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Observations on the Paper Entitled Solutions of the Homogeneous Cubic Equation with Six Unknowns $(w^2+p^2-z^2)(w-p)=(k^2+2)(x+y)R^2$

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Abstract

This paper illustrates the process of obtaining different integer solutions to the homogeneous cubic equation with six unknowns. $(w^2+p^2-z^2)(w-p)=(k^2+2)(x+y)R^2$

Keywords: Homogeneous cubic, Cubic with six unknowns, Integer solutions.

Introduction

While making a survey on higher degree Diophantine equations, the homogeneous cubic equation with six unknowns given in [1] came to our reference in which the authors have obtained three patterns of integer solutions. However, there are other choices of integer solutions which we exhibit in this paper.

Method of analysis

The homogeneous cubic equation with six unknowns to be solved is

$$(w^2+p^2-z^2)(w-p)=(k^2+2)(x+y)R^2 \quad (1)$$

Introduction of the linear transformations

$$x=v+1, y=v-1, z=u, w=u+v, p=u-v, u\neq v, v\neq 1 \quad (2)$$

in (1) leads to

$$u^2+2v^2=(k^2+2)R^2 \quad (3)$$

The above equation (3) is solved through different ways and thus, one obtains different sets of integer solutions to (1).

Way 1:

It is seen that (3) is satisfied by

$$u=k(k^2+2), v=(k^2+2), R=(k^2+2) \quad (4)$$

In view of (2),the corresponding integer solutions to (1) are given by

$$x=k^2+3, y=k^2+1, z=k(k^2+2), w=(k+1)(k^2+2), p=(k-1)(k^2+2), R=k^2+2$$

Way 2:

(3) is written as

$$(k^2+2)R^2-2v^2=u^2=u^2*1 \quad (5)$$

Assume

$$u=(k^2+2)a^2-2b^2 \quad (6)$$

Write the integer 1 on the R.H.S. of (5) as

$$1=\frac{(\sqrt{k^2+2}+\sqrt{2})(\sqrt{k^2+2}-\sqrt{2})}{k^2} \quad (7)$$

Substituting (6) & (7) in (5) and employing the method of factorization, consider

$$\sqrt{k^2+2}\,R+\sqrt{2}\,v=\frac{(\sqrt{k^2+2}+\sqrt{2})(\sqrt{k^2+2}\,a+\sqrt{2}\,b)^2}{k}$$

Equating the coefficients of corresponding terms, note that

$$R=\frac{(k^2+2)a^2+2b^2+4ab}{k}, v=\frac{(k^2+2)a^2+2b^2+2(k^2+2)ab}{k} \quad (8)$$

Since our aim is to obtain integer solutions, taking in (6) $a=kA, b=kB$ & (8) and

using (2),the corresponding integer solutions to (1) are as below:

$$x=k[(k^2+2)A^2+2B^2+2(k^2+2)AB]+1,$$
$$y=k[(k^2+2)A^2+2B^2+2(k^2+2)AB]-1,$$
$$z=k^2[(k^2+2)A^2-2B^2],$$
$$w=(k^2+k)(k^2+2)A^2+2B^2(k-k^2)+2k(k^2+2)AB,$$
$$p=(k^2-k)(k^2+2)A^2-2B^2(k+k^2)-2k(k^2+2)AB,$$
$$R=k[(k^2+2)A^2+2B^2+4AB]$$

Way 3:

Write (3) as

$$(k^2+2)R^2-u^2=2v^2 \quad (9)$$

Assume V as

$$v=(k^2+2)a^2-b^2 \quad (10)$$

Write the integer 2 on the R.H.S. of (9) as

$$2=(\sqrt{k^2+2}+k)(\sqrt{k^2+2}-k) \quad (11)$$

Following the procedure as in Way 2, the corresponding integer solutions to (1) are given by

$$x=(k^2+2)a^2-b^2+1, y=(k^2+2)a^2-b^2-1, z=k(k^2+2)a^2+k\,b^2+2(k^2+2)ab,$$
$$w=(k+1)(k^2+2)a^2+(k-1)\,b^2+2(k^2+2)ab, p=(k-1)(k^2+2)a^2+(k+1)\,b^2+2(k^2+2)ab,$$
$$R=(k^2+2)a^2+b^2+2kab$$

Generation of solutions:

Let (u_0, v_0, R_0) be any given integer solution to (3). We illustrate below the method of obtaining a general formula for generating sequence of integer solutions based on the given solution.

Illustration (i):

Let $R_1=3R_0,\ u_1=h-3u_0,\ v_1=h-3v_0$ (12)

be the second solution of (3), Substituting (12) in (3) & performing a few calculations, we have

$h = 2u_0 + 4v_0$, and then

$$u_1 = -u_0 + 4v_0$$

$$v_1 = 2u_0 + v_0$$

This is written in the form of matrix as

$$\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}^t = M \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}^t \quad (13)$$

where$M = \begin{pmatrix} -1 & 4 \\ 2 & 1 \end{pmatrix}$, and 't' is the transpose

Repeating the above process, the general solution $(u_n, v_n)$to (3) is given by

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix}^t = M^n \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}^t \quad (14)$$

To find M^n, the eigen values of M are$\alpha = 3, \beta = -3$

We know that

$$M^n = \frac{\alpha^n}{\alpha - \beta}(M - \beta I) + \frac{\beta^n}{\beta - \alpha}(M - \alpha I)$$

Using the above formula, we have

$$M^n = \begin{pmatrix} 3^{n-1}(1 + 2(-1)^n) & 2.3^{n-1}(1 - (-1)^n) \\ 3^{n-1}(1 - (-1)^n) & 3^{n-1}(2 + (-1)^n) \end{pmatrix}$$

In view of (2), the general solution to (1) is given by

$x_n = 3^{n-1}(1 - (-1)^n)u_0 + 3^{n-1}(2 + (-1)^n)v_0 + 1$

$y_n = 3^{n-1}(1 - (-1)^n)u_0 + 3^{n-1}(2 + (-1)^n)v_0 - 1$

$z_n = 3^{n-1}(1 + 2(-1)^n)u_0 + 2.3^{n-1}(1 - (-1)^n)v_0$

$R_n = 3^n R_0$

$w_n = 3^{n-1}(2 + (-1)^n)u_0 + 3^{n-1}(4 - (-1)^n)v_0$

$p_n = 3^n(-1)^n u_0 - 3^n(-1)^n v_0$, n=1,2,3…….

Where

$u_n = 3^{n-1}(1 + 2(-1)^n)u_0 + 2.3^{n-1}(1 - (-1)^n)v_0$

$v_n = 3^{n-1}(1 - (-1)^n)u_0 + 3^{n-1}(2 + (-1)^n)v_0$

Illustration (ii):

Let $v_1 = (k^2 + 1)v_0$

$$u_1 = h + (k^2 + 1)u_0$$

$$R_1 = h - (k^2 + 1)R_0$$

Repeating the process as in the illustration (i) the corresponding general solution to (1) is given by

$x_n = (k^2 + 1)^n v_0 + 1$

$y_n = (k^2 + 1)^n v_0 - 1$

$z_n = \left(\frac{\alpha^n + \beta^n}{2}\right)u_0 + \frac{\sqrt{k^2 + 2}}{2}(\alpha^n - \beta^n)R_0$

$w_n = \left(\frac{\alpha^n + \beta^n}{2}\right)u_0 + \frac{\sqrt{k^2 + 2}}{2}(\alpha^n - \beta^n)R_0 + (k^2 + 1)^n v_0$

$p_n = \left(\frac{\alpha^n + \beta^n}{2}\right)u_0 + \frac{\sqrt{k^2 + 2}}{2}(\alpha^n - \beta^n)R_0 - (k^2 + 1)^n v_0$

Where

$u_n = \left(\frac{\alpha^n + \beta^n}{2}\right)u_0 + \frac{\sqrt{k^2 + 2}}{2}(\alpha^n - \beta^n)R_0$

$v_n = (k^2 + 1)^n v_0$

Illustration (iii):

Let $u_1 = k^2 u_0$

$$v_1 = h + k^2 v_0$$

$$R_1 = h - k^2 R_0$$

Repeating the process as in the illustration (i) the corresponding general solution to (1) is given by

$$x_n = \left(\frac{\alpha^n + \beta^n}{2}\right) u_0 + \frac{\sqrt{2k^2+4}}{4}(\alpha^n - \beta^n) R_0 + 1$$

$$y_n = \left(\frac{\alpha^n + \beta^n}{2}\right) u_0 + \frac{\sqrt{2k^2+4}}{4}(\alpha^n - \beta^n) R_0 - 1$$

$$z_n = k^{2n} u_0$$

$$w_n = k^{2n} u_0 + \left(\frac{\alpha^n + \beta^n}{2}\right) u_0 + \frac{\sqrt{2k^2+4}}{4}(\alpha^n - \beta^n) R_0$$

$$p_n = k^{2n} u_0 - \left(\frac{\alpha^n + \beta^n}{2}\right) u_0 + \frac{\sqrt{2k^2+4}}{4}(\alpha^n - \beta^n) R_0$$

Where

$$u_n = k^{2n} u_0$$

$$v_n = \left(\frac{\alpha^n + \beta^n}{2}\right) u_0 + \frac{\sqrt{2k^2+4}}{4}(\alpha^n - \beta^n) R_0$$

Way 4:

In view of (3),

$$u^2 = (k^2+2)R^2 - 2v^2 \qquad (15)$$

Introducing the linear transformation

$$R = X + 2T,\ v = X + (k^2+2)T,\ u = kU \qquad (16)$$

in (3), it is written as

$$X^2 = 2(k^2+2)T^2 + U^2 \qquad (17)$$

which is satisfied by the system of double equations as in case(a) & case(b)

Case (a):

$$\mathrm{X} + \mathrm{U} = 2(k^2+2)T$$

$$X - U = T$$

Solving these two linear equations, we get

$$X = \frac{(2k^2+5)T}{2}$$

$$U = \frac{(2k^2+3)T}{2}$$

put T=2s then we get the integer solution of X and U are as

$$X = (2k^2+5)s$$

$$U = (2k^2+3)s$$

Substituting the values of X, U, T in (16), we get the non-trivial integer solutions of equation (1) are given by

$$x = (4k^2+9)s + 1$$

$$y = (4k^2 + 9)s - 1$$

$$z = (2k^3 + 3k)s$$

$$R = (2k^2 + 9)s$$

$$w = (2k^3 + 4k^2 + 3k + 9)s$$

$$p = (2k^3 - 4k^2 + 3k - 9)s$$

Case (b):

$$\mathrm{X} + \mathrm{U} = (k^2 + 2)T$$

$$X - U = 2T$$

Repeating the process as in the case (a) the corresponding solution to (1) is given by

$x = (3k^2 + 8)s + 1$

$y = (3k^2 + 8)s - 1$

$z = k^3 s$

$R = (k^2 + 8)s$

$w = (k^3 + 3k^2 + 8)s$

$p = (k^3 - 3k^2 - 8)s$

Way 5:

In view of (17), for k=4, we have

$$X^2 = 36T^2 + U^2$$

which is satisfied by

$6T = 2rs$

$U = r^2 - s^2$

$X = r^2 + s^2$

Put$r = 3\bar{R}$in the above equations we get

$T = \bar{R}s, U = 9\bar{R}^2 - s^2, X = 9\bar{R}^2 + s^2$

In view of (2), the non-zero distinct integer solutions to (1) are given by

$x = 9a^2 + s^2 + 18as + 1$

$y = 9a^2 + s^2 + 18as - 1$

$z = 36a^2 - 4s^2$

$R = 9a^2 + s^2 + 2as$

$w = 45a^2 - 3s^2 + 18as$

$p = 27a^2 - 5s^2 - 18as$

Reference:

[1] M. A. Gopalan, N. Thirunraiselvi, K. Agalya, Solutions of the Homogeneous Cubic Equation with six unknowns $(w^2 + p^2 - z^2)(w - p) = (k^2 + 2)(x + y)R^2$, Jamal Academic Research Journal: An Interdisciplinary Special Issue, Pp: 273-278, February 2016.