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A Study on the Hyperbola $y^2 = 87x^2 - 78$

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ABSTRACT

The binary quadratic equation $y^2 = 87x^2 - 78$ is considered for obtaining its integral solutions. A few interesting properties among the solutions are presented. Employing the integral solutions of the equation under consideration. a few remarkable observations are illustrated.

Keywords: Binary quadratic, pell equation, integral solutions, hyperbola, parabola, second order Ramanujan numbers.

Introduction

Any non-homogeneous binary quadratic equation of the form $y^2 - Dx^2 = 1$, where D is a given positive non-square integer, requiring integer solutions for x and y is called Pellian equation (also known pell-Fermat equation). In cartesian co-ordinates, the equation has the form of a hyperbola. The pellian equation has infinitely many distinct integer solutions as long as D is not a perfect square and namely, the solution with x, y positive integers of smallest possible size. One may refer [1-12] for a few choices of Pellian equations along with their corresponding integer solutions.

The solutions to Pellian equations have long been of interest to mathematics. Even small values of D can lead to fundamental solutions which are quite large. For example, when D=61, the fundamental solution is (1766319049, 226153980). The above results motivated us to search for integer solutions to other choices of Pellian equation. This paper concerns with the Pellian equation $y^2 = 87x^2 - 78$, a few interesting properties among the solutions are presented. Employing the integral solutions of the equation under consideration. The process of obtaining second order Ramanujan numbers is illustrated with numerical examples.

Method of analysis:

The hyperbola represented by the non-homogeneous quadratic equation under consideration is

$$y^2 = 87x^2 - 78 \qquad (1)$$

The smallest positive integer solution is $x_0 = 1, y_0 = 3$

Corresponding Pell equation, $y^2 = 87x^2 - 78$

$$\tilde{x}_0 = 3, \tilde{y}_0 = 28$$

If (x_n, y_n) represents the general solution of (1), then

$$\tilde{x}_n = \frac{1}{2\sqrt{D}} gn \qquad (2)$$

$$\tilde{y}_n = \frac{1}{2} fn \qquad (3)$$

$$f_n = (28 + 3\sqrt{87})^{n+1} + (28 - 3\sqrt{87})^{n+1}$$

$$g_n = (28 + 3\sqrt{87})^{n+1} - (28 - 3\sqrt{87})^{n+1}$$

A few numerical solutions to (1) are presented in table below:

Table: Numerical solutions

n	x_n	y_n
1	**37**	**345**
2	**2071**	**19317**
3	**115939**	**1081407**
4	**6490513**	**60539475**
5	**363352789**	**3389129193**

Observations:

$$y_n \equiv o\ (\mathrm{mod}\ 3)\ n = 0,1,\ldots$$

$\Rightarrow$ A few interesting relations among the solutions are given below:

$$234x_{n+3} + 234x_{n+1} - 13104x_{n+2} = 0$$

$$39y_{n+1} + 364x_{n+1} - 13x_{n+2} = 0$$

$$39y_{n+2} + 13x_{n+1} - 364x_{n+2} = 0$$

$$39y_{n+3} + 364x_{n+1} - 20731x_{n+2} = 0$$

$$56x_{n+2} - x_{n+1} - x_{n+3} = 0$$

$$6552y_{n+1} + 61113x_{n+1} - 39x_{n+3} = 0$$

$\Rightarrow$ Expressions representing square integers:

$$f_{n^2} = \frac{6439x_{2n+2} - x_{2n+4} + 4368}{2184}$$

$$f_{n^2} = \frac{1073x_{2n+2} - y_{2n+3} + 728}{364}$$

$$f_{n^2} = \frac{60059x_{2n+2} - y_{2n+4} + 40742}{20371}$$

$$f_{n^2} = \frac{29x_{2n+3} - 115y_{2n+2} + 728}{364}$$

$$f_{n^2} = \frac{29x_{2n+4} - 6439y_{2n+2} + 40742}{20371}$$

$$f_{n^2} = \frac{1073x_{2n+4} - 6439y_{2n+3} + 728}{364}$$

$$f_{n^2} = \frac{60059x_{2n+4} - 6439y_{2n+4} + 26}{13}$$

$$f_{n^2} = \frac{y_{2n+3} - 37y_{2n+2} + 234}{117}$$

$$f_{n^2} = \frac{y_{2n+4} - 2071y_{2n+2} + 13104}{6552}$$

$$f_{n^2} = \frac{1073y_{2n+4} - 60059y_{2n+3} + 6786}{3393}$$

$\Rightarrow$ Expressions representing cubical integers:

$$f_{n^3} = \frac{6439x_{3n+3} - x_{3n+5} + 19317x_{n+1} - 3x_{n+3}}{2184}$$

$$f_{n^3} = \frac{29x_{3n+3} - y_{3n+3} + 87x_{n+1} - 3y_{n+1}}{13}$$

$$f_{n^3} = \frac{1073x_{3n+3} - y_{3n+4}}{364} + 3\left(\frac{1073x_{n+1} - y_{n+2}}{364}\right)$$

$$f_{n^3} = \frac{60059x_{3n+3} - y_{3n+5}}{20371} + 3\left(\frac{60059x_{n+1} - y_{n+3}}{20371}\right)$$

$$f_{n^3} = \frac{58x_{3n+4} - 230y_{3n+3}}{728} + 3\left(\frac{58x_{n+2} - 230y_{n+1}}{728}\right)$$

$$f_{n^3} = \frac{29x_{3n+5} - 6439y_{3n+3}}{20371} + 3\left(\frac{29x_{n+3} - 6439y_{n+1}}{20371}\right)$$

$$f_{n^3} = \frac{1073x_{3n+5} - 6439y_{3n+4}}{364} + 3\left(\frac{1073x_{n+3} - 6439y_{n+2}}{364}\right)$$

$$f_{n^3} = \frac{60059x_{3n+5} - 6439y_{3n+5}}{13} + 3\left(\frac{60059x_{n+3} - 6439y_{n+3}}{13}\right)$$

$$f_{n^3} = \frac{y_{3n+4} - 37y_{3n+3}}{117} + 3\left(\frac{y_{n+2} - 37y_{n+1}}{117}\right)$$

$$f_{n^3} = \frac{y_{3n+5} - 2071y_{3n+3}}{6552} + 3\left(\frac{y_{n+3} - 2071y_{n+1}}{6552}\right)$$

$\Rightarrow$ Expressions representing biquadratic integers:

$$f_{n^4} = \frac{6439x_{4n+4} - x_{4n+6}}{2184} + 4\left(\frac{6439x_{2n+2} - x_{2n+4} + 4368}{2184}\right) - 2$$

$$f_{n^4} = \frac{29x_{4n+4} - y_{4n+4}}{13} + 4\left(\frac{29x_{2n+2} - y_{2n+2} + 26}{13}\right) - 2$$

$$f_{n^4} = \frac{1073x_{4n+4} - y_{4n+5}}{364} + 4\left(\frac{1073x_{2n+2} - y_{2n+3} + 728}{364}\right) - 2$$

$$f_{n^4} = \frac{60059x_{4n+4} - y_{4n+6}}{20371} + 4\left(\frac{60059x_{2n+2} - y_{2n+4} + 40742}{20371}\right) - 2$$

$$f_{n^4} = \frac{115x_{4n+6} - 6439x_{4n+5}}{39} + 4\left(\frac{115x_{2n+4} - 6439x_{2n+3} + 78}{39}\right) - 2$$

$$f_{n^4} = \frac{115y_{4n+5} - 1073x_{4n+5}}{13} + 4\left(\frac{115y_{2n+3} - 1073x_{2n+3} + 26}{13}\right) - 2$$

$$f_{n^4} = \frac{29x_{4n+6} - 6439y_{4n+4}}{20371} + 4\left(\frac{29x_{2n+4} - 6439y_{2n+2} + 40742}{20371}\right) - 2$$

$$f_{n^4} = \frac{1073x_{4n+6} - 6439y_{4n+5}}{364} + 4\left(\frac{1073x_{2n+4} - 6439y_{2n+3} + 728}{364}\right) - 2$$

$$f_{n^4} = \frac{y_{4n+6} - 2071y_{4n+4}}{6552} + 4\left(\frac{y_{2n+4} - 2071y_{2n+2} + 13104}{6552}\right)$$

$$f_{n^4} = \frac{1073y_{4n+6} - 60059y_{4n+5}}{3393} + 4\left(\frac{1073y_{2n+4} - 60059y_{2n+3} + 6786}{3393}\right) - 2$$

⇒Employing linear combinations among the solutions, one obtains solutions to other choices of hyperbolas

Choices	Hyperbola	X and Y
1.	$9X^2 - Y^2 = 54756$	$X = 115x_{n+1} - x_{n+2}, Y = \sqrt{87}(x_{n+2} - 37x_{n+1})$
2.	$9X^2 - Y^2 = 171714816$	$X = 6439x_{n+1} - x_{n+3}, Y = \sqrt{87}(x_{n+3} - 2071x_{n+1})$
3.	$9X^2 - Y^2 = 6084$	$X = 29x_{n+1} - y_{n+1}, Y = \sqrt{87}(y_{n+1} - 3x_{n+1})$
4.	$9X^2 - Y^2 = 4769856$	$X = 1073x_{n+1} - y_{n+2}, Y = \sqrt{87}(y_{n+2} - 345x_{n+1})$
5.	$9X^2 - Y^2 = 14{,}93{,}91{,}95{,}076$	$X = 60059x_{n+1} - y_{n+3}, Y = \sqrt{87}(y_{n+3} - 19317x_{n+1})$
6.	$9X^2 - Y^2 = 54756$	$X = (115x_{n+3} - 6439x_{n+2}), Y = \sqrt{87}(37x_{n+3} - 2071x_{n+2})$
7.	$1521X^2 - 676Y^2 = 3224422656$	$X = 58x_{n+2} - 230y_{n+1}, Y = \sqrt{87}(37y_{n+1} - 3x_{n+2})$
8.	$9X^2 - Y^2 = 6084$	$X = 115y_{n+2} - 1073x_{n+2}, Y\sqrt{87}(37y_{n+2} - 345x_{n+2})$
9.	$X^2 - Y^2 = 4769856$	$X = 180127(x_{n+2} - y_{n+3}), Y = \sqrt{87}(37y_{n+3} - 19137x_{n+2})$
10.	$9X^2 - Y^2 = 497931692$	$X = 29_{n+3} - 6439y_{n+1}, Y = \sqrt{87}(2071y_{n+1} - 3x_{n+3})$
11.	$9X^2 - Y^2 = 4769856$	$X = 1073_{n+3} - 6439y_{n+2}, Y = \sqrt{87}(2071y_{n+2} - 345x_{n+3})$
12.	$9X^2 - Y^2 = 6084$	$X = 60059x_{n+3} - 6439y_{n+3}, Y = \sqrt{87}(2071y_{n+3} - 19371x_{n+3})$
13.	$841X^2 - Y^2 = 46049796$	$X = y_{n+2} - 37y_{n+1}, Y = \sqrt{87}(115y_{n+1} - y_{n+2})$
14.	$841X^2 - Y^2 = 144412160256$	$X = y_{n+3} - 2071y_{n+1}, Y = \sqrt{87}(6439y_{n+1} - y_{n+3})$
15.	$X^2 - Y^2 = 46049796$	$X = 1073y_{n+3} - 60059y_{n+2}, Y = \sqrt{87}(6439y_{n+2} - 115y_{n+3})$

⇒Employing linear combinations among the solutions, one obtains solutions to other choices of parabolas

Choices	parabola	X and Y
1.	$X - 3Y^2 = 12$	$X = \frac{115x_{2n+2} - 56x_{2n+1} + 6}{3}, Y = \sqrt{87}(x_{n+2} - 37x_{n+1})$

2.	$14309568X - Y^2 = 57238272$	$X = 6439x_{2n+2} - x_{2n+4} + 4368, Y = \sqrt{87}(x_{n+3} - 2071x_{n+1})$
3.	$507X - Y^2 = 6084$	$X = \frac{29x_{2n+2} - y_{2n+2} + 26}{13}, Y = \sqrt{87}(-3x_{n+1} + y_{n+1})$
4.	$3276X - Y^2 = 4769856$	$X = \frac{1073x_{2n+2} - y_{2n+3} + 728}{364}, Y = \frac{\sqrt{87}(y_{n+2} - 345x_{n+1})}{1092}$
5.	$9X - Y^2 = 733356$	$X = 60059x_{2n+2} - y_{2n+4} + 40742, Y = (y_{n+3} - 19317x_{n+1})\sqrt{87}$
6.	$9X - Y^2 = 1404$	$X = 115x_{2n+4} - 6439x_{2n+3} + 78, Y = \sqrt{87}(37x_{n+3} + 2071x_{n+2})$
7.	$9X - Y^2 = 13104$	$X = 29x_{2n+3} - 115y_{2n+2} + 728, Y = \sqrt{87}(37y_{n+1} - 3x_{n+2})$
8.	$9X^2 - Y^2 = 468$	$X = 115y_{2n+3} - 1073x_{2n+3}, Y = \sqrt{87}(37y_{n+2} - 345x_{n+2})$
9.	$1092X - Y^2 = 4769856$	$X = 180127(x_{2n+3} - y_{2n+4}) + 2184, Y = \sqrt{87}(37y_{n+3} - 19317x_{n+2})$
10.	$9X - Y^2 - 733356$	$X = 29x_{2n+4} - 6439y_{2n+2} + 40742, Y = \sqrt{87}(2071y_{n+1} - 3x_{n+3})$
11.	$9X - Y^2 = 13104$	$X = 1073x_{2n+4} - 6439y_{2n+3} + 728, Y = \sqrt{87}(2071y_{2n+2} - 345x_{n+3})$
12.	$117X - Y^2 = 6084$	$X = 60059x_{2n+4} - 6439y_{2n+4} + 26, Y = \sqrt{87}(2071y_{n+3} - 19371x_{n+3})$
13.	$98397X - Y^2 = 46049796$	$X = y_{2n+3} - 37y_{n+2} + 234, Y = \sqrt{87}(115y_{n+1} - y_{n+2})$
14.	$5510232X - Y^2 = 144421160256$	$X = y_{2n+4} - 2071y_{2n+2} + 13104, Y = \sqrt{87}(6439y_{n+1} - y_{n+3})$
15.	$3393X - Y^2 = 46049796$	$X = 1073y_{2n+4} - 60059y_{2n+3} + 6786, Y = \sqrt{87}(6439y_{n+2} - 115y_{n+3})$

⇒ *Considerin suitable values of* x_n&y_n,*one generates* 2^{nd} order Ramanujan numbers with base integers as real integers.

For illustration, consider

$y_1 = 345 = 1*345 = 3*115 = 5*69 = 23*15$

Now $1*345 = 3*115$
$\Rightarrow (1+345)^2 + (3-115)^2 = (1-345)^2 + (3+115)^2 = 132260$

$1*345 = 5*69$
$\Rightarrow (1+345)^2 + (5-69)^2 = (1-345)^2 + (5+69)^2 = 123812$

$1*345 = 23*15$
$\Rightarrow (1+345)^2 + (23-15)^2 = (1-345)^2 + (23+15)^2 = 119780$

$3*115 = 5*69$
$\Rightarrow (3+115)^2 + (5-69)^2 = (3-115)^2 + (5+69)^2 = 18020$

$3*115 = 23*15$
$\Rightarrow (3+115)^2 + (23-15)^2 = (3-115)^2 + (23+15)^2 = 13988$

$5*69 = 23*15$
$\Rightarrow (5+69)^2 + (23-15)^2 = (5-69)^2 + (23+15)^2 = 5540$

Thus **132260 , 123812 , 119780 , 18020 ,13988 , 5540** represent 2^{nd} order Ramanujan numbers.

⇒ Considering suitable values of x_n&y_n , one generates 2^{nd}order Ramanujan number with base integers as Gaussian integers.

For illustration, consider again

Now,

$1*345 = 3*115$
$\Rightarrow (1+345i)^2 + (3-115i)^2 = (1-345i)^2 + (3+115i)^2 = -132240$

1*345 = 5*69
$\Rightarrow (1+345i)^2+(5-69i)^2=(1-345i)^2+(5+69i)^2= -123760$

1*345 = 23*15
$\Rightarrow (1+345i)^2+(23-15i)^2=(1-345i)^2+(23+15i)^2=-118720$

3*115 = 5*69
$\Rightarrow (3+115i)^2+(5-69i)^2=(3-115i)^2+(5+69i)^2= -17952$

3*115 = 23*15
$\Rightarrow (3+115i)^2+(23-15i)^2=(3-115i)^2+(23+15i)^2= -12912$

5*69 = 23*15
$\Rightarrow (5+69i)^2+(23-15i)^2=(5-69i)^2+(23+15i)^2= -4432$

Note that **-132240, -123760 , -118720 , -17952 , -12912 ,-4432** represent 2nd order Ramanujan numbers with base integers as Gaussian integers.

CONCLUSION:

In this paper, an attempt has been to obtain non zero distinct integer solutions to the binary quadratic Diophantine equation in title representing hyperbolas. As there are varieties of hyperbolas, the readers may search for other forms of hyperbolas to obtain integer solutions for the corresponding hyperbolas.

References:

[1] Telang S.J., “Number Theory ”, Tata Mcgraw hillPublishiing company ltd, New Delhi.(2000)

[2] Thomas Koshly, “Elementary Number Theory Applications ”, Academics Press,(2005)

[3] Gopalan M.A and Janaki G, “ Observations on $y^2=3x^2+1$”, Acta ciancia Indica, XXXIVM,No.2., (2008),693-696.

[4] Gopalan M.A and Yamuna R.S., “Remarkable Observations on the binary quadratic equation”, Impact J.Sci.Tech., vol.4.,(2010), 61-65.

[5] Gopalan M.A and Sangeetha G, “A Remarkable Observations on $y^2=10x^2+1$”, Impact J.Sci.Tech., vol.4.,(2010), 103-106.

[6] Gopalan M.A and Palanikumar R, “ Observations on $y^2=12x^2+1$”, Antarctia J.Math., 8(2)(2011), 149-152.

[7] Gopalan M.A and Geetha K, “ Observations on the Hyperbola $y^2=18x^2+1$”, RETELL, An Inter-disciplinary Research Journal , Vol.13,No.1,(2012), 81-83.

[8] Gopalan M.A and Vijayalakshmi S, Usha rani T.R and Malliga S, “ Observations on $y^2=12x^2-3$”,Bessel Journal of Math, 2(3), (2012), 153-158.

[9] Gopalan M. A ., Sumathi G and Vidhyalakshmi S, “ Observations on $y^2=26x^2+1$ ”,Bessel Journal of Math, Vol.4.Issue 1, (2014),21-25.

[10] J.Shanthi, P.Deepalakshmi, M.A.Gopalan., “A Study on the positive pell equation $y^2=42x^2+7$”, International research Journal of educational and Technology , volume 1., Issue 5,pages 107-118, January 2021.

[11] S.Vidhyalakshmi, M.A.Gopalan, J.Shanthi , “The quadratic equation $y^2=15x^2-6$”, IJAR, 1(4), 108-110,2015.

[12] S.Vidhyalakshmi, J.Shanthi, M.Devi, M.A.Gopalan,” A study on the hyperbola $y^2=87x^2+1$”, International Research Journal of Education and Technology , volume.5., issue 03, pages 18-32, March (2023).