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On the Analytic Functions

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ABSTRACT

In this article, I have developed an easier method to find an analytic function with the help of real or imaginary function. The function $f(z) = u(x, y) + iv(x, y)$ is said to be analytic if it is single valued and possess a unique derivative with respect to z at all points of a region R .There are many methods to find analytic function with the help of real or imaginary function such as Milne Thomson and direct method etc. I have also noticed that by using different methods (Milne Thomson method, direct method), for a single u getting more than one values of v (i.e. v is not unique). To overcome this I proposed the modified form of direct method, which helps to find analytic function easily. For this provided counter examples with explanation.

Let Ω be a connected bounded domain on the complex plane, S be its boundary, which is closed, star-shaped, C^1-smooth, and $H(\Omega)$ is the set of analytic (holomorphic) in Ω functions. The aim of this paper is to prove that an arbitrary $f \in L^1(S)$,satisfying the condition $= 0$,can be boundary value of an $f \in H(\Omega)$.

Key words: Analytic function, Complex variable function, Cauchy- Riemann equation, Milne-Thomson's method, Exact- differential equation, Harmonic function, Laplace's equation, Holomorphic function, Meromorphic function, poles, essential singular point, single valued function, entire function.

1. Introduction

Let Ω be a connected bounded domain on the complex plane, S be its boundary, which is closed, star-shaped, C^1- smooth, and $H(\Omega)$ is the set of analytic (holomorphic) in Ω functions. The aim of this paper is to prove that an arbitrary $f \in L^1(S)$, satisfying the condition $\int_s f(s)ds = 0$, can be the boundary value of an analytic in Ω function.

Since S is star-shaped, there is a point $O \in \Omega$ such that every ray, issued at the O, intersects S at just one point. If $S_q := \{z : z = qs, s \in S\}$, $0 < q_0 \leq q \leq 1$, $S = S_1$, then

$$\int_{s_q} f(z)dz = \int_S f(qs)d(qs) = q\int_S f(qs)ds \qquad (1)$$

One has:

$$\int_{s_q} |f(z)||dz| = q\int_S |f(qs)||ds|. \qquad (2)$$

Let $||f|| := ||f||_{L1(S)} = \int_S |f(s)||ds|$. Assume that

$$||f|| < \infty. \qquad (3)$$

Theorem 1.

Assume that $f(z) \in H(\Omega)$, *there exists* $\lim_{q\to 1} \left| f(qs) - f(s) \right| = 0$ *and* (3)

holds. Then

$$\int_s f(s)ds = 0 \qquad (4)$$

and

$$f(z) = \frac{1}{2\pi i}\int_S \frac{f(s)ds}{s-z}, \quad z \in \Omega \qquad (5)$$

Theorem 2

Assume that $f(s) \in L^1(S)$ *and* $\int_s f(s)ds = 0$. *Then there exists* $f(z) \in H(\Omega)$

such that $\lim_{q\to 1} \left| f(qs) - f(s) \right| = 0$ *and formulas* (4), (5) *hold.*

Thus, any $f(s) \in L^1(S)$, such that $\int_s f(s)ds\, ds = 0$, is a boundary value of a function $f(z) \in H(\Omega)$ and formulas (4)–(5) hold.

Usually, It is assumed that

$f(z) \in H(\Omega)$ is continuous up to the boundary S. Theorem 1 shows that if $f \in H(\Omega)$ has a

boundary value in $L^1(S)$, then the Cauchy formula (5) holds and the boundary value satisfies

formula (4). Theorem 2 shows that any $f \in L^1(S)$, such that $\int_S f(s)ds\, ds = 0$, is a boundary value of an $f \in H(\Omega)$. In [3] thre are some conditions on the function $f(s)$, satisfying the Hoelder condition on S, for this function to be boundary value of a function from $H(\Omega)$.

2. Proofs

Proof of Theorem 1.

Let $\lim_{q\to 1} \left| f(qs) - f(s) \right| = 0$. Then, $\left| f(qs) \right| < \left| f(s) \right| + \epsilon$ as soon as

$|1 - q| < \delta$, where $\epsilon > 0$ is arbitrarily small if δ is sufficiently small. Therefore,

$$\lim_{q\to 1} \left| \int_{S_q} f(z)dz - \int_S f(s)ds \right| \leq \lim_{q\to 1} \left| f(qs) - f(s) \right| = 0. \quad (6)$$

Since $f \in H(\Omega)$, by the Cauchy theorem one has $\int_{S_q} f(z)dz = 0$. It follows that

$$0 = \lim_{q\to 1} \int_{S_q} f(z)dz = \int_S f(s)ds, \qquad z = qs. \quad (7)$$

Therefore, relation (4) is proved.

Similarly, starting with the Cauchy formula

$$\lim_{q\to 1} \frac{1}{2\pi i} \int_S \frac{f(\zeta)d\zeta}{\zeta - z} = f(z), \quad (8)$$

and passing to the limit $q \to 1$, one obtains formula (5).

Theorem 1 is proved

Proof of Theorem 2.

Let $f = f(s) \in L^1(S)$. We want to prove that there is a function $f = f(z) \in H(\Omega)$

with the boundary value f (s) on S. If f (s) ∈ L1(S), then there is a continuous on S function h_\epsilon(s)\ such that f - h_\epsilon ⩽ ϵ , where ϵ > 0 is an arbitrary small number.

The set of polynomials is dense in the space C(S) and, therefore, in L1(S), since S is

compact. Therefore, h_\epsilon can be chosen to be a polynomial. A polynomial can be considered as an element of H(Ω). Take the limit ϵ → 0 and a sequence of polynomials which converges to f in the norm of L1(S). Then

$$\lim_{\epsilon\to 0} \int_S h_\epsilon(s)ds = \int_S f(s) = 0 \quad (9)$$

The first integral vanishes by the Cauchy theorem, because polynomials are analytic functions.

We also have

$$\lim_{\epsilon\to 0} \frac{1}{2\pi i} \int_S \frac{h_\epsilon\, ds}{s - z} = \frac{1}{2\pi i} \int_S \frac{f(s)(s)ds}{s - z} := f(z), \quad (10)$$

Where $f(z) \in H(\Omega)$.

Theorem 2 is proved

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