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Existence and Uniqueness Results and has at Least One Solution for Nonlinear Fractional Differential Equation with Two or More Terms of Fraction Order

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ABSTRACT

The work addressed in this research is toshow the existence and uniqueness resultsand has at least one solution for initial value problems for nonlinear fractional differential equations with two or more terms of fractional orders by considering some fractional problems

Keywords: Fixed Point, Fixed Point Theorem, Fractional Calculus, Fractional Derivative, Fractional Differential Equation, Caputo Fractional Derivative, Riemann Liouville Fractional Derivative

1. Introduction

Fixed point theory serves as an essential tool for various branches of mathematical analysis and its applications. Fixed point theory is a developing branch of mathematics related with functional analysis and topology. Fractional derivatives: In applied mathematics and mathematical analysis, a fractional derivative is a derivative of any arbitrary order, real or complex. Fractional deferential equation also known as extraordinary differential equations are a generalization of differential equations through the application of fractional calculus. Fractional calculus is a branch of mathematical analysis that studies the several different possibilities of defining real number powers or complex number powers of the differentiationoperator*D.* Another option for computing fractional derivatives is the Caputo fractional derivative. It was introduced by Michele Caputo in his 1967 research, in contrast to the Riemann – Liouville fractional derivative, when solving differential equation using Caputo's definition, it is not necessary to define the fractional order initial value conditions.Modelling by fractional operators in most have been to be better than modelling by ordinary derivative operators, fractional operators have been used in a modelling of many physical systems such as rheology, viscoelasticity, porus structures, chemical physics, electrochemistry, and many other branches of science (Miller and Ross,1993, Kilbas*et al*, 2006 and Zhai, 2013). There exist many articles about the existence and uniqueness of solutions of problems containing operators of fractional orders and using fixed point techniques (Zhai, 2017, Marasi and Afshari, 2018 , Hussain, *et al*, 2020, and Marasi and Ayadi, 2021), but research studies on the existence and uniqueness of solutions for FDEs with two or more terms of fractional operators is limited. Fujita, (1990)studied the following specific Cauchy problem:

$$\frac{\partial^\alpha}{\partial t} w(t,x) = \frac{\partial^\beta}{\partial t} w(t,x), \qquad 1 \le \alpha, \quad \beta \le 2,$$

and established some existence and uniqueness results, where $\alpha\ and\ \beta$ are real numbers. In (Kosmatov, 2009), the researcher considered an initial value problem of the following form:

$$D^\alpha y(t) = g\left(t, D^\beta y(t)\right), \qquad 0 < t \le 1,$$

$$y^{(k)}(0) = \mu_k, \qquad 0 \le k \le n-1, k \in Z,$$

where$n - 1 < \beta < \alpha < n$, and studied existence and uniqueness of related solutions.The present research work focused on the general form of fractional differential equation given by (Marasi and Aydi, 2021) in establishing new existence and uniqueness of positive results for initial value problems for nonlinear fractional differential equations with two or more terms of fractional orders via a fixed point technique

2. Discussion of the Methodology

For convenience, we provide some necessary and useful definitions on fractional calculus theory which we need throughout the research work

Definition1

The Riemann-Liouville fractional integral of order q for a continuous function f is defined as

$$\mathrm{I}^q f(t) = \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f(s)ds, \qquad q > 0,$$

Provided that the right-hand side is point-wise defined on $(0,\infty)$, where $\Gamma(.)$is the gamma function, which is defined by

$$\Gamma(q) = \int_0^\infty t^{q-1}e^{-t}\,dt$$

Definition 2

For n-times continuously differentiable function $f:(0,\infty) \to \mathbb{R}$,the Caputo derivative of order $q > 0$ is defined as:

$${}^cD^q f(t) = \frac{1}{\Gamma(n-q)}\int_0^t (t-s)^{n-q-1} f^{(n)}(s)ds,$$

$n-1 < q < n, n = [q]+1$

where$[q]$denotes the integer part of the real number q.

The following lemmastransforms the general form of fractional differential equation given by (Marasi and Aydi, 2021) to an integral equation

Lemma 1

Let $q > 0$, then the differential equation

$${}^cD^q h(t) = 0$$

has solutions $h(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3 + \cdots + c_{n-1}t^{n-1}$

here$c_i \in \mathbb{R}, i = 0,1,2,\dots,n-1$ and $n = [q]+1$

Lemma 2

Let $q > 0$, then

$$I_a^q\,{}^cD_a^q h(t) = h(t) + c_0 + c_1(t-a)^1 + c_2(t-a)^2 + \cdots + c_{n-1}(t-a)^{n-1}$$

In particular when a = 0

$$I^q\,{}^cD^q h(t) = h(t) + c_0 + c_1 t + c_2 t^2 + c_3 t^3 + \cdots + c_{n-1}t^{n-1}$$

Proof:

By the definition

$$I_a^q\,{}^cD_a^q h(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1} D_a^\alpha h(s)ds$$

$$I_a^q\,{}^cD_a^q h(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\frac{1}{\Gamma(n-\alpha)}\left[\int_a^s (s-\tau)^{n-\alpha-1}h^n(t)\right]ds$$

$$I_a^q\,{}^cD_a^q h(t) = \frac{1}{\Gamma(\alpha)\Gamma(n-\alpha)}\int_a^s h^n(\tau)d\tau\int_a^t (t-s)^{\alpha-1}(s-t)^{n-q-1}\,ds$$

$$= \frac{1}{\Gamma(\alpha)\Gamma(n-\alpha)}\int_a^s h^n(\tau)d\tau\frac{\alpha-1}{n-\alpha}\int_a^t (t-s)^{\alpha-2}(s-t)^{n-\alpha}\,ds$$

$$= \frac{1}{\Gamma(\alpha)\Gamma(n-\alpha)}\int_a^s h^n(\tau)d\tau\frac{(\alpha-1)(\alpha-1)\dots 2\times 1}{(n-\alpha)(n-\alpha-1)\dots(n-2)}\int_a^t (s-t)^{n-2}\,ds$$

$$= \frac{1}{\Gamma(n)}\int_\alpha^t (t-\tau)^{n-1}\,h^n(\tau)d\tau$$

$$= \frac{1}{\Gamma(n)}\int_\alpha^t (t-\tau)^{n-1}\,h^{n-1}(\tau)d\tau$$

$$= -\frac{h^{n-1}(\alpha)}{\Gamma(n)}(t-\alpha)^{n-1} - \frac{h^{n-2}(\alpha)}{\Gamma(n-1)}(t-\alpha)^{n-2} + \frac{1}{\Gamma(n-2)}\int_{\alpha}^{t}(t-\tau)^{n-3}\,h^{(n-2)}(\tau)d\tau$$

$$= -\frac{h^{n-1}(\alpha)}{\Gamma(n)}(t-\alpha)^{n-1} - \frac{h^{n-2}(\alpha)}{\Gamma(n-1)}(t-\alpha)^{n-2} + \cdots + \frac{h'(\alpha)}{\Gamma(2)}(t-\alpha) - \frac{h(\alpha)}{\Gamma(1)} + h(t)$$

$$= h(t) + c_0 + c_1(t-a)^1 + c_2(t-a)^2 + \cdots + c_{n-1}(t-a)^{n-1}$$

Where

$$c_i = \frac{h^i(\alpha)}{\Gamma(i+1)}, i = 0,1,2,3,\ldots,n-1$$

$$c_i = \frac{h^i(\alpha)}{\Gamma(i+1)}, i = 0,1,2,3,\ldots,n-1$$

Lemma 3

Let $y(t) \in C([0,1])$ a function $u(t) \in C^2([0,1],\mathbb{R})$be a solution of linear sequential fractional differential equation

$$({}^{c}D^{q} + k^{c}D^{q-1})u(t) = y(t), \tag{1}$$

with boundary values $u(0) = 0$ and

$$\sum_{i=1}^{m-1} \alpha_i\, u(\xi_i) = \beta\int_0^{\eta}\frac{(\eta-s)^{q-1}}{\Gamma(q)}\;u(s)ds$$

has the unique solution given by

$$u(t) = \frac{(e^{-kt}-1)}{\Delta}\left(\beta\int_0^{\eta}\frac{(\eta-s)^{q-1}}{\Gamma(q)}\left(\int_0^{s} e^{-k(s-x)}\times\left(\int_0^{x}\frac{(x-\tau)^{q-2}}{\Gamma(q-1)}y(\tau)d\tau\right)dx\right)ds - \sum_{i=1}^{m-1}\alpha_i\int_0^{\xi_i} e^{-k(\xi_i-s)}\left(\int_0^{s}\frac{(s-x)^{q-2}}{\Gamma(q-1)}y(x)dx\right)ds\right) + \int_0^{t} e^{-k(t-s)}\left(\int_0^{s}\frac{(s-x)^{q-2}}{\Gamma(q-1)}y(x)dx\right)ds,$$

where (2)

$$\Delta = \sum_{i=1}^{m-1}\alpha_i\left(e^{-k\xi_i}-1\right) - \beta\int_0^{\eta}\frac{(\eta-s)^{q-1}}{\Gamma(q)}e^{-ks}ds + \frac{\beta\eta^{q}}{\Gamma(q-1)} \neq 0, \tag{3}$$

Proof: For $q \in (1,2]$, we consider the following linear fractional differential equation

$$1 < q \leq 2({}^{c}D^{q} + k^{c}D^{q-1})u(t) = y(t), \tag{4}$$

where ${}^{c}D^{q}$ denotes the Caputo fractional derivative of order q, we can write its solution as

$$u(t) + k^{c}D^{q-1}u(t) = \frac{1}{\Gamma(q)}\int_0^{t}(t-s)^{q-1}\,y(s)ds + c_o + c_1 t, \tag{5}$$

wherec_0 and c_1 are arbitrary constants. Now (5) can be expressed as

$$u(t) = -k\int_0^{t}u(s)ds + \frac{1}{\Gamma(q)}\int_0^{t}(t-s)^{q-1}\,y(s)ds + c_o + c_1 t, \tag{6}$$

Differentiating (6) we obtain

$$u'(t) = -ku(t) + \frac{1}{\Gamma(q-1)}\int_0^{t}(t-s)^{q-2}\,y(s)ds + c_{1,} \tag{7}$$

which can alternately be written as

$$(u(t)e^{kt})' = e^{kt}\left(\frac{1}{\Gamma(q-1)}\int_0^{t}(t-s)^{q-2}\,y(s)ds + c_1\right) \tag{8}$$

Integrating from 0 to t, we have

$$u(t) = Ae^{-kt} + \int_0^{t} e^{-k(t-s)}\int_0^{s}\frac{(s-x)^{q-2}}{\Gamma(q-1)}y(x)dxds + B. \tag{9}$$

Using the data $u(0) = 0$ in (9), we find that $A = -B$. thus, (9) takes the form

$$u(t) = A(e^{-kt} - 1) + \int_0^t e^{-k(t-s)} \int_0^s \frac{(s-x)^{q-2}}{\Gamma(q-1)} y(x)dxds. \qquad (10)$$

Using the condition

$$\sum_{i=1}^{m-1} \alpha_i u(\xi_i) = \beta \int_0^\eta \frac{(\eta - s)^{q-1}}{\Gamma(q)} u(s)ds$$

Hence in (10), we obtain

$$A = \frac{1}{\Delta}\left(\beta \int_0^\eta \frac{(\eta-s)^{q-2}}{\Gamma(q-1)} \left(\int_0^s e^{-k(t-s)} \left(\int_0^s \frac{(x-\tau)^{q-2}}{\Gamma(q-1)} y(\tau)d\tau\right)dx\right)ds\right) - \sum_{i=1}^{m-1} \alpha_i \int_0^{\xi_i} e^{-k(\xi-s)} \left(\int_0^s \frac{(s-x)^{q-2}}{\Gamma(q-1)} y(x)dx\right)ds,$$

where$\Delta$is given by (3). Substituting the value of A in (10), we get the solution (2). The converse follows by direct computation. This completes the proof. In the next lemma, we present some estimates that we need in the sequel.

We also present the following lemma involving fixed point theorem which will be useful in proving an existence and uniqueness theorem of the solution of general form of fractional differential equation given by (H. R. Marasi and H. Aydi, 2021).

Lemma 4

For $y \in C([0,1], \mathbb{R})$ with $\|y\| = \sup_{t\in[0,1]} |y(t)|$ we have

$$(i) \left|\left(\int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)} \left(\int_0^s e^{-k(s-x)} \left(\int_0^x \frac{(x-\tau)^{q-2}}{\Gamma(q-1)} y(w)d\tau\right)dx\right)ds\right)\right| \le \frac{\eta^{2q-2}}{(k\Gamma(q))^2}(k\eta + e^{-k\eta} - 1)\|y\|.$$

$$(ii) \left|\sum_{i=1}^{m-1} \alpha_i \int_0^{\xi_i} e^{-k(\xi_i - s)} \left(\int_0^s \frac{(s-x)^{q-2}}{\Gamma(q-1)} y(x)dx\right)ds\right| \le \sum_{i=1}^{m-1} |\alpha_i| \xi_i^{q-1} \left(1 - e^{-k\xi_i}\right) \frac{\|y\|}{k\Gamma(q),}$$

$$(iii) \left|\left(\int_0^t e^{-k(t-s)} \left(\int_0^s \frac{(s-x)^{q-2}}{\Gamma(q-1)} y(x)dx\right)dx\right)\right| \le \frac{1}{k\Gamma(q)}(1 - e^{-k})\|y\|, \qquad (11)$$

For computational convenience, we set

$$\mathrm{P} = \sup_{t\in[0,1]} \frac{|e^{-kt}-1|}{|\Delta|} = \frac{|e^{-k}-1|}{|\Delta|} \qquad (12)$$

$$\tilde{\mathrm{P}} = \sup_{t\in[0,1]} \frac{|ke^{-kt}|}{|\Delta|} = \frac{ke^{-k}}{|\Delta|} A_1 = P\Delta_1 + \frac{(1-e^{-k})}{k\Gamma(q)}, \qquad A_2 = \tilde{P}\Delta_1 + \frac{(2-e^{-k})}{\Gamma(q)} \qquad (13)$$

$$\omega = \zeta(1 + \lambda_o + \gamma_o) \qquad (14)$$

$$\gamma_o = \sup_{t\in I} \left|\int_0^t \gamma(t,s)ds\right|, \qquad \lambda_o = \sup_{t\in I} \left|\int_0^t \lambda(t,s)ds\right| \qquad (15)$$

$$\Delta_1 = |\beta| \frac{\eta^{2q-2}}{(k\Gamma(q))^2}(k\eta + e^{-k\eta} - 1) + \sum_{i=1}^{m} |\alpha_i| \xi_i^{q-1}(1 - e^{-k\xi}) \frac{1}{k\Gamma(q)} \qquad (16)$$

This result is based on the Banach Fixed Point theorem (M. Sebawe, 2002).

We also again provide some definitions and a fixed point thereon involving Banach Fixed Point theorem and Schauder's Fixed Point theorem

Definition 3:

Banach Fixed Point Theorem is an important tool in the theory of metric spaces; it guarantees the existence and uniqueness of fixed points of a certain self-map of metric spaces, and provide constructive methods to find those fixed points

The Banach fixed point theorem states that *"a contraction mapping on a complete Metric space has a unique fixed point". In this theorem, consider a Metric space X = (X, d), where X ≠ ∅. Suppose that X is completes and let T: X → X to be a contraction on X. Then T has precisely one fixed point.*

Definition 4:

The Schauder's Fixed Point Theorem is an extension of the Brouwer fixed point theorem to topological vector spaces, which may be of infinite dimension. It asserts that if *K* is a nonempty convex closed subset of a Hausdorff topological vector space *V* and *T* is a continuous mapping of *K* into itself such that *T*(*K*) is contained in a compact subset of *K*, then *T* has a fixed point.

A consequence, called Schaefer's fixed point theorem, is particularly useful for proving existence solutions to nonlinear partial differential equations. Schaefer's theorem is in fact a special case of the far reaching Leray – Schauder's theorem which was proved earlier by Julius Schauder's and Jean Leray.

The statement is as follows:

Let T be a continuous and compact mapping of a Banach space X into itself, such that the set $\{x \in X: x := \lambda Tx \; for \; some \; 0 \leq \lambda \leq 1\}$ is bounded. Then T has a fixed point

3. Results and Discussion

The existence and uniqueness results and has at least one solution for two or more termsnonlinear fractional differential equations of fractional orders by considering some fractional problems were established as follows:

Theorem 1 Let Assume that $f: [0,1] \times \mathbb{R}^{n+3} \to \mathbb{R}$ is a continuous function satisfying the condition (C1)

$|f(t, x, y, w, u_1, u_2, \dots, u_n) - f(t, x', y', w', v_1, v_2, \dots, v_n)| \leq L_1|x - x'| + L_2|y - y'| + L_3|w - w'| + d_1|u_1 - v_1| + d_2|u_2 - v_2| + \cdots + d_n|u_n - v_n|$, for all $t \in [0,1]$ and $x, y, w, x', y', w', u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_n \in \mathbb{R}^{n+3}$,

where$L_i, d_j > 0, \forall_i = 1,2,3, \forall_j = 1,2, \dots, n$ are Lipschitz constants

Then the Fractional differential equation has a unique solution if

$$\left(\Lambda_1 + \Lambda_2 \sum_{i=1}^{n} \frac{1}{\Gamma(2-\beta_i)}\right)\omega < 1,$$

Theorem 2:

Let$f: [0,1] \times \mathbb{R}^{n+3} \to \mathbb{R}$ be a continuous function satisfying the condition of theorem 1 (C_1) and $(C_2)|f(t, x, y, w, u_1, \dots, u_n)| \leq \mu(t), \forall \, (t, x, y, w, u_1, \dots, u_n) \in [0,1] \times \mathbb{R}^{n+3}$ where $\mu \in C([0,1], \mathbb{R}^+)$

Then, the fractional differential equation has at least one solution on [0, 1]. If

$$\left(P + \bar{P} \sum_{i=1}^{n} \frac{1}{\Gamma(2-\beta_i)}\right)\Delta_1 \omega < 1,$$

Proof. Let

$$\|\mu\| = \sup_{t \in [0,1]} |\mu(t)|,$$

and consider

$$B_R = \{u \in X: \|u\| \leq R\},$$

we fix

$$R \geq \left(\Lambda_1 + \Lambda_2 \sum_{i=1}^{n} \frac{1}{\Gamma(2-\beta_i)}\right)\|\mu\|.$$

We define the function$F_1 \; and \; F_2$ on B_R as

$$(F_1u)(t) = \int_0^t e^{-k(t-s)} \left(\int_0^s \frac{(s-x)^{q-2}}{\Gamma(q-1)} \times f(x, u(x), (\varphi u)(x), (\psi u)(x), {}^{C}D^{\beta_1}u(x), \dots, {}^{C}D^{\beta_n}u(x))dx \right) ds, \quad (17)$$

$$(F_2u)(t) = \frac{(e^{-kt}-1)}{\Delta}\left(|\beta| \int_0^{\eta} \frac{(\eta-s)^{q-1}}{\Gamma(q)} \left(\int_0^s e^{-k(s-x)} \left(\int_0^x \frac{(x-\tau)^{q-2}}{\Gamma(q-1)} \times f(\tau, u(\tau), (\varphi u)(\tau), (\psi u)(\tau), {}^{C}D^{\beta_1}u(x), \dots, {}^{C}D^{\beta_n}u(\tau)d\tau\right) dx\right) ds - \sum_{i-1}^{m-1}|\alpha_i| \int_0^{\xi_i} e^{-k(\xi_i - s)} \left(\int_0^s \frac{(s-x)^{q-2}}{\Gamma(q-1)} \times f(x, u(x), (\varphi u)(x), (\psi u)(x), {}^{C}D^{\beta_1}u(x) \dots, {}^{C}D^{\beta_n}u(x))dx\right) ds\right) \quad (18)$$

for$u, v \in B_R$, using the notation (13), we have

$$\left|(F_1u)\;(t) - (F_2v)\;(t)\right| \leq \left(p\Delta_1 + \frac{1}{k\Gamma(q)}(1 - e^{-k})\right)\|\mu\|$$

$$\left|(F_1u)\;(t) - (F_2v)\;(t)\right| \leq \Lambda_1\|\mu\|, \quad (19)$$

Also

$$|(F_1u)'(t) - (F_2v)'(t)| \leq \left(\check{p}\Delta_1 + \frac{1}{k\Gamma(q)}(2 - e^{-k})\right)\|\mu\|$$

$$|(F_1u)'(t) - (F_2v)'(t)| \leq \Lambda_2\|\mu\|, \quad (19)$$

By the definition of the Caputo fractional derivative with $0 < \beta_i < 1$

which implies that

$$\left| {}^{C}D^{\beta_1}(F_1u + F_2v) \right| \le \int_0^t \frac{(t-s)^{-\beta_i}}{\Gamma(1-\beta_i)} |F'_1u + F'_2v| ds \tag{20}$$

$$\left| {}^{C}D^{\beta_1}(F_1u + F_2v) \right| \le \frac{\Lambda_2}{\Gamma(2-\beta_i)} \|\mu\|. \tag{21}$$

From the above inequalities, we get

$$\|F_1u + F_2v\| = \|F_1u + F_2v\| + \sum_{i=1}^{n} \left\| {}^{C}D^{\beta_i}F_1u + {}^{C}D^{\beta_i}F_2v) \right\| \tag{22}$$

Using equation 3 and 5 in 6, we get

$$\|F_1u + F_2v\| \le \Lambda_1\|\mu\| + \Lambda_2 \sum_{i=1}^{n} \frac{1}{\Gamma(2-\beta_i)} \|\mu\| \tag{23}$$

$$\|F_1u + F_2v\| \le \left(\Lambda_1 + \Lambda_2 \sum_{i=1}^{n} \frac{1}{\Gamma(2-\beta_i)} \right) \|\mu\| \le R \tag{24}$$

Thus, $F_1u + F_2v \in B_R$, we prove that F_2 is a contraction mapping.

Let $u, v \in B_R$, we have

$$\begin{aligned} |F_2u(t) - F_2v(t)| \le \sup_{t[0,1]} \frac{|e^{-kt}-1|}{|\Delta|} \Bigg(|\beta| \int_0^{\eta} \frac{(\eta-s)^{q-1}}{\Gamma(q)} \Bigg(\int_0^{s} e^{-k(s-x)} \Bigg(\int_0^{x} \frac{(x-\tau)^{q-2}}{\Gamma(q-1)} \\ \times \left| f(\tau, u(\tau), (\varphi u)(\tau), (\psi u)(\tau), ({}^{C}D^{\beta_1}u)(\tau), \dots, ({}^{C}D^{\beta_n}u)(\tau)) \right. \\ \left. - f(\tau, v(\tau), (\varphi v)(\tau), (\psi v)(\tau), ({}^{C}D^{\beta_1}v)(\tau), \dots, ({}^{C}D^{\beta_n}v)(\tau)) \right| d\tau \Bigg) dx \Bigg) ds \\ - \sum_{i-1}^{m-1} |\alpha_i| \int_0^{\xi_i} e^{-k(\xi_i - s)} \Bigg(\int_0^{s} \frac{(s-x)^{q-2}}{\Gamma(q-1)} \\ \times \Big(\big| f(x, u(x), (\varphi u)(x), (\psi u)(x), {}^{C}D^{\beta_1}u(x), \dots, {}^{C}D^{\beta_n}u(x)) \\ - f(x, v(x), (\varphi v)(x), (\psi v)(x), {}^{C}D^{\beta_1}v(x), \dots, {}^{C}D^{\beta_n}v(x) \big| \Big) dx \Bigg) ds \Bigg) \\ \le p\Delta_1 \zeta (1 + \gamma_o + \lambda_o) \|u - v\| \end{aligned}$$

$$|F_2u(t) - F_2v(t)| \le p\Delta_1\omega\|u - v\| \tag{25}$$

Also,

$$|F_2'u(t) - F_2'v(t)| \le \bar{p}\Delta_1 \zeta (1 + \gamma_o + \lambda_o) \|u - v\|$$

$$|F_2'u(t) - F_2'v(t)| \le \bar{p}\Delta_1\omega\|u - v\| \tag{26}$$

Substitute equation (26) into (20)

which implies that

$$\left| {}^{C}D^{\beta_1}F_2u(t) - {}^{C}D^{\beta_i}F_2v(t) \right| \le \int_0^t \frac{(t-s)^{-\beta_i}}{(1-\beta_i)} \bar{p}\Delta_1\omega\|u - v\| ds$$

$$\left| {}^{C}D^{\beta_1}F_2u(t) - {}^{C}D^{\beta_i}F_2v(t) \right| \le \frac{1}{\Gamma(2-\beta_i)} \bar{p}\Delta_1\omega\|u - v\| \tag{27}$$

From the above inequalities, we have

$$\|F_2u - F_2v\| = \|F_2u - F_2v\| + \sum_{i=1}^{n} \left\| {}^{C}D^{\beta_1}F_2u - {}^{C}D^{\beta_1}F_2v) \right\| \tag{28}$$

Substitute equation (25) and (27) in to equation (28)

$$\|F_2u - F_2v\| \le p\Delta_1\omega\|u - v\| + \sum_{i=1}^{n} \frac{1}{\Gamma(2-\beta_i)} \bar{p}\Delta_1\omega\|u - v\| \tag{29}$$

$$\|F_1u - F_2v\| \leq \left(P + \tilde{P}\sum_{i=1}^{n}\frac{1}{\Gamma(2-\beta_i)}\right)\Delta_1\omega\|u-v\| \qquad (30)$$

Where

$$\|u - v\| < 1 \qquad (31)$$

Using equation 31 in to (30)

As:

$$\left(P + \tilde{P}\sum_{i=1}^{n}\frac{1}{\Gamma(2-\beta_i)}\right)\Delta_1\omega < 1$$

$F_2$is a contraction mapping. Then the Fractional Differential Equation has at least one solution on [0, 1]. This completes the proof.

4. Fractional Problems

Example 1: Consider the following fractional differential equation; first we show the existence and uniqueness of positive solution

$$(.^C D^q + k.^C D^{q-1})u(t) = f\left(t, u(t), (\phi u)(t), (\psi u)(t), ^C D^{\beta_i}u(t), \ldots, .^C D^{\beta_n}u(t)\right) \quad (32)$$

$t \in [0,1]$

Subject to: initial value problem

$u(0) = 0,$

Where

$$q = \frac{9}{5},\quad k = \frac{1}{11},\quad \alpha_1 = \frac{1}{6},\quad \alpha_2 = \frac{1}{3},\quad \alpha_3 = \frac{1}{5},\quad \xi_1 = \frac{1}{4},\quad \xi_2 = \frac{1}{3},\quad \xi_3 = \frac{1}{2},\quad \eta = \frac{1}{30},\quad \beta_1 = \frac{1}{2},\quad \beta_2 = \frac{3}{4},\quad \beta = 1$$

with

$$\gamma_o = \frac{e^2 - 1}{4}$$

and

$$\lambda_o = \frac{e^3 - 1}{6},$$

Solution: first we show the existence and uniqueness of positive solution

From equation (32)

$$(.^C D^q + k.^C D^{q-1})u(t) = f\left(t, u(t), (\phi u)(t), (\psi u)(t), ^C D^{\beta_i}u(t), \ldots, .^C D^{\beta_n}u(t)\right) \quad (32)$$

Let

$$\omega = f\left(t, u(t), (\phi u)(t), (\psi u)(t),\ ^cD^{\frac{1}{2}}u(t),\ ^cD^{\frac{3}{4}}u(t)\right), \qquad (33)$$

By Equating Equation (32) and (33), we have

$$(.^C D^q + k.^C D^{q-1})u(t) = \omega \qquad (34)$$

$$\sum_{i=1}^{m-1}\alpha_1 u(\xi_1) + \alpha_2 u(\xi_2) + \alpha_3 u(\xi_3) = \beta\int_0^\eta \frac{(\eta - s)^{q-1}}{\Gamma(q)}u(s)ds, \qquad (35)$$

From equation (34) and (35)

$$\left(\ ^cD^{\frac{9}{5}} + \frac{1}{11}\ ^cD^{\frac{4}{5}}\right)u(t) = \omega, \quad t \in [0,1]$$

$$u(0) = 0$$

$$\frac{1}{6}u\left(\frac{1}{4}\right) + \frac{1}{3}u\left(\frac{1}{3}\right) + \frac{1}{5}u\left(\frac{1}{2}\right) = \int_0^{\frac{1}{30}} \frac{\left(\frac{1}{30} - s\right)^{\frac{4}{5}}}{\Gamma\left(\frac{9}{5}\right)}u(s)ds$$

Now, by choosing different value of $f\left(t, u(t), (\phi u)(t), (\psi u)(t),\ ^cD^{\frac{1}{2}}u(t),\ ^cD^{\frac{3}{4}}u(t)\right)$

$$f\left(t, u(t), (\phi u)(t), (\psi u)(t), {}^{c}D^{\frac{1}{2}}u(t), {}^{c}D^{\frac{3}{4}}u(t)\right)$$
$$= \frac{2|u(t)|}{9(1+|u(t)|)} + \frac{2e^{-\frac{\pi t}{2}}sin\left(\frac{\pi}{2}t\right)}{11(1+t^3)}\left((\phi u)(t) + \frac{\left|{}^{c}D^{\frac{1}{2}}u(t)\right|}{1+\left|{}^{c}D^{\frac{1}{2}}u(t)\right|}\right) + \frac{2cost+e^t}{17(1+t^2)}\left((\phi u)(t) + \frac{\left|{}^{c}D^{\frac{3}{4}}u(t)\right|}{1+\left|{}^{c}D^{\frac{3}{4}}u(t)\right|}\right)$$

We have that

$$\left|f\left(t, u(t), (\phi u)(t), (\psi u)(t), {}^{c}D^{\frac{1}{2}}u(t), {}^{c}D^{\frac{3}{4}}u(t)\right) - f\left(t, v(t), (\phi v)(t), (\psi v)(t), {}^{c}D^{\frac{1}{2}}v(t), {}^{c}D^{\frac{3}{4}}v(t)\right)\right|$$
$$\leq \frac{2}{9}|u(t)-v(t)| + \frac{2}{11}|\emptyset u(t) - \emptyset v(t)| + \frac{2}{17}|\psi u(t) - \psi v(t)| + \frac{2}{11}\left|{}^{c}D^{\frac{1}{2}}u(t) - {}^{c}D^{\frac{1}{2}}v(t)\right| + \frac{2}{17}\left|{}^{c}D^{\frac{3}{4}}u(t) - {}^{c}D^{\frac{3}{4}}v(t)\right|$$

with the given value, it is found that

$$\zeta_1 = \frac{2}{9}, \qquad \zeta_2 = \frac{2}{11}, \qquad \zeta = \frac{2}{9},$$

where

$$\Delta = \sum_{i=1}^{m-1}\alpha_i\left(e^{-k\xi_i}-1\right) - \beta\int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)}e^{-ks}ds + \frac{\beta\eta^q}{\Gamma(q-1)} \neq 0,$$

$\Delta \approx -5.6919 \times 10^{-3} \neq 0,$

Recall that

$$\Delta_1 = |\beta|\frac{\eta^{2q-2}}{\left(k\Gamma(q)\right)^2}(k\eta + e^{-k\eta} - 1) + \sum_{i=1}^{m}|\alpha_i|\,\xi_i^{q-1}\left(1-e^{-k\xi}\right)\frac{1}{k\Gamma(q)}$$

$\Delta_1 \approx 3.7867 \times 10^{-3,}$

and

$$\mathrm{P} = \sup_{t\in[0,1]}\frac{|e^{-kt}-1|}{|\Delta|} = \frac{|e^{-k}-1|}{|\Delta|}$$

$$\Lambda_1 = P\Delta_1 + \frac{(1-e^{-k})}{k\Gamma(q)},$$

$$\Lambda_1 = \left(\sup_{t\in[0,1]}\frac{|e^{-kt}-1|}{|\Delta|} = \frac{|e^{-k}-1|}{|\Delta|}\right)|\beta|\frac{\eta^{2q-2}}{\left(k\Gamma(q)\right)^2}(k\eta + e^{-k\eta} - 1) + \sum_{i=1}^{m}|\alpha_i|\,\xi_i^{q-1}\left(1-e^{-k\xi}\right)\frac{1}{k\Gamma(q)} + \frac{(1-e^{-k})}{k\Gamma(q)},$$

$\Lambda_1 \approx 1.4022$

To find Λ_2

$$\Lambda_2 = \tilde{P}\Delta_1 + \frac{(2-e^{-k})}{\Gamma(q)}$$

Where

$$\tilde{\mathrm{P}} = \sup_{t\in[0,1]}\frac{|ke^{-kt}|}{|\Delta|} = \frac{ke^{-k}}{|\Delta|}$$
$$\Lambda_2 = \left(\sup_{t\in[0,1]}\frac{|ke^{-kt}|}{|\Delta|} = \frac{ke^{-k}}{|\Delta|}\right)\Delta_1 + \frac{(2-e^{-k})}{\Gamma(q)}$$

$\Lambda_2 \approx 1.3659,$

To find ω

where

$\omega = \zeta(1 + \lambda_o + \gamma_o)$

$\omega = \zeta(1 + \frac{e^3-1}{6}, + \frac{e^2-1}{4})$

$\omega \approx 0.39268$

Finally we have that

$\Delta \approx -5.6919 \times 10^{-3} \neq 0$

$\Delta_1 \approx 3.7867 \times 10^{-3,}$

$\Lambda_1 \approx 1.4022, \quad \Lambda_2 \approx 1.3654, \quad \omega \approx 0.39263,$

$$\left(\Lambda_1 + \Lambda_2 \sum_{i=1}^{2} \frac{1}{\Gamma(2-\beta_i)}\right)\omega = 0.9206 < 1.$$

Therefore, by theorem (1) there exists a unique solution on [0, 1].

Next, we show that

$$\left(P + \bar{P} \sum_{i=1}^{n} \frac{1}{\Gamma(2-\beta_i)}\right)\Delta_1 \omega < 1,$$

From equation (32) Give that

$$\omega = f\left(t, u(t), (\phi u)(t), (\psi u)(t), {}^{C}D^{\frac{1}{2}}u(t), {}^{C}D^{\frac{3}{4}}u(t)\right), \quad (33)$$

$$\sum_{i=1}^{m-1} \alpha_1 u(\xi_1) + \alpha_2 u(\xi_2) + \alpha_3 u(\xi_3) = \beta \int_0^{\eta} \frac{(\eta - s)^{q-1}}{\Gamma(q)} u(s) ds, \quad (34)$$

By Equating Equation (32) and (33), we have

$$(.^{C} D^{q} + k.^{C} D^{q-1}) u(t) = \omega \quad (35)$$

From equation (34) and (35)

$$\left({}^{c}D^{\frac{9}{5}} + \frac{1}{11}\, {}^{c}D^{\frac{4}{5}}\right) u(t) = \omega, \quad t \in [0,1]$$

$$u(0) = 0$$

$$\frac{1}{6} u\left(\frac{1}{4}\right) + \frac{1}{3} u\left(\frac{1}{3}\right) + \frac{1}{5} u\left(\frac{1}{2}\right) = \int_0^{\frac{1}{30}} \frac{\left(\frac{1}{30} - s\right)^{\frac{4}{5}}}{\Gamma\left(\frac{9}{5}\right)} u(s) ds$$

Now, by choosing different values of $f\left(t, u(t), (\phi u)(t), (\psi u)(t), {}^{C}D^{\frac{1}{2}}u(t), {}^{C}D^{\frac{3}{4}}u(t)\right)$

considering

$$f\left(t, u(t), (\phi u)(t), (\psi u)(t), {}^{C}D^{\frac{1}{2}}u(t), {}^{C}D^{\frac{3}{4}}u(t)\right)$$
$$= \frac{e^{-\pi i}(1 + sin^2(\pi i))|u(t)|}{6(t+6)^3(1+|u(t)|)} + \frac{e^{-\pi i} sin(\pi t)}{7(1+t^2)}\left((\phi u)(t) + \frac{\left|{}^{C}D^{\frac{1}{2}}u(t)\right|}{1 + \left|{}^{C}D^{\frac{1}{2}}u(t)\right|}\right) + \frac{1 + sin^2(\pi t)}{8(1+t^2)}\left((\psi u)(t) + \frac{\left|{}^{C}D^{\frac{3}{4}}u(t)\right|}{1 + \left|{}^{C}D^{\frac{3}{4}}u(t)\right|}\right)$$

We have

$$\left| f\left(t, u(t), (\phi u)(t), (\psi u)(t), {}^{C}D^{\frac{1}{2}}u(t), {}^{C}D^{\frac{3}{4}}u(t)\right) - f\left(t, v(t), (\phi v)(t), (\psi v)(t), {}^{C}D^{\frac{1}{2}}v(t), {}^{C}D^{\frac{3}{4}}v(t)\right)\right|$$
$$\leq \frac{1}{6}|u(t) - v(t)| + \frac{1}{7}|\emptyset u(t) - \emptyset v(t)| + \frac{1}{8}|\psi u(t) - \psi v(t)| + \frac{1}{7}\left|{}^{C}D^{\frac{1}{2}}u(t) - {}^{C}D^{\frac{1}{2}}v(t)\right| + \frac{1}{8}\left|{}^{C}D^{\frac{3}{4}}u(t) - {}^{C}D^{\frac{3}{4}}v(t)\right|$$

we have that

$$\zeta = \frac{1}{6}, \quad \zeta_2 = \frac{1}{7}, \quad \zeta_1 = \frac{1}{6},$$

Recall that

$$\Delta = \sum_{i=1}^{m-1} \alpha_i \left(e^{-k\xi_i} - 1\right) - \beta \int_0^{\eta} \frac{(\eta - s)^{q-1}}{\Gamma(q)} e^{-ks} ds + \frac{\beta \eta^q}{\Gamma(q-1)} \neq 0,$$

$\Delta \approx -2.2581 \times 10^{-2}$

where

$$\Delta_1 = |\beta| \frac{\eta^{2q-2}}{\left(k\Gamma(q)\right)^2}(k\eta + e^{-k\eta} - 1) + \sum_{i=1}^{m} |\alpha_i|\, \xi_i^{q-1}\left(1 - e^{-k\xi}\right)\frac{1}{k\Gamma(q)}$$

$\Delta_1 \approx 6.0286 \times 10^{-2,}$

and

$$\text{P} = \sup_{t\in[0,1]} \frac{|e^{-kt}-1|}{|\Delta|} = \frac{|e^{-k}-1|}{|\Delta|}$$

where

$$\Delta = \sum_{i=1}^{m-1} \alpha_i \left(e^{-k\xi_i}-1\right) - \beta \int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)} e^{-ks} ds + \frac{\beta\eta^q}{\Gamma(q-1)} \neq 0,$$

$\Delta = -2.2581 \times 10^{-2}$

$$\text{P} = \sup_{t\in[0,1]} \frac{|e^{-kt}-1|}{|\Delta|} = \frac{|e^{-k}-1|}{-2.2581 \times 10^{-2}}$$

$P \approx 3.8483,$

To find $\tilde{P}$

$$\tilde{\text{P}} = \sup_{t\in[0,1]} \frac{|ke^{-kt}|}{|\Delta|} = \frac{ke^{-k}}{|\Delta|}$$

$$\Delta = \sum_{i=1}^{m-1} \alpha_i \left(e^{-k\xi_i}-1\right) - \beta \int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)} e^{-ks} ds + \frac{\beta\eta^q}{\Gamma(q-1)} \neq 0,$$

$\Delta = -2.2581 \times 10^{-2}$

$$\tilde{\text{P}} = \sup_{t\in[0,1]} \frac{|ke^{-kt}|}{|\Delta|} = \frac{ke^{-k}}{|-2.2581 \times 10^{-2}|}$$

$\tilde{P} \approx 3.6761,$

To find $\tilde{\omega}$

$\omega = \zeta(1 + \lambda_o + \gamma_o)$

$\omega \approx 0.96303,$

$P \approx 3.8483, \quad \tilde{P} \approx 3.6761, \quad \omega \approx 0.96303,$

Thus,

$$\left(P + \tilde{P} \sum_{i=1}^{2} \frac{1}{\Gamma(2-\beta_i)}\right) \Delta_1 \omega = 0.6670 < 1.$$

Therefore, by theorem (2) has at least one solution on [0,1]

Example 2: Consider the following fractional differential equation; first we show the existence and uniqueness of positive solution

$(.^C D^q + k.^C D^{q-1})u(t) = f\left(t, u(t), (\phi u)(t), (\psi u)(t), ^C D^{\beta_1} u(t), \dots, .^C D^{\beta_n} u(t)\right)$

$t \in [0,1]$

Subject to initial value problem

$u(0) = 0,$

Where

$q = \frac{8}{5}, \quad k = \frac{1}{5}, \quad \beta_1 = \frac{1}{4}, \quad \beta_2 = \frac{3}{4}, \quad \alpha_1 = \frac{1}{7}, \quad \alpha_2 = \frac{1}{4}, \quad \alpha_3 = \frac{1}{6}$

$\xi_i = \frac{i}{10}, \quad i = 1,2,3,4, \quad \beta = 1, \quad \eta = \frac{1}{20},$

with

$\gamma_o = \frac{e^2-1}{4}$

and

$\lambda_o = \frac{e^3-1}{6},$

Solution: first we show the existence and uniqueness of positive solution, from equation (32)

$$(.^C D^q + k.^C D^{q-1})u(t) = f\left(t, u(t), (\phi u)(t), (\psi u)(t), ^C D^{\beta_i} u(t), \dots, .^C D^{\beta_n} u(t)\right)$$

let

$$\omega = f\left(t, u(t), (\phi u)(t), (\psi u)(t), {}^{C}D^{\frac{1}{4}}u(t), {}^{C}D^{\frac{3}{4}}u(t)\right), \quad (33)$$

$$\sum_{i=1}^{m-1} \alpha_1 u(\xi_1) + \alpha_2 u(\xi_2) + \alpha_3 u(\xi_3) = \beta \int_0^{\eta} \frac{(\eta - s)^{q-1}}{\Gamma(q)} u(s)ds, \quad (34)$$

By Equating Equation (32) and (33), we have

$$(.^{C}D^{q} + k.^{C}D^{q-1})u(t) = \omega \quad (35)$$

From equation (34) and (35)

$$\left({}^{c}D^{\frac{8}{5}} + \frac{1}{11}\,{}^{c}D^{\frac{3}{5}}\right)u(t) = \omega, \quad t \in [0,1]$$

$$u(0) = 0$$

$$\frac{1}{7}u\left(\frac{1}{10}\right) + \frac{1}{4}u\left(\frac{2}{10}\right) + \frac{1}{6}u\left(\frac{3}{10}\right) = \int_0^{\frac{1}{20}} \frac{\left(\frac{1}{20} - s\right)^{\frac{3}{5}}}{\Gamma\left(\frac{9}{5}\right)} u(s)ds$$

Now, by choosing different value of $f\left(t, u(t), (\phi u)(t), (\psi u)(t), {}^{C}D^{\frac{1}{4}}u(t), {}^{C}D^{\frac{3}{4}}u(t)\right)$

$$f\left(t, u(t), (\phi u)(t), (\psi u)(t), {}^{C}D^{\frac{1}{4}}u(t), {}^{C}D^{\frac{3}{4}}u(t)\right) = \frac{|u(t)|}{3(1 + |u(t)|)} + \frac{e^{-\frac{\pi t}{2}} sin\left(\frac{\pi}{2}t\right)}{13(1 + t^3)}\left((\phi u)(t) + \frac{\left|{}^{C}D^{\frac{1}{4}}u(t)\right|}{1 + \left|{}^{C}D^{\frac{1}{4}}u(t)\right|}\right) + \frac{cost + e^t}{17(1 + t^2)}\left((\phi u)(t) + \frac{\left|{}^{C}D^{\frac{3}{4}}u(t)\right|}{1 + \left|{}^{C}D^{\frac{3}{4}}u(t)\right|}\right)$$

We have that

$$\left|f\left(t, u(t), (\phi u)(t), (\psi u)(t), {}^{C}D^{\frac{1}{4}}u(t), {}^{C}D^{\frac{3}{4}}u(t)\right) - f\left(t, v(t), (\phi v)(t), (\psi v)(t), {}^{C}D^{\frac{1}{4}}v(t), {}^{C}D^{\frac{3}{4}}v(t)\right)\right| \le \frac{1}{3}|u(t) - v(t)| + \frac{1}{13}|\emptyset u(t) - \emptyset v(t)| + \frac{1}{17}|\psi u(t) - \psi v(t)| + \frac{1}{13}\left|{}^{C}D^{\frac{1}{4}}u(t) - {}^{C}D^{\frac{1}{4}}v(t)\right| + \frac{1}{17}\left|{}^{C}D^{\frac{3}{4}}u(t) - {}^{C}D^{\frac{3}{4}}v(t)\right|$$

with the given value, it is found that

$$\zeta_1 = \frac{1}{3}, \quad \zeta_2 = \frac{1}{13}, \quad \zeta = \frac{1}{3},$$

Recall that

$$\Delta = \sum_{i=1}^{m-1} \alpha_i \left(e^{-k\xi_i} - 1\right) - \beta \int_0^{\eta} \frac{(\eta - s)^{q-1}}{\Gamma(q)} e^{-ks} ds + \frac{\beta\eta^q}{\Gamma(q-1)} \neq 0,$$

$\Delta \approx -8.2345 \times 10^{-2}$

where

$$\Delta_1 = |\beta| \frac{\eta^{2q-2}}{\left(k\Gamma(q)\right)^2}(k\eta + e^{-k\eta} - 1) + \sum_{i=1}^{m} |\alpha_i|\, \xi_i^{q-1}\left(1 - e^{-k\xi}\right)\frac{1}{k\Gamma(q)}$$

$\Delta_1 \approx 7.1362 \times 10^{-2,}$

and

$$\mathrm{P} = \sup_{t\in[0,1]} \frac{|e^{-kt} - 1|}{|\Delta|} = \frac{|e^{-k} - 1|}{|\Delta|}$$

$$\Lambda_1 = P\Delta_1 + \frac{(1 - e^{-k})}{k\Gamma(q)},$$

$$\Lambda_1 = \left(\sup_{t\in[0,1]} \frac{|e^{-kt} - 1|}{|\Delta|} = \frac{|e^{-k} - 1|}{|\Delta|}\right)|\beta| \frac{\eta^{2q-2}}{\left(k\Gamma(q)\right)^2}(k\eta + e^{-k\eta} - 1) + \sum_{i=1}^{m} |\alpha_i|\, \xi_i^{q-1}\left(1 - e^{-k\xi}\right)\frac{1}{k\Gamma(q)} + \frac{(1 - e^{-k})}{k\Gamma(q)},$$

$\Lambda_1 \approx 2.8133$

To find Λ_2

$$\Lambda_2 = \tilde{P}\Delta_1 + \frac{(2-e^{-k})}{\Gamma(q)}$$

Where

$$\tilde{\mathrm{P}} = \sup_{t\in[0,1]} \frac{|ke^{-kt}|}{|\Delta|} = \frac{ke^{-k}}{|\Delta|}$$

$$\Lambda_2 = \left(\sup_{t\in[0,1]} \frac{|ke^{-kt}|}{|\Delta|} = \frac{ke^{-k}}{|\Delta|}\right)\Delta_1 + \frac{(2-e^{-k})}{\Gamma(q)}$$

$\Lambda_2 \approx 1.6793,$

To find ω

let

$\omega = \zeta(1 + \lambda_o + \gamma_o)$

$\omega = \zeta(1 + \frac{e^3-1}{6}, + \frac{e^2-1}{4})$

$\omega \approx 0.82275$

Finally we have that

$\Delta \approx -8.2345 \times 10^{-3} \neq 0$

$\Delta_1 \approx 7.1362 \times 10^{-3,}$

$\Lambda_1 \approx 2.8133, \quad \Lambda_2 \approx 1.6793, \quad \omega \approx 0.82275,$

$$\left(\Lambda_1 + \Lambda_2 \sum_{i=1}^{2} \frac{1}{\Gamma(2-\beta_i)}\right)\omega = 0.54470 < 1.$$

Therefore, by theorem (1) there exists a unique solution on [0, 1].

Next we show that

$$\left(P + \bar{P} \sum_{i=1}^{n} \frac{1}{\Gamma(2-\beta_i)}\right)\Delta_1 \omega < 1,$$

Give that

$$(.^C D^q + k.^C D^{q-1})u(t) = f\left(t, u(t), (\phi u)(t), (\psi u)(t), ^C D^{\beta_i}u(t), \dots, .^C D^{\beta_n}u(t)\right) \quad (32)$$

let

$$\omega = f\left(t, u(t), (\phi u)(t), (\psi u)(t),\ ^C D^{\frac{1}{4}}u(t),\ ^C D^{\frac{3}{4}}u(t)\right), \quad (33)$$

$$\sum_{i=1}^{m-1} \alpha_1 u(\xi_1) + \alpha_2 u(\xi_2) + \alpha_3 u(\xi_3) = \beta \int_0^{\eta} \frac{(\eta - s)^{q-1}}{\Gamma(q)} u(s)ds, \quad (34)$$

By Equating Equation (32) and (33), we have

$$(.^C D^q + k.^C D^{q-1})u(t) = \omega \quad (35)$$

From equation (34) and (35)

$$\left(\ ^c D^{\frac{8}{5}} + \frac{1}{11}\ ^c D^{\frac{3}{5}}\right)u(t) = \omega, \quad t \in [0,1]$$

$u(0) = 0$

$$\frac{1}{7}u\left(\frac{1}{10}\right) + \frac{1}{4}u\left(\frac{2}{10}\right) + \frac{1}{6}u\left(\frac{3}{10}\right) = \int_0^{\frac{1}{20}} \frac{\left(\frac{1}{20} - s\right)^{\frac{3}{5}}}{\Gamma\left(\frac{9}{5}\right)} u(s)ds$$

Now, by choosing different values of $f\left(t, u(t), (\phi u)(t), (\psi u)(t),\ ^C D^{\frac{1}{4}}u(t),\ ^C D^{\frac{3}{4}}u(t)\right)$

Considering

$$f\left(t,u(t),(\phi u)(t),(\psi u)(t),{}^{c}D^{\frac{1}{4}}u(t),{}^{c}D^{\frac{3}{4}}u(t)\right)$$
$$=\frac{e^{-\pi i}(1+sin^2(\pi i))|u(t)|}{18(t+6)^3(1+|u(t)|)}+\frac{e^{-\pi i}sin(\pi t)}{21(1+t^2)}\left((\phi u)(t)+\frac{\left|{}^{c}D^{\frac{1}{4}}u(t)\right|}{1+\left|{}^{c}D^{\frac{1}{4}}u(t)\right|}\right)+\frac{1+sin^2(\pi t)}{24(1+t^2)}\left((\psi u)(t)+\frac{\left|{}^{c}D^{\frac{3}{4}}u(t)\right|}{1+\left|{}^{c}D^{\frac{3}{4}}u(t)\right|}\right)$$

We have

$$\left|f\left(t,u(t),(\phi u)(t),(\psi u)(t),{}^{c}D^{\frac{1}{4}}u(t),{}^{c}D^{\frac{3}{4}}u(t)\right)-f\left(t,v(t),(\phi v)(t),(\psi v)(t),{}^{c}D^{\frac{1}{4}}v(t),{}^{c}D^{\frac{3}{4}}v(t)\right)\right|$$
$$\leq\frac{1}{18}|u(t)-v(t)|+\frac{1}{21}|\emptyset u(t)-\emptyset v(t)|+\frac{1}{24}|\psi u(t)-\psi v(t)|+\frac{1}{21}\left|{}^{c}D^{\frac{1}{4}}u(t)-{}^{c}D^{\frac{1}{4}}v(t)\right|+\frac{1}{24}\left|{}^{c}D^{\frac{3}{4}}u(t)-{}^{c}D^{\frac{3}{4}}v(t)\right|$$

Finally, we have that

$$\zeta_1=\frac{1}{18},\qquad \zeta_2=\frac{1}{21},\qquad \zeta=\frac{1}{18}$$

where

$$\Delta=\sum_{i=1}^{m-1}\alpha_i\left(e^{-k\xi_i}-1\right)-\beta\int_0^{\eta}\frac{(\eta-s)^{q-1}}{\Gamma(q)}e^{-ks}ds+\frac{\beta\eta^q}{\Gamma(q-1)}\neq 0,$$

$\Delta\approx -4.5161\times 10^{-2}$

Recall that

$$\Delta_1=|\beta|\frac{\eta^{2q-2}}{\left(k\Gamma(q)\right)^2}(k\eta+e^{-k\eta}-1)+\sum_{i=1}^{m}|\alpha_i|\,\xi_i^{q-1}\left(1-e^{-k\xi}\right)\frac{1}{k\Gamma(q)}$$

$\Delta_1\approx 9.0186\times 10^{-2,}$

and

$$\text{P}=\sup_{t\in[0,1]}\frac{|e^{-kt}-1|}{|\Delta|}=\frac{|e^{-k}-1|}{|\Delta|}$$

where

$$\Delta=\sum_{i=1}^{m-1}\alpha_i\left(e^{-k\xi_i}-1\right)-\beta\int_0^{\eta}\frac{(\eta-s)^{q-1}}{\Gamma(q)}e^{-ks}ds+\frac{\beta\eta^q}{\Gamma(q-1)}\neq 0,$$

$\Delta= -3.5734\times 10^{-2}$

$$\text{P}=\sup_{t\in[0,1]}\frac{|e^{-kt}-1|}{|\Delta|}=\frac{|e^{-k}-1|}{-3.5734\times 10^{-2}}$$

$P\approx 5.92390,$

To find $\tilde{P}$

$$\tilde{\text{P}}=\sup_{t\in[0,1]}\frac{|ke^{-kt}|}{|\Delta|}=\frac{ke^{-k}}{|\Delta|}$$

$$\Delta=\sum_{i=1}^{m-1}\alpha_i\left(e^{-k\xi_i}-1\right)-\beta\int_0^{\eta}\frac{(\eta-s)^{q-1}}{\Gamma(q)}e^{-ks}ds+\frac{\beta\eta^q}{\Gamma(q-1)}\neq 0,$$

$\Delta= -4.5161\times 10^{-2}$

$$\tilde{\text{P}}=\sup_{t\in[0,1]}\frac{|ke^{-kt}|}{|\Delta|}=\frac{ke^{-k}}{|-4.5161\times 10^{-2}|}$$

$\tilde{P}\approx 5.7894,$

To find $\tilde{\omega}$

$\omega=\zeta(1+\lambda_o+\gamma_o)$

$\omega\approx 0.85230,$

Finally,

$P\approx 5.92390,\quad \tilde{P}\approx 5.7894,\quad \omega\approx 0.85230,$

Thus,

$$\left(P + \tilde{P}\sum_{i=1}^{2}\frac{1}{\Gamma(2-\beta_i)}\right)\Delta_1\omega = 0.8850 < 1.$$

Therefore, by theorem (2) the problem has at least one solution on [0,1]

3. Conclusion

Based on the study of the general form of existence and uniqueness results for two-term nonlinear fractional differential equations via a fixed point technique given by (Marasi and Aydi, 2021), new results concerning the existence and uniqueness of positive results for two or more term nonlinear fractional differential equation via a fixed point technique were established and illustrated by considering some fractional problems

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