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A New Look at the Infinite Hotel Paradox

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ABSTRACT

In the 1920s, the German Mathematician David Hilbert proposed a famous thought experiment, to show how just how difficult it is to wrap our minds around the concept of infinity. Infinite hotel, popularly known as "Hilbert's hotel" is a story of an imaginary hotel with infinitely many rooms that exemplifies the peculiar consequences of supposing an actual infinity of objects or events. Since the 1970s it has been used in a variety of arguments, some of them associating to cosmology and others to philosophy and theology. For a long time, it has remained and still remains unknown whether David Hilbert actually designed the thought experiment named after him, or whether it is simply a piece of mathematical folklore. It turns out that Hilbert introduced his hotel thought experiment in a lecture in January 1924, but did not publish it. The counter-intuitive nature of the hotel only became well known in 1947, when George Gamow described it in his book, and it took nearly three more decades until it attracted wide interest in scientific, philosophical, and theological contexts. In this report I have tried to outline the origins and early history of Infinite hotel paradox and also tried to present a new twist on a familiar paradox, linking seemingly contrasting ideas comprehensively. Hilbert's Hotel is a paradox which addresses infinite set comparisons and extends to incorporate ideas from calculus – mainly infinite series.

INTRODUCTION

The variations of Hilbert's Grand Hotel paradox that have been presented below, involve the use of telescoping series which will serve as a means to stimulate robust understanding of the familiar variations of the Hilbert hotel by giving rise to rational conflict, or an uncommon analysis of convergence and divergence of infinite series (alternatively an introduction to these ideas).

FAMILIAR VARIATIONS OF THE HILBERT HOTEL:

Imagine there is a hotel with infinite rooms, the room numbers of which are natural numbers i.e., numbered 1,2,3, 4,… and so forever and you are the manager. At first, it might seem that you could accommodate anyone whoever shows up at any point of time. But, is there a limit, a way to exceed even the infinity of rooms at the Hilbert Hotel.

Let us start by assuming that only one person is allowed in each room and all the rooms present in the hotel are occupied, i.e., there are an infinite number of guests in infinite number of rooms. Now how would you accommodate the guests if someone new shows up and they want a room. The typical mathematical solution involves vacating room number 1 for the new guest, and the previous occupant of room number 1 is relocated to room number 2, and all the other guests are arranged in such a way that each guest is allocated to the room number 1 greater than their initial room. Putting it formally, the solution makes use of the bijection f(n) = n + 1 to advance and accommodate all the guests.

Now if the hotel were to host 'k' ($k \in \mathbb{N}$) new guests, we can simply accommodate the new guests from room numbers 1 to k and relocating the other guests via the bijection f(n) = n + k.

Again, how do we accommodate all the guests if a bus carrying infinitely many people arrives at the hotel requesting rooms for the new guests. From the above discussion, we know what to do with a finite number of people, but what do we do with infinite people? To solve this particular problem, we use the concept of even numbers. We relocate the existing guests to the room number double of their current room number.

For example, the person in room number 1 moves to room number 2 and the person at room number 2 moves to room number 4, 3 to 6, 4 to 8 and so on. Since, number of even natural numbers are infinite, therefore it can successfully accommodate infinite number of guests as all natural numbers and all even natural numbers share one to one correspondence/mapping (1->2, 2->4,…,n->2n). This process makes all the odd numbered rooms free, and the new infinite guests can be lodged at every odd numbered room, in short, we are always going to have a room to accommodate a new guest. In this case, we make the use of the bijection f(n) = 2k.

This solution can be further extended to case where two infinite buses carrying infinite guests arrive at the hotel, the original guests can be accommodated by using the bijection f(n) = 3k, this way the rooms which are a multiple of three are fully accommodated and now the guests of the first bus can occupy the rooms following the rule f(n) = 3k − 1; $where$ K $\in \mathbb{N}$ (rooms numbered 2,5,8, and so on) and the guests of bus 2 occupy the rooms adhering to the rule: f(n) = 3k − 2; where k $\in \mathbb{N} -$

{1} (rooms numbered 4,7,10 and so on). In short for countable number (suppose 'l') of buses carrying infinite passengers in each bus, we relocate the existing guests to t rooms following the bijection:f(n) = (l + 1)k; k ∈ ℕ.

We are not done yet, consider this, you are working at the Hilbert's hotel and outside you see a line of infinitely many buses with infinitely many people inside those buses waiting to book a room at the hotel that has infinitely many rooms. Can we accommodate every person in every bus a room? Well yes, we absolutely can. Around 300 BC, Euclid proved that there are infinitely many prime numbers. So we are going to be using the prime numbers to help us this time. So to accommodate every guest, we need to move every existing person in the hotel to the rooms to the first prime number to the power of their room number. So the person in room 5 is going to move to room numbered 25 = 32, because two is the first prime number, here, the person in room 5 moves to room 32, the person in room 8 is going to move to room 28 = 256 and so on for other rooms. Now, the current guests that were already in the hotel in new rooms. But what do we do for the guests waiting in the infinitely many and infinitely long buses. For the people in the buses, we are going to assign each person in that bus a seat number starting from 1, and each person is going to be assigned a room that is numbered the next prime number to the power of their seat number, for example, in bus 1 (prime number in use is 3) the person in seat number 1 is going to

be assigned the room number 31 = 3, the person in seat 3 will go to the room numbered 33 = 27 , seat 7 in bus1 is assigned room 37 = 2187. And similarly for the other buses we use corresponding prime numbers (bus 2 ->5, bus 3 ->7 and so on). To understand it better let's try to think about the person in seat number 15 of bus 3, the person will go to room numbered 715 = 4.74756151 X 1012. In short, people in bus 1 follow the bijection:f(n) = 3k ; where k is the seat number, for bus 2: f(n) = 5k and we carry this on for subsequent buses. So, each of these prime numbers hen we actually put k into them, since they only have one and the natural numbers of their prime bases as factors we never have rooms overlapping. More generally, we can put the new arrival of bus m with seat number k into the room pk; where p is the (m + 1)th prime number.

In this section we will look at several variations to Hilbert's Hotel that align with the aforementioned results. To connect the above ideas with infinite series, I will introduce an unconventional twist to the various cases discussed before. Suppose that the Infinite Hotel undertakes renovations such that the guests of different rooms are charged distinct amounts for their accommodations based on the room occupied by the occupant. Assume Room number 1, to be the most luxurious and fabulous of all rooms, and it costs 1 Goldcoin. Room number 2 costs 1/2 Goldcoin, room number 3 costs 1/3 Goldcoin, i.e., room number n costs 1/n Goldcoin ∃ n ∈ ℕ. When every room is engaged, what is the total income of the hotel? Since we have a divergent series, and this problem may be used to introduce a well known fact:

The series $\sum_{k=2}^{\infty} \frac{1}{k}$ is divergent (so the revenue should be infinite).

ROOM RENT PROBLEMS

As discussed initially in the classical case, a new guest arrives and the hotel manager decides to give him room number 1. Therefore, the guests are relocated to the next numbered room, as before. Now, since, room number n is more expensive than room number (n + 1), here a new question emerges: As all the guests who have now moved to cheaper rooms, what is the estimated refund that the manager will have to pay? Answer to this is that the occupants have to be refunded the difference in cost between the initial room and the one finally assigned. At first thought, it appears the refund to be rather large, maybe something close to the initial revenue. Instinctively thinking- "If the hotel has to pay each guest then, the amount will get infinitely bigger Paying an infinite number of guests will cost an infinite amount of money." However, on further analysis it turns out that the total refund is 1 Goldcoin as

$$\sum_{l=1}^{n} \left(\frac{1}{l} - \frac{1}{l+1}\right) = 1 - \frac{1}{n+1} \text{ and } \lim_{n\to\infty}\left(1 - \frac{1}{n+1}\right) = 1.$$

In another extension of this variation on the classical problem, we can now ask, what if two guests arrive? Or what if k guests arrive? Addressing the situation where 'k' guests arrive, the occupant of room number n is assigned room number n + k. So, in this case, the refund must be the difference between $\frac{1}{n}$ and $\frac{1}{n+k}$, the total worth of the refund is $1 + \frac{1}{2} + \frac{1}{3} + \cdots \cdot \ \frac{1}{n+k}$ Goldcoins and, the nth partial sum of :

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+k}\right) \text{ is a telescoping sum: } \sum_{l=1}^{n} \left(\frac{1}{l} - \frac{1}{l+k}\right) = \left(1 - \frac{1}{1+k}\right) + \left(\frac{1}{2} - \frac{1}{2+k}\right) + \cdots + \left(\frac{1}{k} - \frac{1}{k+k}\right) + \left(\frac{1}{k+1} - \frac{1}{(k+1)+k}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{(n-1)+k}\right) + \left(\frac{1}{n} - \frac{1}{n+k}\right) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k} - \left(\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{n+k}\right) \text{ for } n \geq k, \text{ and } \quad \lim_{n\to\infty}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k} - \left(\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{n+k}\right)\right) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k}.$$

However, all the earlier calculations – as straightforward as they are – are unnecessary and avoidable if we adopt a smarter way to approach the same problem: Before the guest arrives, every room is paid for. After the guest settles down in his room and refunds have to be issued to other guests, and since all the rooms are already paid for, barring room number 1. Therefore, the refund must be equal to 1 Goldcoin. Such a view may be generalized to make sense of the calculations for the case of k important new guests.

Our next step is to accommodate a bus full of infinitely many guests. As discussed before, the existing guests are asked to move to even-numbered rooms and the new guests to odd numbered ones – then what is the total refund? As expected, now, the refund is now infinite, as-

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{2n}\right)=\sum_{n=1}^{\infty}\frac{1}{2n}=\infty$$

But now a new question arises: Should the refund to infinitely many guests be infinite in total? The above refund is given by-

$$\sum_{n=1}^{\infty}\frac{1}{n}-\sum_{n=1}^{\infty}\frac{1}{2n}$$

which casually can be thought of as '$\infty - \infty$, which is obviously undefined. In the previous case of accommodating k guests the refund was:

$$\sum_{n=1}^{\infty}\frac{1}{n}-\sum_{n=1}^{\infty}\frac{1}{n+k},$$

which is also '$\infty - \infty$, and surprisingly we saw that this sum is equal to: $1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n+k}$

which is a finite amount. Having seen the previous result, it is reasonable to be discontent with having to give an infinite refund, even if we are to accommodate infinite guest.

1. A NEW TASK AT HAND

Now I present a new problem: Can we find a realistic way to accommodate an infinite (but countable) number of new people, in such a way that the total refund for original guests is finite?

The search for a suitable convergent series can sometimes block out the original question. For example, if the new guests are sent to rooms numbered with raised to the power of 2, then the refund is given by:

$$\sum_{n=1}^{\infty}\frac{1}{2^n},$$

seems to work well. Similarly, if we make the new guests occupy rooms numbered with triangular numbers, this can give us a finite amount:

$$\sum_{n=1}^{\infty}\frac{2}{n(n+1)}\leq\sum_{n=1}^{\infty}\frac{2}{n^2}=\frac{\pi^2}{3}.$$

Note that here, the assumption taken into consideration is that existing occupants are asked to leave their rooms to accommodate the new guests – this clearly explains the fact that why each guest receives a refund corresponding to his own room number.

In our previous solutions, the guests were not asked to leave their rooms, but rather were allotted a new room. Therefore, the refund was the difference between the new room and the initial room (and not the cost of the initial room). A different look at convergence and divergence of series is required to understand this situation clearly. The challenge remains: Is it actually possible to relocate guests to accommodate infinitely many new guests such that every guest has his own room and the refund to the existing guests is finite?

The idea behind this solution involves the partitioning the set $\mathbb{N}$, into sets Ak for k = 1, 2, 3, and relocating the guest from room number n in Ak to room number n + k..

Let

$$A_k=\{n\,|\,(k-1)^2+1\leq n\leq k^2\}\text{ for }=1,\,2,\,3,\,\dots.$$

Clearly the sets Ak are disjoint and $\bigcup_{n=1}^{\infty} Ak$= N. Let f : N → N be defined by f (n) = n + k for n ∈ Ak for k = 1, 2, 3, Now we find a function f is well defined and one-to-one. Now, f (1) = 2, f (n) = n + 2 for 2 ≤ n ≤ 4, f (n) = n + 3 for 5 ≤ n ≤ 9, Therefore f (1) = 2, f (2) = 4, f (3) = 5, f (4) = 6, f (5) = 8, f (6) = 9, Note that the range of f is {2, 4, 5, 6, 8, 9, 10, 11, 12, 14 } (the occupied room). Therefore, the person who is in room number n is asked to move to room number f (n) and we can accommodate infinitely many new guests because after this new room arrangement, rooms with numbers in the set {1, 3, 7, 13, 21,. } are vacant. Now the refund would be:

$$\sum_{k=1}^{\infty}\sum_{n=(k-1)^2+1}^{k^2}\left(\frac{1}{n}-\frac{1}{n+k}\right)=\sum_{k=1}^{\infty}\sum_{n=(k-1)^2+1}^{k^2}\frac{k}{n\,(n+k)}.$$

Clearly the series is convergent and thus refund is finite (check calculations below)

$$\sum_{k=1}^{\infty}\sum_{n=(k-1)^2+1}^{k^2}\frac{k}{n\,(n+k)} \le \sum_{k=1}^{\infty}\sum_{n=(k-1)^2+1}^{k^2}\frac{k}{((k-1)^2+1)\left((k-1)^2+1+k\right)}$$
$$=\sum_{k=1}^{\infty}\frac{(2k-1)\,k}{((k-1)^2+1)\left((k-1)^2+1+k\right)}.$$

Now using limit comparison test and comparing with $\sum_{k=1}^{\infty}\frac{1}{k^2}=\sum_{k=1}^{\infty} b_k$ and $\sum_{k=1}^{\infty} a_k$,

where Now, $a_k = \frac{(2k-1)k}{((k-1)^2+1)((k-1)^2+1+k)}$

$$\lim_{k\to\infty}\frac{a_k}{b_k} = \lim_{k\to\infty}\frac{(2k-1)k}{((k-1)^2+1)((k-1)^2+1+k)}k^2$$
$$= \lim_{k\to\infty}\frac{(2-\frac{1}{k})}{((1-\frac{1}{k})^2+\frac{1}{k^2})((1-\frac{1}{k})^2+\frac{1}{k^2}+\frac{1}{k})} = 2.$$

Therefore, since $2 < \infty$ and $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty}\frac{1}{k^2}$ is convergent, then $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty}\frac{(2k-1)k}{((k-1)^2+1)((k-1)^2+1+k)}$ is convergent. Hence $\sum_{k=1}^{\infty}\sum_{n=(k-1)^2+1}^{k^2}\frac{k}{n(n+k)}$ is also convergent by the comparison test because we saw before that

$$\sum_{k=1}^{\infty}\sum_{n=(k-1)^2+1}^{k^2}\frac{k}{n(n+k)} \le \sum_{k=1}^{\infty}\frac{(2k-1)k}{((k-1)^2+1)((k-1)^2+1+k)}.$$

CONCLUSION

The foundational appeal of addressing and finding answers to paradoxes is a strong motivator, and one which can challenge and extend a person's thinking. The paradoxes discussed in this report make use of a familiar context – hotel accommodations and adjustments, bookings and refunds – to formulate ideas of convergence and divergence as they apply to infinite series and sets. An important contribution of the twist is in connecting seemingly disparate occurrences of infinity in undergraduate mathematics under one same roof. The variations and situations discussed offer interesting problems for further research and study, and can be modified, depending on one's disposition and willingness, to create more or less demanding explanations and resolutions. These problems contribute in stimulating mathematical debate, promoting a tradition of speculation and justification, or simply develop a fun mental playground for the curious kind.

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