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Solving Fuzzy Differential Equations using Error Analysis

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Abstract

In this paper, numerical algorithms for solving fuzzy ordinary differential equations are considered. A scheme based on the Modified two-step Simpson method is discussed in detail and this is followed by a complete error analysis.

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1. Introduction

Fuzziness is a basic type of uncertainty in real world. For a deterministic differential equation, the easiest way to introduce fuzziness is to assume that the initial value is a fuzzy variable. Such an equation was first called fuzzy differential equation by Kandel and Byatt. Fuzzy differential equations and initial value problems were regularly treated by O.Kaleva in [9] and [10], S.Seikkala. The organized paper is as follows: In the first three sections, we recall some concepts and introductory materials to deal with the fuzzy initial value problem. Solving numerically the fuzzy differential equation by Modified two-step Simpson method is discussed in section 4. The proposed algorithm is illustrated by an example in the last section.

2. Preliminary

A trapezoidal number u is defined by four real numbers $a < b < c < d$, where the base of the trapezoidal is the interval [a, d] and its vertices at $x = b$, $x = c$. Trapezoidal fuzzy numbers will be written as $u = (a, b, c, d)$. The membership function for the trapezoidal fuzzy number $u = (a, b, c, d)$ is defined as the following:

$$u(x) = \begin{cases} \frac{x-a}{b-a}, & a \le x \le b \\ 1, & b \le x \le c \\ \frac{x-d}{c-d}, & c \le x \le d \end{cases} \qquad (1)$$

we will have : $u > 0$ if $a > 0$; $u > 0$ if $b > 0$; $u > 0$ if $c > 0$; & $u > 0$ if $d > 0$.

Let us denote R_F by the class of all fuzzy subsets of R (i.e. $u : R \to [0,1]$) satisfying the following properties:

(i) $\forall u \in R_F$, u is normal, i.e. $\exists x_0 \in R$ with $u(x_0) = 1$;
(ii) $\forall u \in R_F$, u is convex fuzzy set (i.e. $u(tx + (1-t)y) \ge min\{u(x), u(y)\}$, $\forall t \in [0,1]$, $x, y \in R$);
(iii) $\forall u \in R_F$, u is upper semi continuous on R;
(iv) $\overline{\{x \in R;\ u(x) > 0\}}$ is compact, where $\overline{A}$ denotes the closure of A.

Then R_F is called the space of fuzzy numbers Obviously $R \subset R_F$. Here $R \subset R_F$ is understood as $R = \{\chi_{\{x\}}; x \text{ is usual real number}\}$. We define the r-level set, $x \in R$;

$$[u]_r = \{x \setminus u(x) \ge r\}, \quad 0 \le r \le 1; \qquad (2)$$

Clearly, $[u]_0 = \{x \setminus u(x) > 0\}$ is compact,

which is a closed bounded interval and we denote by $[u]_r = [\underline{u}(r), \bar{u}(r)]$. It is clear that the following statements are true,

1. $\underline{u}(r)$ is a bounded left continuous non decreasing function over [0,1],
2. $\bar{u}(r)$ is a bounded right continuous non increasing function over [0,1],
3. $\underline{u}(r) \le \bar{u}(r)$ for all $r \in (0,1]$,

for more details see [2],[3].

Let $D: R_F \times R_F \to R_+ \cup \{0\}$, $D(u,v)=\mathrm{Sup}_{r\in[0,1]} \max\{|\underline{u}(r)-\underline{v}(r)|, |\bar{u}(r)-\bar{v}(r)|\}$, be *Hausdorff distance* between fuzzy numbers, where $[u]_r = [\underline{u}(r), \bar{u}(r)]$, $[v]_r = [\underline{v}(r), \bar{v}(r)]$. The following properties are well-known :

$D(u + w, v + w) = D(u, v), \ \forall\ u,v,w \in R_F,$

$D(k.u, k.v) = |k|D(u, v), \ \forall\ k \in R,\ u,v \in R_F,$

$D(u + v, w + e) \le D(u,w) + D(v,e), \ \forall\ u, v, w, e \in R_F$

and (R_F, D) is a complete metric space.

3. Fuzzy Initial Value Problem

Consider a first-order fuzzy initial value differential equation is given by

$$\begin{cases} y'(t) = f(t, y(t)), & t \in [t_0, T] \\ y(t_0) = y_0 \end{cases} \qquad (3)$$

where y is a fuzzy function of t, $f(t, y)$ is a fuzzy function of the crisp variable t and the fuzzy variable y , y' is the fuzzy derivative of y and $y(t_0) = y_0$ is the trapezoidal shaped fuzzy number.

We denote the fuzzy function y by $y = [\underline{y}, \overline{y}]$. It means that the r-level set of y(t) for

$t \in [t0, T]$ is

$$[y(t)]^r = [\underline{y}(t; r), \bar{y}(t; r)], \qquad [y(t_0)]^r = [\underline{y}(t^0; r), \bar{y}(t^0; r)], \ r \in (0,1]$$

we write $f(t, y) = [\underline{f}(t, y), \overline{f}(t, y)]$ and

$\underline{f}(t, y) = F[t, \underline{y}, \bar{y}]$, $\overline{f}(t, y) = G[t, \underline{y}, \bar{y}]$.

Because of $y' = f(t, y)$ we have

$$\underline{f}(t, y(t); r) = F[t, \underline{y}(t; r), \bar{y}(t; r)] \qquad (4)$$

$$\overline{f}(t, y(t); r) = G[t, \underline{y}(t; r), \bar{y}(t; r)] \qquad (5)$$

By using the extension principle, we have the membership function

$$f(t, y(t))(s) = \sup\{y(t)(\tau) \mid s = f(t, \tau)\}, \ s \in R \qquad (6)$$

so fuzzy number $f(t, y(t))$. From this it follows that

$$[f(t, y(t))]_r = [\underline{f}(t, y(t); r), \overline{f}(t, y(t); r)], \qquad r \in (0,1], \qquad (7)$$

where

$$\underline{f}(t, y(t); r) = \min\{f(t, u) \mid u \in [y(t)]_r\} \qquad (8)$$

$$\overline{f}(t, y(t); r) = \max\{f(t, u) \mid u \in [y(t)]_r\}. \qquad (9)$$

4. Modified Two-Step Simpson Method for solving Fuzzy Differential Equation

Let $Y = [\underline{Y}, \bar{Y}]$ be the exact solution and $y = [\underline{y}, \bar{y}]$ be the approximated solution of the fuzzy initial value problem (3). Let $[Y(t)]_r = [\underline{Y}(t ; r), \bar{Y}(t ; r)]$, $[y(t)]_r = [\underline{y}(t ; r), \bar{y}(t ; r)]$.

Throughout this argument, the value of r is fixed. Then the exact and approximated solution at t_n are respectively denoted by

$[Y(t_n)]_r = [\underline{Y}(t_n ; r), \bar{Y}(t_n ; r)]$, $[y(t_n)]_r = [\underline{y}(t_n ; r), \bar{y}(t_n ; r)]$ $(0 \le n \le N)$.

The grid points at which the solution is calculated are

$h = \frac{T - t_0}{N}, t_i = t_0 + i, \ 0 \le i \le N$. Then we obtain,

$$\underline{Y}(t_{n+1}; r) = \underline{Y}(t_{n-1}; r) + \frac{h}{3} F\left[t_{n-1}, \underline{Y}(t_{n-1}; r), \overline{Y}(t_{n-1}; r)\right] + \frac{4h}{3} F\left[t_n, \underline{Y}(t_n; r), \overline{Y}(t_n; r)\right] + \frac{h}{3} F\begin{bmatrix} t_{n+1}, \underline{Y}(t_n; r) + hF\left[t_n, \underline{Y}(t_n; r), \overline{Y}(t_n; r)\right], \\ \overline{Y}(t_n; r) + hG\left[t_n, \underline{Y}(t_n; r), \overline{Y}(t_n; r)\right] \end{bmatrix} \qquad (10)$$ and

$$\overline{Y}(t_{n+1}; r) = \overline{Y}(t_{n-1}; r) + \frac{h}{3} G\left[t_{n-1}, \underline{Y}(t_{n-1}; r), \overline{Y}(t_{n-1}; r)\right] + \frac{4h}{3} G\left[t_n, \underline{Y}(t_n; r), \overline{Y}(t_n; r)\right] + \frac{h}{3} G\begin{bmatrix} t_{n+1}, \underline{Y}(t_n; r) + hF\left[t_n, \underline{Y}(t_n; r), \overline{Y}(t_n; r)\right], \\ \overline{Y}(t_n; r) + hG\left[t_n, \underline{Y}(t_n; r), \overline{Y}(t_n; r)\right] \end{bmatrix} \qquad (11)$$

Also we have

$$\underline{y}(t_{n+1};r) = \underline{y}(t_{n-1};r) + \frac{h}{3}F\left[t_{n-1}, \underline{y}(t_{n-1};r), \bar{y}(t_{n-1};r)\right] + \frac{4h}{3}F\left[t_n, \underline{y}(t_n;r), \bar{y}(t_n;r)\right] + \frac{h}{3}F\begin{bmatrix} t_{n+1}, \underline{y}(t_n;r) + hF\left[t_n, \underline{y}(t_n;r), \bar{y}(t_n;r)\right], \\ \bar{y}(t_n;r) + hG\left[t_n, \underline{y}(t_n;r), \bar{y}(t_n;r)\right] \end{bmatrix} \quad (12)$$

and

$$\bar{y}(t_{n+1};r) = \bar{y}(t_{n-1};r) + \frac{h}{3}G\left[t_{n-1}, \underline{y}(t_{n-1};r), \bar{y}(t_{n-1};r)\right] + \frac{4h}{3}G\left[t_n, \underline{y}(t_n;r), \bar{y}(t_n;r)\right] + \frac{h}{3}G\begin{bmatrix} t_{n+1}, \underline{y}(t_n;r) + hF\left[t_n, \underline{y}(t_n;r), \bar{y}(t_n;r)\right], \\ \bar{y}(t_n;r) + hG\left[t_n, \underline{y}(t_n;r), \bar{y}(t_n;r)\right] \end{bmatrix} \quad (13)$$

Clearly, $\underline{y}(t;r)$ and $\bar{y}(t;r)$ converge to $\underline{Y}(t;r)$ and $\bar{Y}(t;r)$, respectively whenever $h \to 0$ [4].

5. Numerical Results

In this section, the exact solutions and approximated solutions are obtained by Modified two-step Simpson method are plotted in Figures 1, 2, 3 & 4.

Example 5.1

Consider the initial value problem [11] $\begin{cases} y'(t) = t\,f(t), & t \in [0,1] \\ y(0) = \left[\left(1.5 + r\sqrt{e},\ 5.5 - 0.5\,r\sqrt{e}\right)\right]. \end{cases}$

The exact solution at $t = 0.1$ is given by

$$Y(1;r) = \left[\left(1.5 + r\sqrt{e}\right)e\ ,\left(\ 5.5 - 0.5\,r\sqrt{e}\right)e\right], \quad 0 \le r \le 1.$$

Using iterative solution of Modified two-step Simpson method, we have

$$\underline{y}(0;r) = 1.5 + r\sqrt{e}, \quad \bar{y}(0;r) = 5.5 - 0.5\,r\sqrt{e}.$$

and by

$$\underline{y}^{(0)}(t_{i+1};r) = \underline{y}(t_i;r) + h\underline{y}(t_i;r) + \frac{h^2}{2}\underline{y}(t_i;r)$$

$$\bar{y}^{(0)}(t_{i+1};r) = \bar{y}(t_i;r) + h\bar{y}(t_i;r) + \frac{h^2}{2}\bar{y}(t_i;r),$$

where $i = 0, 1, \ldots, N-1$ and $h = \frac{1}{N}$. Now, using these equations as an initial guess for following iterative solutions respectively,

$$\underline{y}(t_{i+1};r) = \underline{y}(t_{i-1};r) + \frac{h}{3}\underline{y}(t_{i-1};r) + \frac{4h}{3}\underline{y}(t_i;r) + \frac{h}{3}\left[\underline{y}(t_i;r) + h\underline{y}(t_i;r)\right]$$

$$\bar{y}(t_{i+1};r) = \bar{y}(t_{i-1};r) + \frac{h}{3}\bar{y}(t_{i-1};r) + \frac{4h}{3}\bar{y}(t_i;r) + \frac{h}{3}\left[\bar{y}(t_i;r) + h\bar{y}(t_i;r)\right]$$

and $j = 1, 2, 3$. Thus, we have $\underline{y}(t_i;r) = \underline{y}^{(3)}(t_i;r)$ and $\bar{y}(t_i;r) = \bar{y}^{(3)}(t_i;r)$, for $i = 1 \ldots N$. Therefore, $\underline{Y}(1;r) \approx \underline{y}^{(3)}(1;r)$ and $\bar{Y}(1;r) \approx \bar{y}^{(3)}(1;r)$ are obtained.

Table 2 & 3, shows estimation of approximated solution and error for different values of $r \in [0,1]$ and h.

Table 1: Exact solution

Modified Simpson Method	
r	Trapezoidal fuzzy number
0	1.507519 , 5.527569
0.1	1.673217 , 5.444719
0.2	1.838916 , 5.361870
0.3	2.004614 , 5.279021
0.4	2.170313 , 5.196172
0.5	2.336011 , 5.113323

0.6	2.501710 , 5.030474
0.7	2.667408 , 4.947624
0.8	2.833107 , 4.864775
0.9	2.998805 , 4.781926
1	3.164504 , 4.699076

Table 2: Approximated solution for different values of r & *h*

Trapezoidal fuzzy number		
h / *r*	0.1	0.01
0	1.525143 , 5.592189	1.502499 , 5.509164
0.1	1.692778 , 5.508371	1.667646 , 5.426594
0.2	1.860414 , 5.424553	1.832793 , 5.344021
0.3	2.028049 , 5.340735	1.997940 , 5.261442
0.4	2.195685 , 5.256918	2.163088 , 5.178869
0.5	2.363320 , 5.173100	2.328235 , 5.096300
0.6	2.530956 , 5.089283	2.493381 , 5.013729
0.7	2.698592 , 5.005465	2.658527 , 4.931149
0.8	2.866227 , 4.921647	2.823673 , 4.848575
0.9	3.033863 , 4.837829	2.988820 , 4.766006
1	3.201499 , 4.754011	3.153966 , 4.683436

Table 3: E**rror for different values of r & *h***

Trapezoidal fuzzy number		
h / *r*	0.1	0.01
0	0.082244	0.023424
0.1	0.083213	0.023696
0.2	0.084180	0.023972
0.3	0.085150	0.024253
0.4	0.086118	0.024527
0.5	0.087087	0.024799
0.6	0.088055	0.025074
0.7	0.089024	0.025356
0.8	0.089993	0.025634
0.9	0.090961	0.025905
1	0.091930	0.026178

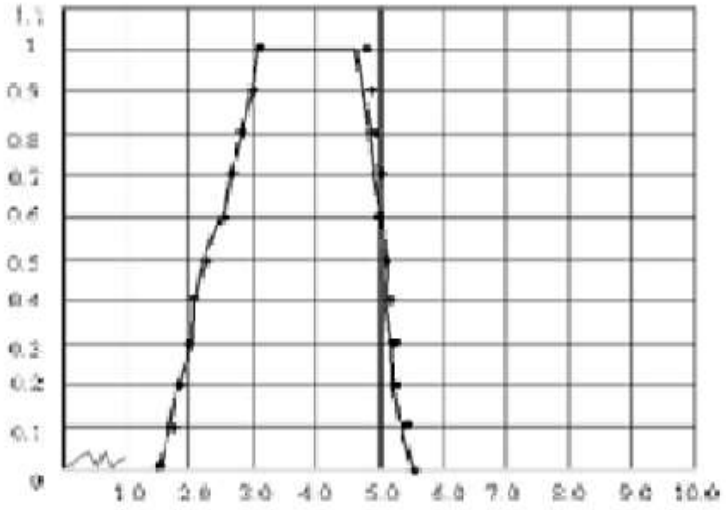

Figure 1 : h = 0.1

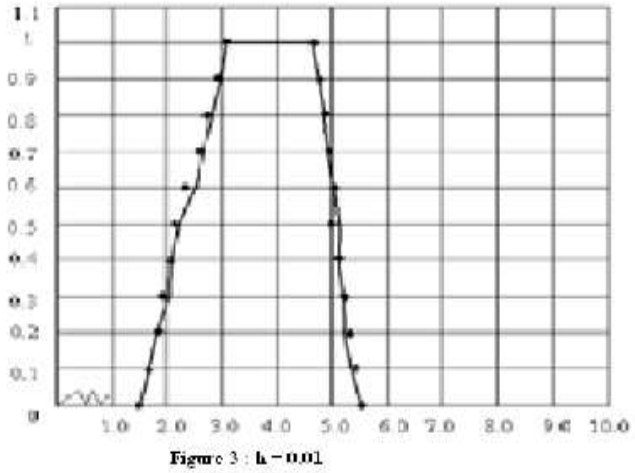

Figure 3 : h = 0.01

Mod.Simpson •
Exact ——

Graphical Representation of exact and approximated solution

Conclusion:

By minimizing the step size h, the solution by exact method and Modified two-step Simpson method almost coincides.

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