International Journal of Research Publication and Reviews

Journal homepage: www.ijrpr.com ISSN 2582-7421

Observations on Homogeneous Bi-Quadratic Equation with Five Unknowns $(x^4 - y^4) + 2(x-y)(x^3+y^3) = 36(z^2 - w^2)P^2$

S. Mallika [1], M. A. Gopalan [2]

[1]Assistant Professor, Department of Mathematics, Shrimati Indira Gandhi College, Affiliated to Bharathidasan University, Trichy-620 002, Tamil Nadu, India.

[2] Professor, Department of Mathematics, Shrimati Indira Gandhi College, Affiliated to Bharathidasan University, Trichy-620 002, Tamil Nadu, India.

ABSTRACT

This paper concerns with the problem of determining non-trivial integral solutions of the homogeneous bi-quadratic equation with five unknowns given by $(x^4 - y^4) + 2(x-y)(x^3+y^3) = 36(z^2 - w^2)P^2$.We obtain infinitely many non-zero integer solutions of the equation by introducing the linear transformations.

Keywords: Bi-quadratic equation with five unknowns, integral solutions, homogeneous bi-quadratic, Linear Transformations.

Introduction

The theory of Diophantine equations offers a rich variety of fascinating problems. In particular, bi-quadratic diophantine equations, homogeneous and non-homogeneous have aroused the interest of numerous mathematicians since antiquity[1-5]. In this context, one may refer [6-14] for various problems on the bi-quadratic diophantine equations with five variables. However, often we come across homogeneous bi-quadratic equations and as such one may require its integral solution in its most general form. It is towards this end, this paper concerns with the problem of determining non-trivial integral solutions of the homogeneous equation with five unknowns given by $(x^4 - y^4) + 2(x-y)(x^3+y^3) = 36(z^2 - w^2)P^2$

Method of analysis:

The homogeneous bi-quadratic diophantine equation with five unknowns under consideration is

$$(x^4 - y^4) + 2(x-y)(x^3+y^3) = 36(z^2 - w^2)P^2 \quad (1)$$

Introduction of the linear transformations

$$x = u + 2v, y = u - 2v, z = 2uv + 1, w = 2uv - 1 \quad (2)$$

in (1) leads to

$$u^2 + 8v^2 = 9P^2 \quad (3)$$

which can be solved through different methods in view of (2), different patterns of integer solutions to (1) are obtained.

Pattern.1

Assume

$$P = a^2 + 8b^2 \quad (4)$$

Write 9 as

$$9 = (1 + i\sqrt{8})(1 - i\sqrt{8}) \quad (5)$$

Using (4) and (5) in (3) and employing the method of factorization ,we get

$$(u + \sqrt{8}iv)(u - \sqrt{8}iv) = (1 + i\sqrt{8})(1 - i\sqrt{8})(a + i\sqrt{8}b)^2(a - i\sqrt{8}b)^2$$

On equating the positive and negative factors, we have ,

$$(u + i\sqrt{8}v) = (1 + i\sqrt{8})(a + i\sqrt{8}b)^2$$

$$(u - \sqrt{8}iv) = (1 - i\sqrt{8})(a - i\sqrt{8}b)^2$$

On equating real and imaginary parts, we get,

$$u = u(a,b) = a^2 - 8b^2 - 16ab$$

$$v = v(a,b) = a^2 - 8b^2 + 2ab$$

Substituting the values of $u$and v in (2) ,we get the values of $x, y, z$and w as

$$\left.\begin{array}{l} x = x(a,b) = 3a^2 - 24b^2 - 12ab \\ y = y(a,b) = -a^2 + 8b^2 - 20ab \\ z = z(a,b) = 2a^4 - 96a^2b^2 - 28a^3b + 224ab^3 + 128b^4 + 1 \\ w = w(a,b) = 2a^4 - 96a^2b^2 - 28a^3b + 224ab^3 + 128b^4 - 1 \end{array}\right\} \quad (6)$$

Thus (4) and (6) represent non-zero distinct integer solutions of (1).

Note:1 It is worth to mention here that 9 may be represented as follows:

$$9 = \frac{(7+2i\sqrt{8})(7-2i\sqrt{8})}{3^2} \quad (7)$$

Following the analysis as that of Pattern I ,one may obtain different set of integer solution to (1) as

$$x = x(A,B) = 33A^2 - 264B^2 - 12AB$$

$$y = y(A,B) = 9A^2 - 72B^2 - 180AB$$

$$z = z(A,B) = 252A^4 + 16128B^4 - 12096A^2B^2 + 612A^3B - 4896AB^3 + 1$$

$$w = w(A,B) = 252A^4 + 16128B^4 - 12096A^2B^2 + 612A^3B - 4896AB^3 - 1$$

$$P = P(A,B) = 9A^2 + 72B^2$$

Pattern. II

Rewrite the equation (3) as

$$9P^2 - 8v^2 = u^2 * 1 \quad (8)$$

Write 1 as

$$1 = (3+\sqrt{8})(3-\sqrt{8}) \quad (9)$$

Assume

$$u = 9a^2 - 8b^2 = (3a+\sqrt{8}b)(3a-\sqrt{8}b) \quad (10)$$

Using (10) and (9) in (8) and using the method of factorization, we get

$$(3P+\sqrt{8}v)(3P-\sqrt{8}v) = (3a+\sqrt{8}b)^2(3a-\sqrt{8}b)^2\big((3+\sqrt{8})(3-\sqrt{8})\big)$$

Equating the positive and negative parts, we have

$$(3P+\sqrt{8}v) = (3+\sqrt{8})(3a+\sqrt{8}b)^2 \quad (11)$$

$$(3P-\sqrt{8}v) = (3-\sqrt{8})\,(3a-\sqrt{8}b)^2 \quad (12)$$

Equating the coefficients of rational and irrational parts, either in (11) or in (12),we get

$$P = 9a^2 + 8b^2 + 16ab \quad (13)$$

$$v = 9a^2 + 8b^2 + 18ab \quad (14)$$

Substituting the values of u, v ,the non-zero distinct integral solutions of (1) are obtained as

$$x = x(a,b) = 27a^2 + 8b^2 + 36ab$$

$$y = y(a,b) = -9a^2 - 24b^2 - 36ab$$

$$z = z(a,b) = 162a^4 - 128b^4 + 324a^3b - 288ab^3 + 1$$

$$w = w(a,b) = 162a^4 - 128b^4 + 324a^3b - 288ab^3 - 1$$

$$P = P(a,b) = 9a^2 + 8b^2 + 16ab$$

Note:**2** It is worth to mention here that in (8),1 may also be represented as follows:

$$1 = \frac{(9+2\sqrt{8})(9-2\sqrt{8})}{7^2}$$

Following the same procedure as above, replacing a by $7A$ and b by $7B$,one may get different set of solutions as below.

$$x(A,B) = 693A^2 - 168B^2 + 756AB$$

$$y(A,B) == 189A^2 - 616B^2 - 756AB$$

$$z(A,B) = 111132A^4 - 87808B^4 + 333396A^3B - 296352AB^3 + 1$$

$$w(A,B) = 111132A^4 - 87808B^4 + 333396A^3B - 296352AB^3 - 1$$

$P(A,B) = 189A^2 + 168B^2 + 224AB$

Pattern. III

Rewrite the equation (3) as

$u^2 + 8v^2 = 9*P^2*1$ (15)

Write 1 as

$1 = \frac{(7+3i\sqrt{8})(7-3i\sqrt{8})}{11^2}$ (16)

Using (4),(5) and (16) in (15) and employing the method of factorization, we get,

$$(u+i\sqrt{8}v)(u-i\sqrt{8}v) = (1+i\sqrt{8})(1-i\sqrt{8})(a+ib\sqrt{8})^2(a-ib\sqrt{8})^2\frac{(7+3i\sqrt{8}((7-3i\sqrt{8})}{11^2}$$

Equating the positive parts and negative parts, we get,

$(u+i\sqrt{8}v) = (1+i\sqrt{8})\frac{7+3i\sqrt{8})}{11}(a+ib\sqrt{8})^2$ (17)

$(u-i\sqrt{8}v) = (1-i\sqrt{8})\frac{(7-3i\sqrt{8})}{11}(a-ib\sqrt{8})^2$ (18)

Equating real and imaginary parts of either (17) or (18), we get,

$u = \frac{-17a^2+136b^2-160ab}{11}$

$v = \frac{10a^2-80b^2-34ab}{11}$

Substituting the values of u,v and assuming $a = 11A, b = 11B$,the non-zero distinct integral solutions of (1) are obtained as

$$x = x(A,B) = 33A^2 - 264B^2 - 2508AB$$

$$y = y(A,B) = -407A^2 + 3256B^2 - 1012AB$$

$$z = z(A,B) = 242[-170A^4 - 10880B^4 + 8160A^2B^2 - 1022A^3B + 8176AB^3] + 1$$

$$w = w(A,B) = 242[-170A^4 - 10880B^4 + 8160A^2B^2 - 1022A^3B + 8176AB^3] - 1$$

$$P = P(A,B) = 121[A^2 + 8B^2]$$

Note:3 1 can also be represented as

$1 = \frac{(7+2i\sqrt{8})(7-2i\sqrt{8})}{9^2}$ (19)

Using (4),(5) and (19) in (15) ,and employing the same procedure as above ,we get the solution as

$$x = x(a,b) = a^2 - 8b^2 - 20ab$$

$$y = y(a,b) = -3a^2 + 24b^2 - 12ab$$

$$z = z(a,b) = -2a^4 + 96a^2b^2 - 28a^3b + 224ab^3 - 128b^4 + 1$$

$$w = w(a,b) = -2a^4 + 96a^2b^2 - 28a^3b + 224ab^3 - 128b^4 - 1$$

$$P = P(a,b) = a^2 + 8b^2$$

Note:4 It is worth to mention here that 1 may be represented as follows:

$1 = \frac{(1+i\sqrt{8})(1-i\sqrt{8})}{9}$ (20)

Using (4),(5) and (20) in (15) ,and employing the same procedure as above , replacing a by $3A$ and b by $3B$,we get the solution as

$$x = x(A,B) = -9A^2 + 72B^2 - 180AB$$

$$y = y(A,B) = -33A^2 + 264B^2 - 12AB$$

$$z = z(A,B) = -252A^4 - 16128B^4 + 12096A^2B^2 + 612A^3B - 4896AB^3 + 1$$

$$w = w(A,B) = -252A^4 - 16128B^4 + 12096A^2B^2 + 612A^3B - 4896AB^3 - 1$$

$$P = P(A,B) = 9A^2 + 72B^2$$

Pattern. IV

Introducing the linear transformations

$P = X + 8T, v = X + 9T$ (21)

in (3), it leads to

$X^2 = u^2 + 72T^2$ (22)

which is satisfied by

$$\left.\begin{aligned} u &= 72r^2 - s^2 \\ X &= 72r^2 + s^2 \\ T &= 2rs \end{aligned}\right\} \quad (23)$$

Using (21),(23) and (2),we obtained the non-zero distinct integer solution of (1) as

$$x(r,s) = 216r^2 + s^2 + 36rs$$

$$y(r,s) = -72r^2 - 3s^2 - 36rs$$

$$z(r,s) = 10368r^4 - 2s^4 + 2592r^3s - 36rs^3 + 1$$

$$w(r,s) = 10368r^4 - 2s^4 + 2592r^3s - 36rs^3 - 1$$

$$P(r,s) = 72r^2 + s^2 + 16rs$$

Note:5 It is to be noted that (22) may be represented as

$$(X + u)(X - u) = 72T^2$$

Then we have the system of double equation as shown in the following Table.1

Table.1 System of double equations

System	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$X + u$	$36T^2$	$18T^2$	$12T^2$	$6T^2$	$4T^2$	$2T^2$	T^2	72	36	18	12	6	4	2
$X - u$	2	4	6	12	18	36	72	T^2	$2T^2$	$4T^2$	$6T^2$	$12T^2$	$18T^2$	$36T^2$

Employing the procedure as above in each of the above representations, the corresponding solutions to (1) are represented below.

Solution for system:1

$$x(k) = 54k^2 + 18k + 1$$

$$y(k) = -18k^2 - 18k - 3$$

$$z(k) = 648k^4 + 324k^3 - 18k - 1$$

$$w(k) = 648k^4 + 324k^3 - 18k - 3$$

$$P(k) = 18k^2 + 8k + 1$$

Solution for system:2

$$x(k) = 27k^2 + 18k + 2$$

$$y(k) = -9k^2 - 18k - 6$$

$$z(k) = 162k^4 + 162k^3 - 36k - 7$$

$$\backslash w(k) = 162k^4 + 162k^3 - 36k - 9$$

$$P(k) = 9k^2 + 8k + 2$$

Solution for system:3

$x(k) = 18k^2 + 18k + 3$

$y(k) = -6k^2 - 18k - 9$

$z(k) = 72k^4 + 108k^3 - 54k - 17$

\$w(k) = 72k^4 + 108k^3 - 54k - 19$

$P(k) = 6k^2 + 8k + 3$

Solution for system:4

$x(k) = 9k^2 + 18k + 6$

$y(k) = -3k^2 - 18k - 18$

$z(k) = 18k^4 + 54k^3 - 108k - 71$

$w(k) = 18k^4 + 54k^3 - 108k - 73$

$P(k) = 3k^2 + 8k + 6$

Solution for system:5

$x(k) = 6k^2 + 18k + 9$

$y(k) = -2k^2 - 18k - 27$

$z(k) = 8k^4 + 36k^3 - 162k - 161$

$w(k) = 8k^4 + 36k^3 - 162k - 163$

$P(k) = 2k^2 + 8k + 9$

Solution for system:6

$x(k) = 3k^2 + 18k + 18$

$y(k) = -k^2 - 18k - 54$

$z(k) = 2k^4 + 18k^3 - 324k - 647$

$w(k) = 2k^4 + 18k^3 - 324k - 649$

$P(k) = k^2 + 8k + 18$

Solution for system:7

$x(k) = 6k^2 + 36k + 36$

$y(k) = -2k^2 - 36k - 108$

$z(k) = 8k^4 + 72k^3 - 1296k - 2591$

$w(k) = 8k^4 + 72k^3 - 1296k - 2593$

$P(k) = 2k^2 + 16k + 36$

Solution for system:8

$x(k) = 2k^2 + 36k + 108$

$y(k) = -6k^2 - 36k - 36$

$z(k) = -8k^4 - 72k^3 + 1296k + 2593$

$w(k) = -8k^4 - 72k^3 + 1296k + 2591$

$P(k) = 2k^2 + 16k + 36$

Solution for system:9

$x(k) = k^2 + 18k + 54$

$y(k) = -3k^2 - 18k - 18$

$z(k) = -2k^4 - 18k^3 + 324k + 649$

$w(k) = -2k^4 - 18k^3 + 324k + 647$

$P(k) = k^2 + 8k + 18$

Solution for system:10

$x(k) = 3k^2 + 18k + 27$

$y(k) = -6k^2 - 18k - 9$

$z(k) = -8k^4 - 36k^3 + 162k + 163$

$w(k) = -8k^4 - 36k^3 + 162k + 161$

$P(k) = 2k^2 + 8k + 9$

Solution for system:11

$x(k) = 3k^2 + 18k + 18$

$y(k) = -9k^2 - 18k - 6$

$z(k) = -18k^4 - 54k^3 + 108k + 73$

$w(k) = -18k^4 - 54k^3 + 108k + 71$

$P(k) = 3k^2 + 8k + 6$

Solution for system:12

$x(k) = 6k^2 + 18k + 9$

$y(k) = -18k^2 - 18k - 3$

$z(k) = -72k^4 - 108k^3 + 54k + 19$

\ $w(k) = -72k^4 - 108k^3 + 54k + 17$

$P(k) = 6k^2 + 8k + 3$

Solution for system:13

$x(k) = 9k^2 + 18k + 6$

$y(k) = -27k^2 - 18k - 2$

$z(k) = -162k^4 - 162k^3 + 36k + 9$

\ $w(k) = -162k^4 - 162k^3 + 36k + 7$

$P(k) = 9k^2 + 8k + 2$

Solution for system:14

$x(k) = 18k^2 + 18k + 3$

$y(k) = -54k^2 - 18k - 1$

$z(k) = -648k^4 - 324k^3 + 18k + 3$

$w(k) = -648k^4 - 324k^3 + 18k + 1$

$P(k) = 18k^2 + 8k + 1$

Note:6 In addition to these the following quintuples also satisfy (1).

i) $(111k, -253k, -12922k^2 + 1, -12922k^2 - 1, 89k)$

ii) $(39k, -73k, -952k^2 + 1, -952k^2 - 1, 27k)$

iii) $(33k, -47k, -280k^2 + 1, -280k^2 - 1, 19k)$

iv) $(33k, -39k, -108k^2 + 1, -108k^2 - 1, 17k)$

v) $(39k, -33k, 108k^2 + 1, 108k^2 - 1, 17k)$

vi) $(47k, -33k, 280k^2 + 1, 280k^2 - 1, 19k)$

vii) $(73k, -39k, 952k^2 + 1, 952k^2 - 1, 27k)$

viii) $(253k, -111k, 12922k^2 + 1, -12922k^2 - 1, 89k)$

Pattern. V

Write (3) as

$$u^2 - P^2 = 8(P^2 - v^2)$$

$$\Rightarrow (u+P)(u-P) = 8(P+v)(P-v) \tag{24}$$

which can be expressed in the form of ratio as

$$\frac{(u+P)}{8(P-v)} = \frac{(P+v)}{(u-P)} = \frac{A}{B} \quad , \quad (B \neq 0)$$

This is equivalent to following system of equations

$$Bu + 8Av + P(B - 8A) = 0$$

$$-Au + Bv + P(B + A) = 0$$

Solving these two equations using cross multiplication method, we get the values of u, v and T

as

$$u = 8A^2 - B^2 + 16AB$$

$$v = 8A^2 - B^2 - 2AB$$

$$P = 8A^2 + B^2$$

Substituting the above values of u and v in (2) we obtain the nonzero distinct integer solution of (1) as

$$x(A,B) = 24A^2 - 3B^2 + 12AB$$

$$y(A,B) = -8A^2 + B^2 + 20AB$$

$$z(A,B) = 128A^4 + 2B^4 - 96A^2B^2 + 224A^3B - 28AB^3 + 1$$

$$w(A,B) = 128A^4 + 2B^4 - 96A^2B^2 + 224A^3B - 28AB^3 - 1$$

$$P(A,B) = 8A^2 + B^2$$

Note:7 It is observed that (24) may also be represented as in the following cases:

$i)\ \frac{(u+P)}{8(P+v)} = \frac{(P-v)}{u-P} = \frac{A}{B} \quad (B \neq 0)$

$ii)\ \frac{(u-P)}{8(P-v)} = \frac{(P+v)}{u+P} = \frac{A}{B} \quad (B \neq 0)$

$iii)\ \frac{(u-P)}{8(P+v)} = \frac{P-v}{u+P} = \frac{A}{B} \quad (B \neq 0)$

$iv)\ \frac{(u+P)}{8(P-v)} = \frac{(P+v)}{u-P} = \frac{A}{B} \quad (B \neq 0)$

$v)\ \frac{(u+P)}{4(P+v)} = \frac{2(P-v)}{u-P} = \frac{A}{B} \quad (B \neq 0)$

$vi)\ \frac{(u-P)}{4(P-v)} = \frac{2(P+v)}{u+P} = \frac{A}{B} \quad (B \neq 0)$

$vii)\ \frac{(u-P)}{4(P+v)} = \frac{2(P-v)}{u+P} = \frac{A}{B} \quad (B \neq 0)$

$viii)\ \frac{(u+P)}{4(P-v)} = \frac{2(P+v)}{u+P} = \frac{A}{B} \quad (B \neq 0)$

$ix)\ \frac{(u+P)}{2(P+v)} = \frac{4(P-v)}{u-P} = \frac{A}{B} \quad (B \neq 0)$

$x)\ \frac{(u-P)}{2(P-v)} = \frac{4(P+v)}{u+P} = \frac{A}{B} \quad (B \neq 0)$

$xi)\ \frac{(u-P)}{2(P+v)} = \frac{4(P-v)}{u+P} = \frac{A}{B} \quad (B \neq 0)$

$xii)\ \frac{(u+P)}{2(P-v)} = \frac{4(P+v)}{u-P} = \frac{A}{B} \quad (B \neq 0)$

Employing the procedure as above in each of the above representations, we can get the different sets of distinct non-zero integer solutions to (1).

Conclusion:

In this paper, a search is made for obtaining different choice of integer solutions to homogeneous bi-quadratic diophantine equation with five unknowns $(x^4 - y^4) + 2(x-y)(x^3 + y^3) = 36(z^2 - w^2)P^2$. To conclude, as bi-quadratic equations are rich in variety, the researchers may search for integer solutions to the other types of bi-quadratic equations with variables greater than or equal to five.

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