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Minimum Surface Area of a Cake with Bigger Portion in Cylindrical Shape and Remaining Portion in Shape of a Spherical Cap and Minimum Distance Between a Point and a Curve

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ABSTRACT

A cake is in the shape of a cylinder extended by spherical cap. Volume of the spherical cap is calculated in terms of its depth. With given volume of the stuff in the cake its minimum surface area is determined. Maximum volume of a cylinder surmounted by a cone and maximum volume of cylinder surmounted an inverted hemisphere are also separately determined.

INTRODUCTION

Many textbooks on Differential Calculus [1]deals with optimisation problems, viz, evaluation of minimum surface area and of maximum volume subject to some constraints.SN Maitra[1 to 6] published papers related to Lagrange's Multiplier. To begin with, we solve a textbook unsolved problem(S. Narayan) of optimisation. A pyramid with vertex at (0,0,C) is inscribed in an ellipsoid

$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{x^2}{c^2}=1$ (1)

and the sides parallel to the elliptic section. Show that the maximum volume of the pyramid is $\frac{64}{81}abc$.

SOLUTION TO TO THE PROBLEM

Let x,y,z be the coordinates of one corner of the pyramid base;(c-z) becomes its height. Then volume of the pyramid inscribed in ellipsoid [1] is

V=$\frac{4}{3}xy(c-z)$ (2)

In order to find the maximum volume of the pyramid inscribed,drawing the relevant figure we choose function F and Lagrange's Multiplierλ such that using (1) and (2),we get

F=$\frac{4}{3}xy(c-z)+\lambda(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{x^2}{c^2})$ (3)

$\frac{\delta F}{\delta x}=\frac{4}{3}y(c-z)-\lambda\frac{2x}{a^2}=0$ (4)

$\frac{\delta F}{\delta y}=\frac{4}{3}x(c-z)-\lambda\frac{2y}{b^2}=0$ (5)

$\frac{\delta F}{\delta z}=-\frac{4}{3}xy-\lambda\frac{2z}{c^2}=0$ (6)

Multiplying (4),(5), (6) respectively by x,y ,z and adding and then using (1),one gets

$\frac{2}{3}xy(c-z)+\frac{2}{3}xy(c-z)-\frac{2}{3}xyz=\lambda(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{x^2}{c^2})$

Or, $\lambda=\frac{4}{3}xyc-2xyz$ (7)

Substituting (7) into (4), one gets

$\frac{2}{3}y(c-z)=\frac{x}{a^2}(\frac{4}{3}xyc-2xyz)$

Or,$\frac{x^2}{a^2}=\frac{c-z}{2c-3z}$ (8)

$\frac{y^2}{b^2}=\frac{c-z}{2c-3z}$ (9)

Adding (8) and (9) and using (1) we get

$\frac{2(c-z)}{2c-3z}+\frac{z^2}{c^2}$= 1

Or, $\frac{2(c-z)}{2c-3z}=1-\frac{z^2}{c^2}$

Or, $\frac{2(c-z)}{2c-3z}=\frac{c^2-z^2}{c^2}$

$\frac{2}{2c-3z}=\frac{(c+z)}{c^2}$

Or, -3zc+$2c^2-3z^2+2zc=2c^2$

Or, $3z^2+zc=0$

0r,z=$\frac{-c}{3}$ (10)

which(negative sign) indicates that z-coordinate is below the centre of the sphere.

Because (10),equations (8) and (9) give

$\frac{x^2}{a^2}$=$\frac{y^2}{b^2}$= $\frac{4}{9}$ (11)

By virtue of (10) and (11), equation (1) gives the maximum volume of the inscribed pyramid as

$V_{max} = \frac{64}{81}abc$ (12)

As far as introduction is concerned, let us solve another textbook problem[1] herein: Find the minimum distance from point (3,4,15) to the cone $x^2 + y^2 = 4z^2$.

Solution the problem:

Let us consider a point (x,y,z) on the cone of given equation which is constraint equation

$x^2 + y^2 = 4z^2$ (13)

The distance S of the point (3,4,15) to this point on the cone is given by

S^2= $(x-3)^2+(y-4)^2+(z-15)^2$ (14)

Using (13) and (14) and Lagrange's Multiplier λ we form function F such that

F= $(x-3)^2+(y-4)^2+(z-15)^2+\lambda(x^2 + y^2-4z^2)$

$\frac{\delta F}{\delta x}=0, (x-3)+\lambda x$ =0 (15)

$\frac{\delta F}{\delta y}=0, (y-4)+\lambda y$=0 (16)

$\frac{\delta F}{\delta z}=0, (z-15)-4\lambda z$=0 (17)

The above four equations lead to

$\frac{x-3}{x}=\frac{y-4}{y}=\frac{z-15}{-4z}=-\lambda$ (18)

Or, $\frac{3}{x}=\frac{4}{y}$ *Or, y=*$\frac{4}{3}x$*. From second equation of (18),*

-4yz+16z=yz-15y

Or,16z= 5y(z-3)

Y= $\frac{16z}{5(z-3)}$ and x=$\frac{12z}{5(z-3)}$ (19)

Substituting (19) in(13), one gets

$(\frac{16z}{5(z-3)})^2+(\frac{12z}{5(z-3)})^2=4z^2$

Or,$(z-3)^2$=*4*

0r z=5 or 1 *(20)*

which due to (19) gives

Y= 8 and x=6 (21)

0r,

5Y=- 8 and 5x=-6 (22)

Value (20) suggests that the minimum distance will occur when z=5 and the maximum when z=1.

Hence putting the values of x, y and z from (21) and (20) in (14) is obtained the required minimum distance:

S^2= $(6-3)^2 + (8-4)^2 + (5-15)^2 = 125$

Or, S_{min} =$5\sqrt{5}$ (23)

Furthermore, some problems of maximization are done in textbooks of Calculus and in Google search such as maximization of volume /surface area of right circular cylinder and right circular cone inscribed in a sphere. In this feature an attempt has been made to solve some maximization problems, that have not yet appeared in textbooks or been published elsewhere , though, not that cumbersome.

MAXIMIZATION OF VOLUME OF A PRALLELOPIPED SURMOUNTED BY A PYRAMID INSCRIBED IN A SPHERE

Let 2x, 2y,2z be dimensions of the base and height of the parallelopiped surmounted by a pyramid of height h inscribed in a sphere of given radius R where x,y,z are the coordinates of one corner of the parallel opiped with reference to the centre of the sphere as the origin. h=R-z.Then volume of this combination is given by

V=8xyz+$\frac{4xy}{3}$(R-z) (24)

subject to the constraint equation of the sphere

$x^2 + y^2 + z^2 = R^2$ (25)

With the help of Lagrange's Multiplierλ we introduce Function F so that

F=8xyz+$\frac{4xy}{3}$ (R-z) +$\lambda(R^2 - x^2 - y^2 - z^2)$ (26)

$\frac{\delta F}{\delta x} = \frac{20zy}{3}+\frac{4yR}{3}$ -2λx =0 (27)

$\frac{\delta F}{\delta y} = \frac{20zx}{3}+\frac{4xR}{3}$ -2λy =0 (28)

$\frac{\delta F}{\delta z} = \frac{20xy}{3}-2\lambda$z =0 (29)

Equating (26) and (27) is obtained

x=y (30)

Using (30) in (29) one gets

$\frac{10x^2}{3z}$=λ (31)

Using (30) and (31) in (28) is gotten

$10z^2$ +2zR=$10x^2$ (32)

Using (32) and(30) in (2) is obtained(x=y)

$10x^2 + 10y^2 + 10z^2 = 10R^2$

0r, $30z^2 + 4zR = 10R^2$

Or, , $15z^2 + 2zR - 5R^2 = 0$

z=$\frac{(\sqrt{76}-1)R}{15} = \frac{(7.72)R}{15}$ *(33)*

The optimum value of the height is

z_{opt}*=.515R* *(34)*

Because of (30),(34) and (25),we get optimum values of the dimensions of the base:

$2x^2 = 2y^2$ =R^2 -${z_{opt}}^2$ = R^2 $-(.515)^2\, R^2$ = $.7348R^2$

Or, x=y=.607R

Or, x_{opt}=y_{opt}=$\frac{3R}{5}$, z_{opt}=$\frac{R}{2}$ (approximately) (35)

without loss of generality and subtlety .

Applying (35) in (24) is acquired the maximum volume of the above combination of the parallelopiped and pyramid inscribed in the sphere:

$V_{max} = \frac{42}{25}R^3$ (36)

MAXIMUM VOLUME OF A CYLINDER SURMOUNTED BY A RIGHT CIRCULAR CONE INSCRIBED IN A SPHERE

Let a cylinder be surmounted by a cone of height h be inscribed in a sphere of radius R.In that case by geometry, the centre of the cylinder(mid point) coincides with the centre of the sphere.Let the common radius of the cylinder and cone be r. Then by geometry the height of the cylinder is 2(R-h).

$r^2 = R^2 - (R-h)^2 = 2Rh - h^2$ (37)

Volume V of the combination of the cylinder and cone due to (37) as depicted above is given by

V=$2\pi(2Rh-h^2)(R-h)+\frac{\pi}{3}(2Rh-h^2)h$

Or,V=$\frac{\pi}{3}(2Rh-h^2)(6R-5h)$ (38)

For maximum or minimum of V, we have

$\frac{\delta V}{\delta h}=\frac{\pi}{3}\{(6R-5h)(2R-2h)-5(2Rh-h^2)\}=0$

$\frac{\delta V}{\delta h}=\frac{\pi}{3}\{-5(2Rh-h^2)+2(6R-5h)(R-h)\}=0$

Or,$15h^2-32Rh+12R^2=0$

With $\frac{d^2V}{dh^2}<0$

Or,$\frac{h}{R}=\frac{32\pm\sqrt{32^2-720}}{30}=\frac{16\pm\sqrt{16^2-180}}{15}=\frac{16\pm\sqrt{76}}{15}=.485$

h=$\frac{R}{2}$ (approximately) (39)

which gives the value of h for maximum volume of the combination of the cylinder and the cone which on account of (38) yields

$V_{max}=\frac{7\pi}{8}R^3$ (40)

MINIMUM SURFACE AREA OF A SPHERICAL CAPE WITH A GIVEN VOLUME OF IT

Let us consider a cake/bread in shape of cylinder surmounted by a spherical cape of height cut off from a sphere of radius parallel to its diametric plane.At the outset we shall find out with the help of Calculus volume and surface area of the above cap.Let us consider a circular disc at a distance x of thickness dx from the centre of the sphere. Then radius r of the spherical cap by geometry is calculated as

$r^2 = 2Rh - h^2$ (41)

Volume of the cap is

$v_1=\int_0^h \pi(2Rx-x^2)dx=\pi(Rh^2-\frac{h^3}{3})$

which due to elimination of R by use of (41) becomes

$v_1=\frac{hr^2}{2}+\frac{h^3}{6}$ (42)

In order to compute surface area of the spherical cap, let the lines joining the centre of the sphere to an element dx subtends an angle dθ and let radius R make an angle θ with the axis of the disk while $angle\ \theta$ varies from 0 to $\alpha\ to\ cover\ the\ cap\ whose$ surface area can be obtained as

$S_1=\int_0^\alpha 2\pi R sin\theta . R\, d\theta=\int_0^\alpha 2\pi R^2 sin\theta . d\theta=2\pi R^2(1-cos\alpha)$ (43)

where $cos\alpha=\frac{R-h}{R}$ (44)

so that (43) reduces to the form (vide (41))

$S_1=2\pi Rh$

Or, $S_1=\pi(r^2+h^2)$ (45)

adding (42) and (45) the total volume V and surface area S of the cake are given by

V= $\pi(r^2h+\frac{hr^2}{2}+\frac{h^3}{6})$

Or, V== $\pi(\frac{3hr^2}{2}+\frac{h^3}{6})$ *(46)*

$S=2\pi rh+\pi(2r^2+h^2)$ (47)

Involving Lagrange's Multiplier $\lambda\ and\ equation\ (46) and (47), function\ F\ can\ be\ written\ as$

F=$\pi(2rh+2r^2+h^2)+\lambda\{V-\pi\left(\frac{3hr^2}{2}+\frac{h^3}{6}\right)\}$ (48)

$\frac{\delta F}{\delta r} = \pi(2h + 4r) - 3\pi\lambda rh = 0$

Or, $\lambda = \frac{2h+4r}{3rh}$ (49)

$\frac{\delta F}{\delta h} = \pi(2r + 2\text{h}) - \lambda(\frac{3r^2}{2} + \frac{h^2}{2}) = 0$

$\lambda = \frac{4(h+r)}{3r^2+h^2}$ (50)

Equating (49) to (50) we get

$\frac{2h+4r}{3rh} = \frac{4(h+r)}{3r^2+h^2}$

Or,12r$h^2 + 12r^2h = 6r^2h + 12r^3 + 2h^3 + 4\text{r}h^2$

Or, $6r^3 + h^3 - 4\text{r}h^2 - 3r^2h$=0

Or, $6p^3 - 3p^2 - 4p + 1$=0 (51)

where p=$\frac{r}{h}$

Or, $6p^2(p-1) + 3p(p-1) - (p-1)$=0

Or,(p-1){ $6p^2 + 3p - 1$}=0

Or, p=1, p=$\frac{-3\pm\sqrt{9+24}}{12}$ =$\frac{1}{4}$(approx)

Or, h=r ,h= $\frac{r}{4}$ (52)

In view of (47) , (52) suggests that the minimum surface area occurs when

h=$\frac{r}{4}$ (53)

and is given by

S_{min}=$\frac{41}{16}\pi r^2$ (54)

the maximum surface area when

h=r (55)

S_{max}=5πr^2 (56)

MAXIMUM VOLUME OF A CYLINDER SURMOUNTED BY AN INVERTED HEMISPHERE INSCRIBED IN A SPHERE

If r is the common radius of the cylinder and the hemisphere inscribed in a sphere of radius R, then height of the cylinder is ,by geometry,

h= $\sqrt{R^2 - r^2}$ +R -r (57)

Hence volume of the inscribed cylinder surmounted by the hemisphere is

Or,V=π$r^2(\sqrt{R^2 - r^2}$ +R-r)+ $\frac{2\pi r^3}{3}$ (58)

Or,V=π$r^2(\sqrt{R^2 - r^2}$ +R)− $\frac{\pi r^3}{3}$

For maximum/minimum of V,we can write

$\frac{\delta V}{\delta r}$=$2\pi r\left(\sqrt{R^2 - r^2} + R\right) - \pi r^2$ - $\frac{1}{\sqrt{R^2-r^2}}\pi r^3$=0

Or,2$(R^2 - r^2) - r^2 + (2R - r)\sqrt{R^2 - r^2}$=0

Or, Or,$(2R^2 - 3r^2) = -(2R - r)\sqrt{R^2 - r^2})$

Squaring both sides, one gets

$(2R^2 - 3r^2)^2 = \{(2R - r)\sqrt{R^2 - r^2})\}^2$

Or,9$r^4 - 12R^2r^2$+4R^4== (4R^2-4Rr+r^2)$(R^2 - r^2)$

Or,9$r^4 - 12R^2r^2$+4R^4== $4R^4 - 4R^3r$+$R^2r^2 - 4R^2r^2 + 4Rr^3$-r^4

*10*$r^3 - 9R^2r$+4R^3-4Rr^2=0 (59)

*10*$q^3 - 4q^2$-9$q + 4$=0 (60)

where q=$\frac{r}{R}$<1 .Adopting a process of approximation, we can solve this cubical equation without loss of generality and of sufficient accuracy, for which let

q=1-μ , $\mu \ll 1$ (61)

such that (60) reduces to the form

$10(1-\mu)^3 - 4(1-\mu)^2 - 9(1-\mu) + 4 = 0$ (62)

Or, expanding binomially and neglecting square and other higher powers of μ is obtained

$10(1-3\mu) - 4(1-2\mu) - 9(1-\mu) + 4 = 0$

0r, $\mu = \frac{1}{13}$

In consequence of which (61) gives

r=$\frac{12}{13}R$ (63)

which on substitution in (58) leads to the maximum volume

$V = \pi r^2(\sqrt{R^2-r^2} + R) - \frac{\pi r^3}{3}$

$V = \pi(\frac{12R}{13})^2(\sqrt{R^2 - (\frac{12R}{13})^2} + R) - \frac{\pi(\frac{12R}{13})^3}{3}$

$= \pi\frac{144}{169}(\frac{18}{13} - \frac{12}{39})R^3$

$= \pi(\frac{144.}{169}\frac{42}{39})R^3$

$V_{max} = \pi(\frac{143.}{169}\frac{42}{39})R^3$ (approx.)

Or, $V_{max} = \pi(\frac{144.}{168}\frac{42}{39})R^3 = \pi\frac{12}{13}R^3$ (approx.) (64)

MIXIMUM SURFACE AREA OF A CYLINDER SURMOUNTED BY AN INVERTED HEMISPHERE INSCRIBED IN A SPHERE

In the light of equation (57), surface area S of the cylinder surmounted by an inverted hemisphere inscribed in a sphere of radius R is given by

$S = 2\pi r(\sqrt{R^2-r^2} + R - r) + 2\pi r^2 + \pi r^2 = 2\pi r(\sqrt{R^2-r^2} + R) + \pi r^2$ (65)

For maxima or minima of S,

$\frac{dS}{dr} = 2\pi\{(\sqrt{R^2-r^2} + R) - \frac{r^2}{\sqrt{R^2-r^2}} + r\}$

$= 2\pi\,(R^2 - 2r^2 + (R+r)\sqrt{R^2-r^2}) = 0$ (66)

$\frac{d^2S}{dr^2} = 2\pi\,(-4r + \sqrt{R^2-r^2} - \frac{r(r+R)}{\sqrt{R^2-r^2}})$

Or, $\frac{2\pi}{\sqrt{R^2-r^2}}\{-4r\sqrt{R^2-r^2} + (R^2-r^2) - r(r+R)\}$

Or, $\frac{2\pi}{\sqrt{R^2-r^2}}\{-4r\sqrt{R^2-r^2} + (R^2-2r^2) - rR)\}$ *(67)*

From (66) is obtained

$R^2 - 2r^2 = -(R+r)\sqrt{R^2-r^2}$

Squaring both sides and simplifying one gets

$R^4 - 4r^2R^2 + 4r^4 = (R^2 + 2Rr + r^2)(R^2 - r^2)$

Or, $5r^4 + 2r^3R - 4r^2R^2 - 2rR^3 = 0$

$5r^3 + 2r^2R - 4rR^2 - 2R^3 = 0$ *(68)*

Denoting $\mu = \frac{r}{R}$<1, leading to $\mu = 1 - \epsilon, \epsilon > 0$ (69)

$5\mu^3 + 2\mu^2 - 4\mu - 2 = 0$

Or, $5(1-\epsilon)^3 + 2(1-\epsilon)^2 - 4(1-\epsilon) - 2 = 0$ *(70)*

Expanding binomially neglecting square and other higher powers of ϵ,we get

$5(1-3\epsilon) + 2(1-2\epsilon) - 4(1-\epsilon) - 2 = 0$

Or, $\epsilon = \frac{1}{15}$ *(71)*

Or,by virtue of (69), we can write

$\mu = \frac{14}{15}$ Or,r= $\frac{14}{15}R$ for which $\frac{d^2S}{dr^2}$<0 (72)

in consequence of which,by use of (65), is found the maximum surface area of the combination:

$S_{max} = \frac{28}{225}\pi(\sqrt{29}\ +1)\ R^2+\pi(\frac{14}{15}R)^2$

$=\frac{28\pi}{225}\{(\sqrt{29}\ +8)\ R^2$ *(73)*

MAXIMUM SURFACE AREA OF A PYRAMID INSCRIBED IN A SPHERE

Let the coordinates of one corner of the base of the pyramid with respect to the centre of a sphere as the origin be (x,y) and height z , inscribed in a sphere of radius R.Then surface area of the pyramid is given by

S=4xy+2x$\sqrt{y^2+z^2}$+2y$\sqrt{x^2+z^2}$ (74)

For maximum or minimum of S, choosing function F and Lagrange's Multiplierλ,

F=4xy+2x$\sqrt{y^2+z^2}$+2y$\sqrt{x^2+z^2}+\lambda(R^2-x^2-y^2-z^2)$ (75)

$\frac{\delta F}{\delta x} = 4y + 2\sqrt{y^2+z^2} + \frac{2xy}{\sqrt{x^2+z^2}} + \lambda(-2x)$ =0 (76)

$\frac{\delta F}{\delta y} = 4x + 2\sqrt{x^2+z^2} + \frac{2xy}{\sqrt{y^2+z^2}} + \lambda(-2y) = 0$ (77)

Combining (77) and (76) or by accounting for symmetry is obtained

x=y (78)

$\frac{\delta F}{\delta z}$=$\frac{2xz}{\sqrt{y^2+z^2}} + \frac{2yz}{\sqrt{x^2+z^2}}$ -2zλ=0

which because of (78) becomes

$\frac{2x}{\sqrt{x^2+z^2}}$=$\lambda$ (79)

Using (78) and (79) in (76) or(77), we get

$2x + \sqrt{x^2+z^2} + \frac{x^2}{\sqrt{x^2+z^2}} - \frac{2x^2}{\sqrt{x^2+z^2}} = 0$

Or,2x+$\frac{z^2}{\sqrt{x^2+z^2}}$=0

Squaring and simplifying,

$z^4 - 4x^2z^2$-$4x^4 = 0$ (80)

Or,$\frac{z^2}{x^2}$=$\frac{4\pm\sqrt{32}}{2}$=$2\pm2\sqrt{2}$ (81)

$\frac{z}{x}$=$\sqrt{2+2\sqrt{2}}$ =$\sqrt{4.82}$ =2.2=$\frac{11}{5}$ (approx.) (82)

Or, x=$y = \frac{5}{11}z$ *(83)*

Using (82) and equation of the sphere

$2(\frac{5z}{11})^2 + z^2$=R^2

z^2=$\frac{121}{171}R^2$ (84)

Optimum values : x^2=y^2=$\frac{25}{121}z^2 = \frac{25}{171}R^2$ (85)

Using (74) and (84) we determine the maximum and minimum surface areas of the pyramid in the sphere.

S=$4x^2 + 4x\sqrt{x^2+z^2}$=$4x^2(1+\sqrt{1+\frac{z^2}{x^2}}\ )$

=$\frac{100R^2}{171}$ (1+$\sqrt{1+\frac{121}{25}}$)

S_{max}= $\frac{100R^2}{171}$ (1+$\sqrt{\frac{146}{25}}$) *(86)*

Or, may be with minor error

S_{max}= $\frac{100R^2}{170}$ (1+$\sqrt{\frac{144}{25}}$) *=2R^2*

MAXIMUM/MINIMUM SURFACE AREA OF A RIGHT CIRCULAR CONE INSCRIBED IN A SPHERE

If h be the height of a cone inscribed in a sphere of radius R,by geometry curved surface area S of the cone inscribed in a sphere can written as

S=$\pi\sqrt{2Rh-h^2}(\sqrt{2Rh}+\sqrt{2Rh-h^2})$

S=$\pi\{\sqrt{4R^2h^2-2Rh^3}+2Rh-h^2\}$ (87)

where its radius r=$\sqrt{2Rh-h^2}$ and lateral height l=$\sqrt{2Rh}$

For maxima / minima of of the surface area,

$\frac{dS}{dh}=\pi\{2(R-h)+\frac{4R^2h-3Rh^2}{\sqrt{4R^2h^2-2Rh^3}}\}=0$

Or, $4(R-h)^2(4R^2-2Rh)=(4R^2-3Rh)^2$ *(88)*

Simplifying we get

$-8Rh(R-h)^2+(4R^2-4Rh)^2=(4R^2-3Rh)^2$

Or, $-8Rh(R-h)^2=(4R^2-3Rh)^2-(4R^2-4Rh)^2$=$(8R^2-7Rh)Rh$

Or,$-8(R^2+h^2-2Rh)$ =$8R^2-7Rh$

Or, $16R^2-23Rh+8h^2$=0 (89)

Letting $\mu=\frac{R}{h}$ <1 (90)

(89) becomes

$16\mu^2-23\mu+8$=0 (91)

$\mu=\frac{23-\sqrt{17}}{32}=\frac{23-\sqrt{17}}{32}$=$\frac{18.8}{32}$ =$\frac{47}{80}$ (approx.) (92)

By observation, R$\leq h<2R$ *which implies* $\frac{1}{2}\leq\mu$ <1 so that because of (90)

h_{opt}=$\frac{80}{47}R$ (93)

Or with sufficient accuracy

h_{opt}=$\frac{80}{48}R$

Or, approximately h_{opt}=$\frac{5}{3}R$ (94)

using which in (87) we obtain the maximum surface area of the cone inscribed in the sphere as

S_{max}=$\pi\sqrt{R^2-(h-R)^2}(\sqrt{2Rh}+\sqrt{R^2-(h-R)^2}$

$=\pi\sqrt{\frac{5}{9}R^2}\,(\sqrt{\frac{10}{3}R^2}+\sqrt{\frac{5}{9}R^2}\,)$

$=\pi(\frac{5}{9}+\frac{5}{3}\sqrt{\frac{2}{3}})R^2$

$=\pi\frac{5}{9}(1+3\sqrt{\frac{2}{3}})R^2$ (95)

MAXIMUM CURVED SURFACE AREA OF A CIRCULAR CYLINDER INSCRIBED IN A SPHERE

If r and h be the radius and height of a cylinder inscribed in a sphere of radius R, area of the curved surface s is by geometry (height of the cylinder=$2\sqrt{(R^2-r^2)}$)

S =$4\pi r\sqrt{(R^2-r^2)}$

Or,S=$4\pi\sqrt{(R^2r^2-r^4)}$ (96)

where centre (middle point) of the cylinder coincides with that of the sphere.

For maxima/minima of S

$\frac{dS^2}{dr^2}$==$16\pi^2(R^2-2r^2)=0$ (97)

Or,r_{opt}=$\frac{R}{\sqrt{2}}$ (98)

Using (98) in (96) one gets the maximum curved surface area as

S_{max}=$4\pi\frac{R}{\sqrt{2}}\sqrt{R^2-(\frac{R}{\sqrt{2}})^2}$

Or, S_{max}=$2\pi R^2$ (99)

MAXIMUM CURVED SURFACE AREA OF A RIGHT CIRCULAR CONE INSCRIBED IN A SPHERE

If h and r be the height and radius of a cone inscribed in a sphere of radius R, the curved surface area S of the cone is given by

S= $\pi r\sqrt{r^2+h^2}$ *(100)*

Since by geometry, the height can be expressed as

h=R+ $\sqrt{R^2-r^2}$ *(101)*

Combining (100) and (101), is obtained

S= $\pi r\sqrt{2R^2+2R\sqrt{R^2-r^2}}$ *(102)*

S^2=$2\pi^2(2R^2r^2$ $+2R\sqrt{R^2r^4-r^6})$

For maxima /minima of S ie S^2

$\frac{dS^2}{dr^2}$=$2\pi^2(2R^2$ $+R\frac{2R^2r^2-3r^4}{\sqrt{R^2r^4-r^6}})=0$

Or, 2R=-$\frac{2R^2-3r^2}{\sqrt{R^2-r^2}}$

Squaring and simplifying we have

$4R^2(R^2-r^2)$= $4R^4-12R^2r^2$ $+9r^4$

Or,$9r^4=8R^2r^2$

$r_{opt}=\frac{2\sqrt{2}}{3}R$ (103)

using which in (101) we get

$h_{opt}=\frac{4}{3}R$ (104)

Using (103) and (104) in (100) is obtained the maximum surface area of the cone inscribed in the sphere:

S_{max}= $\pi\frac{2\sqrt{2}}{3}R\sqrt{(\frac{2\sqrt{2}}{3}\mathrm{R})^2+(\frac{4}{3}R)^2}$

Or, S_{max}= $\pi\frac{8\sqrt{3}}{9}R^2$ *(105)*

MAXIMUM CURVED SURFACE AREA OF A RIGHT CIRCULAR CYLINDER SURMOUNTED BY AN INVERTED HEMISPHERE INSCRIBED IN A SPHERE

Let h and r be the height and radius of a cylinder surmounted by an inverted hemisphere, obviously of the same radius inscribed in a sphere of radius R.Then by geomentry , curved surface area S of combination of the cylinder and hemisphere is

S= $2\pi rh+2\pi r^2$ (106)

where h= $2\sqrt{R^2-r^2}$

so that

S= $4\pi\sqrt{R^2r^2-r^4}$ $+2\pi r^2$ (107)

For maximum / minimum of S, differentiating (107) w.r.t r^2 and equating to zero

$\frac{dS}{dr^2}$=$\frac{2\pi(R^2-2r^2)}{\sqrt{R^2r^2-r^4}}+2\pi$ =0

Or$R^2r^2-r^4$=$(R^2-2r^2)^2$

Or, $R^4-5R^2r^2+5r^4$=*0*

0r, $r^2=\frac{5\pm\sqrt{5}}{10}R^2$ *(108)*

Or,r=$\sqrt{\frac{5\pm\sqrt{5}}{10}}$R

Or, r^2=$\frac{7}{10}R^2$ and r^2=$\frac{3}{5}R^2$ (approx.) (109)

With r= $\sqrt{\frac{7}{10}}R$, by use of (107), we have

S=2π $(\frac{7}{10})^2(2\sqrt{(\frac{10}{7})^2-1}$ +1) R^2

=$\frac{49\pi}{50}(2\sqrt{\frac{51}{49}}$ +1) R^2

$=\frac{49\pi}{50}(2.05 \ +1)\, R^2$

S=$3\pi R^2$ (approximately) (110)

With r=$\frac{2}{5}$, by use of (107)

S=$2\pi\ (\frac{2}{5})^2(2\sqrt{(\frac{5}{2})^2-1}\ +1)\, R^2$

Or, S_{min}=$\frac{17}{10}\pi R^2$ (approx.)

Because of second part of (109)

S_{max}=$3\pi R^2$ (111)

MAXIMUM SURFACE AREA OF A PARALLELOPIPED INSCRIBED IN A SPHERE

If 2x,2y,2z be the dimensions of a parallelopiped, its surface area is given by

S=8(xy+yz+zx) (112)

Choosing function F and Lagrange's Multiplier$\lambda, we\ introduce$

F=8(xy+yz+zx)+ $\lambda\,(R^2-x^2-y^2-z^2)$

In consequence of symmetry in x, y,z, it can be found that the maximum surface area of the parallelopiped occurs for equal values of thereof dimensions yielding

$x_{opt} = y_{opt} = z_{opt} = \frac{R}{\sqrt{3}}$ (113)

$S_{max} = \frac{8(R^2+R^2+R^2)}{3}$ =$8R^2$ (114)

MAXIMUM SURFACE AREA OF A RECTANGULAR TANK OPENED AT THE TOP INSCRIBED IN A SPHERE

If 2x,2y, 2z be the dimensions of the tank opened at the top , inscribed in a sphere of radius R,its surface area is

S=4xy+8z(x+y) (115)

Choosing function F and Lagrange's Multiplier $\lambda, we\ can\ write$

F= 4xy+8z(x+y)+ $\lambda(R^2$ - $x^2-y^2-z^2)$ (116)

$\frac{\delta F}{\delta x}$=4y+8z-2$\lambda x$=0 (117)

$\frac{\delta F}{\delta y}$=4x+8z-2$\lambda y$=0 (118)

$\frac{\delta F}{\delta z}$=8(x+y) -2$\lambda z$ =0 (119)

(117) and (118) imply

x =y (120)

Combining either of (118) and (119) with (120),

λ=$\frac{8x}{z}$ (121)

Combining(119),(120) and(121) and eliminating λ,

4x+8z=$\frac{16x^2}{z}$ (122)

Or,$4x^2-zx-2z^2=0$

Or, x=$\frac{1+\sqrt{1+32}}{8}z = \frac{17}{20}z$ =y (approx.) (123)

Using (123) in equation of the sphere is obtained

$2x^2+z^2=R^2$

Or, $2(\frac{17}{20}z\,)^2+z^2=R^2$

$\frac{489}{200}z^2=R^2$

Or,$\frac{49}{20}z^2=R^2$ (approx.)

$z_{opt}=\frac{9}{14}$R (approx.) (124)

By use of (123) and (124) is obtained optimum values of x,y and z as

$x_{opt} = y_{opt} = \frac{15}{28}R \quad (approx.)$ (125)

Using (125) and(124) in (115), we get maximum surface area as

S_{max}=4xy+8z(x+y) =$\frac{67}{10}R^2$ (approx.) (126)

MINIMUM/ MAXIMUM DISTANCE FROM A POINT TO A SPHERE /ELLIPSOID

Let us find the maximum/minimum distance from a point(A,B,C) to a sphere of radius R, first by Lagrange's Multiplier and thereafter by geometry.

Taking equation of the sphere as

$x^2 + y^2 + z^2 = R^2$ (127)

In general distance S between (A, B,C) and any point (x,y,z) on the sphere is written as

$S^2 = (x - A)^2 + (y - B)^2 + (z - C)^2$ (128)

Choosing function F and Lagrange's Multiplier λ,

F=$(x - A)^2 + (y - B)^2 + (z - C)^2 + \lambda(x^2 + y^2 + z^2 - R^2)$

$\frac{\delta F}{\delta x} = 2(x - A) - 2\lambda x$=0 (129)

$\frac{\delta F}{\delta y} = 2(y - B) - 2\lambda y$=0 (130)

$\frac{\delta F}{\delta z} = 2(z - C) - 2\lambda z$=0 (131)

Combining the above three equations,

$\frac{x}{A} = \frac{y}{B} = \frac{z}{C} = \lambda$ (132)

which by use of (127),one gives the optimum values of the variables (point on the sphere)

x=$\frac{RA}{\sqrt{A^2+B^2+C^2}}$, $y = \frac{RB}{\sqrt{\sqrt{A^2+B^2+C^2}}}$,$z = \frac{RC}{\sqrt{A^2+B^2+C^2}}$ (133)

Substituting (133) in (128) is obtained the maximum/ minimum distance:

$S^2_{max/min}$=$(\frac{RA}{\sqrt{A^2+B^2+C^2}} - A)^2 + (\frac{RB}{\sqrt{A^2+B^2+C^2}} - B)^2$+$(\frac{RC}{\sqrt{A^2+B^2+C^2}} - C)^2$

=$R^2 - 2R\sqrt{A^2 + B^2 + C^2}$+$A^2 + B^2 + C^2$

=$(R - \sqrt{A^2 + B^2 + C^2})^2$ (134)

Or,$S_{max/min}$=$\pm(R - \sqrt{A^2 + B^2 + C^2})$ *(135)*

The foregoing equations reveal :

The distance between the point(A,B,C) and the centre of the sphere as the line joining these two points is

D=$\sqrt{A^2 + B^2 + C^2}$ (136)

$S_{min} = \sqrt{A^2 + B^2 + C^2}$-R (137)

$S_{max} = \sqrt{A^2 + B^2 + C^2}$+R (138)

Eq.(138) represents the maximum distance as defined above,when the point(A,B,C) is inside or outside the sphere whereas Eq.(137) represents the minimum distance when this point lies outside the sphere.But if the point lies inside the sphere,the minimum distance as defined above is

$S_{min} = R - \sqrt{A^2 + B^2 + C^2}$ (139)

However this aspect of "maxima and minima" is easily evaluated without resorting to Lagrange's Multiplier.

MINIMUM/ MAXIMUM DISTANCE FROM A POINT TO AN ELLIPSOID

Equation of the ellipse is written as

$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ (140)

Distance S between the point(A,B,C) and any point on the ellipsoid (140) is:

$S^2 = (x-A)^2 + (y-B)^2 + (z-C)^2$ (141)

Taking function F and Lagrange's Multiplierλ is obtained

F= $(x-A)^2 + (y-B)^2 + (z-C)^2 + \lambda\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2}\right)$ (142)

$\frac{\delta F}{\delta x} = 2(x-A) - 2\lambda\frac{x}{a^2}$=0 (143)

$\frac{\delta F}{\delta y} = 2(y-B) - 2\lambda\frac{y}{b^2}$=0 (144)

$\frac{\delta F}{\delta z} = 2(z-C) - 2\lambda\frac{z}{c^2}$=0 (145)

Which lead to

x=$\frac{Aa^2}{a^2-\lambda}$, y=$\frac{Bb^2}{b^2-\lambda}$, z=$\frac{Cc^2}{c^2-\lambda}$ (146)

which on substitution in (140) yields

$(\frac{Aa^2}{a^2-\lambda})^2 + (\frac{Bb^2}{b^2-\lambda})^2 + \left(\frac{Cc^2}{c^2-\lambda}\right)^2 = 1$

Or, $A^2a^4(b^2-\lambda)^2(c^2-\lambda)^2$+$B^2b^4(a^2-\lambda)^2(c^2-\lambda)^2$+$C^2c^4(a^2-\lambda)^2(b^2-\lambda)^2$

=$(a^2-\lambda)^2(b^2-\lambda)^2(c^2-\lambda)^2$ (146.1)

Given numerical values of A,B,C and a,b,c the value of λ can be approximately calculated and can be used in (146) to get the coordinates the magic point on the sphere. The distance between this point and point(A,B,C) becomes the minimum distance between the given point and the sphere.

MINIMUM DISTANCE OF A PARABOLA FROM A POINT

Distance S of any point (x,y) on parabola of equation

$y^2 = 4ax$ (147)

from a point (A,B,C) can be written as

$S^2 = (x-A)^2 + (y-B)^2$ (148)

Eliminating x between (147) and (148) is obtained

$S^2 = (\frac{y^2}{4a} - A)^2 + (y-B)^2$ (149)

For maxima/minima of S,

$\frac{dS^2}{dy} = 2\left(\frac{y^2}{4a} - A\right)\frac{y}{2a} + 2(y-B) = 0$

Or,$y^3 - 4a(A-2a)y - 8a^2B$=0 (150)

In order to solve this cubical equation we assume

Y=$m^{\frac{1}{3}} + n^{\frac{1}{3}}$ (151)

so that

$y^3 - 3m^{\frac{1}{3}}n^{\frac{1}{3}}y - (m+n)$=0 (152)

Comparing (150) and (151) is gotten

m+n=$8a^2B$ (153)

mn=$\frac{(4aA-8a^2)^3}{27}$ (154)

m-n =$\pm\sqrt{64a^4B^2 - 4\frac{(4aA-8a^2)^3}{27}}$ (155)

Solving (153) and (154) we get

m=$4a^2B + \sqrt{16a^4B^2 - \frac{(4aA-8a^2)^3}{27}}$ Or $4a^2B - \sqrt{16a^4B^2 - \frac{(4aA-8a^2)^3}{27}}$ (156)

n= $4a^2B - \sqrt{16a^4B^2 - \frac{(4aA-8a^2)^3}{27}}$ Or $4a^2B + \sqrt{16a^4B^2 - \frac{(4aA-8a^2)^3}{27}}$ (157)

Using (156) and (157) in (151) is obtained

Y=$(4a^2B+\sqrt{16a^4B^2-\frac{(4aA-8a^2)^3}{27}})^{\frac{1}{3}}+(4a^2B-\sqrt{16a^4B^2-\frac{(4aA-8a^2)^3}{27}})^{\frac{1}{3}}$ (158)

Using the equation of the parabola in (158) is found

x=$\frac{\{(4a^2B+\sqrt{16a^4B^2-\frac{(4aA-8a^2)^3}{27}})^{\frac{1}{3}}+(4a^2B-\sqrt{16a^4B^2-\frac{(4aA-8a^2)^3}{27}})^{\frac{1}{3}}\}^2}{4a}$ (159)

(158) and (159) express the coordinates the relevant point on the parabola. Joining this point to the given point(A,B,C) leads to the minimum distance S_{min} between the later and the parabola. If the given point lies on the the x-axis ie B=0,(148) with equation of the parabola gives

$S^2=(x-A)^2+4ax$ (160)

In this context for maxima and minima of S,

$\frac{dS^2}{dy}$ =2(x-a)+4a=0

x=-a

which leads (160) to

S_{min}=$\pm(A-a)$ (161)

depending whether A>a orA<a.

MINIMUM DISTANCE FROM A POINT ON THE Y-AXIS TO A PARABOLA

The distance S from a point (0,B) to any point(x,y,z) on the parabola

$y^2=4ax$ (162)

is expressed as

S^2=x^2+$(y-B)^2$ (163)

Eliminating x between (162) and (163) is obtained

$S^2=\frac{y^4}{16a^2}$+$(y-B)^2$ (164)

$\frac{dS^2}{dy}=\frac{y^3}{4a^2}$+2(y-B) =0 (165)

The optimum values of x, y ie the coordinates of the magic point can determined by solving (163) and (165) entailing the relevant minimum distance. However, by trial it is observed that y=4,a=2,B=6 satisfy equation (165).Consequently by use of equation of the parabola ,x=2.Hence substituting these values in (164) we arrive at

S_{min}=2$\sqrt{2}$ (166)

MINIMUM DISTANCE FROM A POINT ON THE X-AXIS TO ANY POINT AN ELLIPSE

Let the given point be(A,0) and equation of the ellipse is

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (167)

The distance S between the given point (A,0) and any point (x,y) on the ellipse (167) is given by

$S^2=(x-A)^2+y^2$

$=(x-A)^2+b^2$(1-$\frac{x^2}{a^2}$) (168)

For maxima/minima of S,

$\frac{dS^2}{dy}=-2A+2x(1-\frac{b^2}{a^2})=0$

Or, x_{opt}=$\frac{Aa^2}{a^2-b^2}$ (169)

which in consequence of (167) gives

$y^2=b^2\{1-(\frac{Aa}{a^2-b^2})^2\}$ (170)

Or, $y_{opt}=b\sqrt{1-(\frac{Aa}{a^2-b^2})^2}$ (171)

Using (168),(169) and (171) is obtained the required minimum distance:

S_{min}^2=$(\frac{Ab^2}{a^2-b^2})^2$+$b^2\{1-(\frac{Aa}{a^2-b^2})^2\}$

$=b^2\{1+(\frac{Ab}{a^2-b^2})^2-(\frac{Aa}{a^2-b^2})^2\}$

$=b^2(1-\frac{A^2}{a^2-b^2})$ (172)

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