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Simple Proof of Cauchy's Residue Theorem

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Abstract.

In this paper, I developed the simple proof of Cauchy's Residue Theorem and Cauchy's Residue Theorem is a powerful tool to evaluate line integrals of analytic functions over closed curves. Cauchy's Residue Theorem can be used to evaluate various types of integrals of real valued functions of real variable.

Keywords: Definite integral, Residue's theorem, Derivatives of high order formula, argument principle, Isolated singularities, Non- isolated singularities, Poles and Zeros of holomorphic functions, Analytic function, Complex variable function, essential singular point, single valued function, entire function, simple closed curve .

Introduction:

From later half of 18th Century to the beginning of 19th century, it started exploring complex function and partial derivative as well as integration property. Complex analysis really as one research field in modern analysis was established in 19th century, the main founders were Cauchy, Riemann and Weierstrass. Cauchy established complex variables functions differential and integral theories. The paper "Report regarding integration limit as imaginary number's definite integral" in 1814, 1825, it established Cauchy integral theorem; in 1826, it proposed residue concept; in 1831, it obtained Cauchy integral formula; in 1846, it found integral and path irrelevance theorem.

"Complex function theory" is normal university mathematics and applied mathematics major required course, meanwhile it is also comprehensive university science and engineering foundation course, is real variable functions calculus promotion and development. Among them, Cauchy integral theorem is the basis of complex function theory, is the key to research on complex function theory, is also most unique creation in 19th century, is one of most harmonious theories in abstract science, complex variables functions' lots of important properties and theorems are directly or indirectly deduced from it. The paper mainly by documents: it gets Cauchy integral theorem actually is the residue theorem that integrand has first order pole in integration region; derivatives of high order formula actually is the residue theorem that integrand has order pole in integration region. Make comparative summary on them, and effective combine with documents, it concludes Cauchy integral theorem and Cauchy integral formula, derivatives of high order formula, residue theorem their deduction relations.

Since Cauchy, complex function theory has already more than 170 years' history. It becomes one of important compositions in mathematics with its perfect theory and exquisite technique. It has ever promoted some disciplines development, and tends to be applied into practical problems as a powerful tool, its basic contents have already become science and engineering many majors required courses.

Now, complex function theory still has some subjects that to be researched, so it will continue to move forward and get more applications.

Cauchy's Residue theorem,

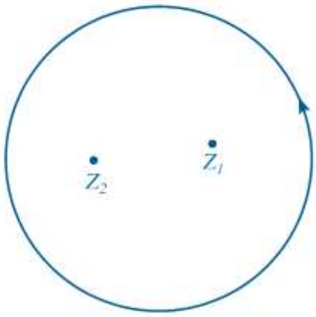
Statement:-

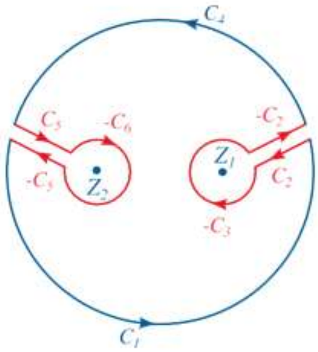
Suppose $f(z)$ is analytic in the region A except for a set of isolated singularities. Also suppose C is a simple closed curve in A that doesn't go through any of the singularities of f and is oriented counterclockwise. Then

$$\int_c f(z)\,dz = 2\pi i \sum residues\ of\ f\ inside\ C \qquad (1)$$

Proof :

The proof is based of the following figures. They only show a curve with two singularities inside it, but the generalization to any number of singularities is straight forward. In what follows we are going to abuse language and say pole when we mean isolated singularity,





i.e. A finite order pole or an essential singularity ('infinite order pole').

The left figure shows the curve C surrounding two poles z_1 *and* z_2 of $\boldsymbol{f}$. The right figure shows the same curve with some cuts and small circles added. It is chosen so that there are no poles of $\boldsymbol{f}$ inside it and so that the little circles around each of the poles are so small that there are no other poles inside them.

The right hand curve is

$$\tilde{c} = c_1 + c_2 - c_3 - c_2 + c_4 + c_5 - c_6 - c_5 \qquad (2)$$

The left hand curve is $c = c_1 + c_4$ Since there are no poles inside $\tilde{c}$ we have, by Cauchy's theorem,

$$\int_{\tilde{c}} f(z)dz = \int_{c_1+c_2-c_3-c_2+c_4+c_5-c_6-c_5} f(z)dz = 0 \qquad (3)$$

Dropping C_2 and C_5, which are both added and subtracted, this becomes

$$\int_{c_1+c_4} f(z)\ dz = \int_{c_3+c_6} f(z)\ dz \qquad (4)$$

Suppose

$$f(z) = \dots\dots\dots\dots\dots + \frac{b_2}{(z-z_1)^2} + \frac{b_1}{z-z_1} + a_0 + a_1(z-z_1) + \dots\dots\dots\dots\dots dz \qquad (5)$$

is the Laurent expansion of f around z_1 then

$$\int_{c_3} f(z)\ dz = \int_{c_3} \dots\dots\dots + \frac{b_2}{(z-z_1)^2} + \frac{b_1}{z-z_1} + a_0 + a_1(z-z_1)\dots\dots\dots dz \qquad (6)$$

$$= 2\pi i b_1$$

$$= 2\pi i Res\,(f, z_1)$$

Similarly

$$\int_{c_6} f(z)dz = 2\pi i Res\,(f, z_2) \qquad (7)$$

Using these residues and the fact that $c = c_1 + c_4$ from equation (4)

$$\int_c f(z) = 2\pi i\,[Res\,(f, z_1) + Res\,(f, z_2)].$$

That proves the residue theorem for the case of two poles. As we said, generalizing to any number of poles is straight forward.

Significance of Cauchy's Residue theorem:

"The Cauchy's Residue theorem is one of the major theorems in complex analysis and will allow us to make systematic our previous somewhat ad hoc approach to computing integrals on contours that surround singularities."

Note:

In complex analysis, the residue theorem, sometimes called Cauchy's residue theorem, is a powerful tool to evaluate line integrals of analytic functions over closed curves; it can often be used to compute real integrals and infinite series as well. It generalizes the Cauchy integral theorem and Cauchy's integral formula. From a geometrical perspective, it can be seen as a special case of the generalized Stokes' theorem.

Cauchy’s Residue Theorem work:

The Residue Theorem relies on what is said to be the most important theorem in Complex Analysis, Cauchy's Integral Theorem. The Integral Theorem states that integrating any complex valued function around a curve equals zero if the function is differentiable everywhere inside the curve.

Applications of residue theorem:

The residue theorem has applications in functional analysis, linear algebra, analytic number theory, quantum field theory, algebraic geometry, Abelian integrals or dynamical systems.

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