}	k_{yyyy}	k_{xx}	k_T	$\bar{N}$
1	39.47842	39.47842	39.47842	-4.00000	157.91367	39.47842
1.1	39.47842	39.47842	39.47842	-4.00000	131.69629	32.92407
1.2	39.47842	39.47842	39.47842	-4.00000	113.34814	28.33704
1.3	39.47842	39.47842	39.47842	-4.00000	100.02093	25.00523
1.4	39.47842	39.47842	39.47842	-4.00000	90.03907	22.50977
1.5	39.47842	39.47842	39.47842	-4.00000	82.36855	20.59214
1.6	39.47842	39.47842	39.47842	-4.00000	76.34486	19.08621
1.7	39.47842	39.47842	39.47842	-4.00000	71.52589	17.88147
1.8	39.47842	39.47842	39.47842	-4.00000	67.60852	16.90213
1.9	39.47842	39.47842	39.47842	-4.00000	64.37944	16.09486
2	39.47842	39.47842	39.47842	-4.00000	61.68503	15.42126

Table 9: Euler Stiffness Coefficients for different Aspect ratios for CCCC Plates

b/a	k_{xxxx}	k_{xxyy}	k_{yyyy}	k_{xx}	k_T	$\bar{N}$
1	405.96388	157.91367	405.96388	-10.28319	1127.75511	109.66982
1.1	405.96388	157.91367	405.96388	-10.28319	944.25701	91.82534

1.2	405.96388	157.91367	405.96388	-10.28319	821.06576	79.84547
1.3	405.96388	157.91367	405.96388	-10.28319	734.98321	71.47427
1.4	405.96388	157.91367	405.96388	-10.28319	672.77601	65.42487
1.5	405.96388	157.91367	405.96388	-10.28319	626.52199	60.92684
1.6	405.96388	157.91367	405.96388	-10.28319	591.27911	57.49961
1.7	405.96388	157.91367	405.96388	-10.28319	563.85291	54.83251
1.8	405.96388	157.91367	405.96388	-10.28319	542.11352	52.71844
1.9	405.96388	157.91367	405.96388	-10.28319	524.60175	51.01549
2	405.96388	157.91367	405.96388	-10.28319	510.29346	49.62407

Results for the Ritz Stiffness coefficients for Buckling Analysis of Thin Rectangular Plates

Results for Ritz Stiffness coefficients as computed, are presented here, from Table 10 and **Error! Reference source not found.**. First of all, the summary of primary stiffness coefficients are presented, then followed by stiffness coefficients generated by considering different aspect ratios.

Table 10: Summary of Ritz Stiffness Coefficients for Plates with different Support Conditions

INTEGRAL	SSSS	CCCC
k_{xx}	2.46740	29.60881
k_{xxxx}	24.35227	1168.90909
k_{xxyy}	24.35227	389.63636
k_{yyyy}	24.35227	1168.90909

Table 11: Ritz Stiffness Coefficients for different Aspect ratios for SSSS Plates

b/a	k_{xxxx}	k_{xxyy}	k_{yyyy}	k_{xx}	k_T	$\bar{N}$
1	24.35227	24.35227	24.35227	2.46740	97.40909	39.47842
1.1	24.35227	24.35227	24.35227	2.46740	81.23689	32.92407
1.2	24.35227	24.35227	24.35227	2.46740	69.91883	28.33704
1.3	24.35227	24.35227	24.35227	2.46740	61.69794	25.00523
1.4	24.35227	24.35227	24.35227	2.46740	55.54063	22.50977
1.5	24.35227	24.35227	24.35227	2.46740	50.80906	20.59214
1.6	24.35227	24.35227	24.35227	2.46740	47.09335	19.08621
1.7	24.35227	24.35227	24.35227	2.46740	44.12076	17.88147
1.8	24.35227	24.35227	24.35227	2.46740	41.70433	16.90213
1.9	24.35227	24.35227	24.35227	2.46740	39.71247	16.09486
2	24.35227	24.35227	24.35227	2.46740	38.05043	15.42126

Table 12: Ritz Stiffness Coefficients for different Aspect ratios for CCCC Plates

b/a	k_{xxxx}	k_{xxyy}	k_{yyyy}	k_{xx}	k_T	$\bar{N}$
1	1168.90909	389.63636	1168.90909	29.60881	3117.09091	105.27578
1.1	1168.90909	389.63636	1168.90909	29.60881	2611.31678	88.19390
1.2	1168.90909	389.63636	1168.90909	29.60881	2273.78073	76.79405
1.3	1168.90909	389.63636	1168.90909	29.60881	2039.28478	68.87425
1.4	1168.90909	389.63636	1168.90909	29.60881	1870.77385	63.18301
1.5	1168.90909	389.63636	1168.90909	29.60881	1746.14815	58.97393
1.6	1168.90909	389.63636	1168.90909	29.60881	1651.67387	55.78318
1.7	1168.90909	389.63636	1168.90909	29.60881	1578.50755	53.31208

1.8	1168.90909	389.63636	1168.90909	29.60881	1520.77549	51.36226
1.9	1168.90909	389.63636	1168.90909	29.60881	1474.46872	49.79831
2	1168.90909	389.63636	1168.90909	29.60881	1436.78409	48.52555

Table 13: Summary of Approximate Critical Buckling Loads from Ritz Method

Aspect Ratio	**SSSS**	**CCCC**
1	39.48	105.28
1.1	32.92	88.19
1.2	28.34	76.79
1.3	25.01	68.87
1.4	22.51	63.18
1.5	20.59	58.97
1.6	19.09	55.78
1.7	17.88	53.31
1.8	16.90	51.36
1.9	16.09	49.80
2	15.42	48.53

1.9 COMPARISM OF THE PRESENT WITH THAT OF PREVIOUS

Results from Testing the Exact Critical Buckling Loads and Values from Ritz Method in The Strong Form of The Governing Equation for Plates of Selected Boundary Conditions.

In this section results obtained by substituting the exact and approximate non-dimensional buckling loads into the Strong Form of the equilibrium equation are presented in Table 1Error**! No text of specified style in document.**

Table 1**Error! No text of specified style in document.**: Results of resultant force from Exact Critical Buckling Loads from this Study

Aspect	**SSSS**	**CCCC**
1	0.00	0.00
1.1	0.00	0.00
1.2	0.00	0.00
1.3	0.00	0.00
1.4	0.00	0.00
1.5	0.00	0.00
1.6	0.00	0.00
1.7	0.00	0.00
1.8	0.00	0.00
1.9	0.00	0.00
2	0.00	0.00

Table 15: Test Results for Approximate Critical Buckling Loads from Ritz Method

Aspect Ratio	**SSSS**	**CCCC**
1	194.82	6234.18
1.1	162.47	5222.63
1.2	139.84	4547.56
1.3	123.40	4078.57
1.4	111.08	3741.55
1.5	101.62	3492.30
1.6	94.19	3303.35
1.7	88.24	3157.02
1.8	83.41	3041.55

1.9	79.42	2948.94
2	76.10	2873.57

Table 16: Comparison of Critical Buckling Loads for SSSS Plate

SSSS Plate			
ASPECT RATIO	**$\bar{N}$ from present study**	**$\bar{N}$ from Ritz**	**Percentage Difference (%)**
1	39.48	39.48	0.00
1.1	32.92	32.92	0.00
1.2	28.34	28.34	0.00
1.3	25.01	25.01	0.00
1.4	22.51	22.51	0.00
1.5	20.59	20.59	0.00
1.6	19.09	19.09	0.00
1.7	17.88	17.88	0.00
1.8	16.90	16.90	0.00
1.9	16.09	16.09	0.00
2	15.42	15.42	0.00

Table 17: Comparison of Critical Buckling Loads for CCCC Plate

CCCC Plate			
ASPECT RATIO	**Nx from present study**	**Nx from Ritz**	**Percentage Difference (%)**
1.00	109.67	105.28	4.01
1.10	91.83	88.19	3.95
1.20	79.85	76.79	3.82
1.30	71.47	68.87	3.64
1.40	65.42	63.18	3.43
1.50	60.93	58.97	3.21
1.60	57.50	55.78	2.99
1.70	54.83	53.31	2.77
1.80	52.72	51.36	2.57
1.90	51.02	49.80	2.39
2.00	49.62	48.53	2.21

1.10 CONCLUSIONS

Based on the results of this research, it can be said that:

Since only the basic governing equations of the plates are used and there are no predetermined functions, the present method overcomes the deficiency of the conventional semi-inverse methods thus serves as a completely rational model in solving plate buckling problems.

The strong form of equilibrium of forces expression for plate derived in this study is satisfactory in determining the deformed shape of thin rectangular plates of various boundary conditions.

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