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An Analytical Approximate Solutions for Bratu-Type Differential Equations

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ABSTRACT.

The primary objective of our research is to explore a novel approach for solving Bratu-Type differential equations. This approach combines the differential transform method (DTM) with Adomian's polynomials. Although the differential transform method (DTM) is a powerful tool, it does have limitations, particularly in dealing with the derivation of differential transforms for exponential nonlinearity. In this thesis, we demonstrate how Adomian's polynomials can effectively address this deficiency, enhancing the DTM. We provide comprehensive illustrations of the application of this hybrid method in solving Bratu-Type differential equations, as well as linear and nonlinear initial value problems (IVPs) through numerous examples. Our numerical experiments showcase the effectiveness of this approach in solving Bratu-Type differential equations, yielding results that closely align with exact solutions. Ultimately, our research confirms the superior accuracy of the hybrid method, which minimizes absolute errors when compared to other existing methods, making it an excellent choice for addressing Bratu-Type differential equations as well as the linear and nonlinear IVPs.

Keywords: Bratu-type differential equation; Differential Transform Method (DTM); Adomian's Polynomials.

1.0 Introduction

Bratu's equation is a nonlinear differential equation that has many applications in mathematics, physics, engineering and other sciences Rashidi, *et. al*, (2020). Moreover, Bratu equation is formulated in the form of a non-linear problem with initial or boundary conditions. In this work, the study will deal with the Bratu type problem in one-dimensional with the initial conditions, which are as follows:

$$u''(x)+\lambda e^{u(x)}=0,\ 0<x<1,\ \lambda>0,\ u(0)=u'(0)=0. \qquad \ldots(1)$$

This problem derives its importance from the first thermal combustion theory, which was created by the simplification of the solid fuel ignition model. Moreover, it appeared in the Chandrasekhar model of the expansion of the universe. It stimulates a thermal reaction process in a rigid material where the process depends on a balance between chemically generated heat and heat transfer by conduction (Wazwaz, 2005). The exact solution was known, which facilitates the application of tests in different methods by comparing with the approximate solutions which showed the accuracy and efficiency of the methods (Ali & Younis, 2022). In real world, many physical and natural phenomena are formulated as differential equations. Most of these differential equations are nonlinear. So there are difficulties in finding the exact or analytical solutions caused by the nonlinear part Ghorbani (2009). Many methods have been proposed to solve or approximate nonlinear differential equations. For examples: Adomian decomposition method (ADM) Tate & Dinde (2019), variational iteration method (VIM) (Wazwaz, 2009), homotopy perturbation method (HPM) Momani & Odibat (2007), differential transform method (DTM) Rashidi, et. al, (2020) and many others, although these methods provides some useful solutions, but these methods need calculations with some restrictions, linearization and transformations; also in some cases to get a good convergence more terms are needed. Nonlinear phenomena play a crucial role in designing more realistic mathematical models to describe the feasibility of nature, so there is a need for a method that can handle exponential nonlinear terms easily without any restrictions, linearization or transformations. Indeed, the so called Differential Transform Method (DTM) which gives a series solutions can overcome some of the above difficulties. The DTM is very effective numerical and analytical method for solving different types of differential equations as well as integral equations. This method converts the differential equations into recurrence relations, and then by Taylor series expansion, with a different approach, it obtains convergent series solutions. The concept of DTM was first introduced by Zhou in 1986 to solve linear and nonlinear initial value problems in electrical circuit analysis Ayaz (2003). In this study, the DTM will be modified to solve the Bratu type equation with exponential nonlinearity. The purpose of this study is to modify the DTM by using the Adomian polynomial's to decompose the exponential nonlinearity.

2.0 Definition of Terms and some Basic Properties

Definition 1

Differential transform of a function $y(x)$ is defined as follows:

$$Y(k)=\frac{1}{k!}\left[\frac{d^k y(x)}{dx^k}\right] \qquad ...(2)$$

where $Y(k)$ is a transformed function. The inverse of $Y(k)$ is express as

$$y(x)=\sum_{k=0}^{\infty}Y(k)x^k \approx Y_N(x)=\sum_{k=0}^{N}Y(k)x^k \qquad ...(3)$$

Definition 2

An equation containing derivatives of one or more dependent variables, with respect to one or more independent variables, is said to be a differential equation (DE).

A differential equation is of the form:

$$y^n(x)+a_1y^{n-1}(x)+...+a_{n-1}y'(x)+a_ny(x)=0 \qquad ...(4)$$

where a_n, $a_{n-1,...,}$ a_1 are real constants with the initial conditions;

$$y(0)=y_0,\ y'(0)=y_1,\ ...,\ y^{(n-1)}(0)=y_{(n-1)} \qquad ...(5)$$

where $y_0,\ y_0',..., y_0^{(n-1)}$ are real constants.

3.0 Basics of the Differential Transform Method

Let the differential transform of an arbitrary function $y=f(x)$ as

$$Y(k)=\frac{1}{k!}\left[\frac{d^k y(x)}{dx^k}\right] \qquad ...(6)$$

where $y(x)$ is the original function and $Y(k)$ is the transformed function. We can write the inverse differential transform of $Y(k)$ as

$$y(x)=\sum_{k0}^{\infty}Y(k)x^k \approx Y_N(x)=\sum_{k=0}^{N}Y(k)x^k \qquad ...(7)$$

The function $y(x)$ can then be written as a finite series with eqn. (7) stated as

$$y(x)=\sum_{k0}^{\infty}Y(k)x^k \qquad ...(8)$$

The following theorems can be derived from eqn. (6), (7) and (8)

i. If $y(x)=u(x)\pm v(x)$, then $Y(k)=U(k)\pm V(k)$
ii. If $y(x)=\alpha u(x)$, then $Y(k)=\alpha U(k)$
iii. If $y(x)=y'(x)$, then $Y(k)=(k+1)Y(k+1)$
iv. If $y(x)=y''(x)$, then $Y(k)=(k+1)(k+2)Y(k+2)$
v. If $y(x)=y^{(n)}(x)$, then $Y(k)=(k+1)(k+2)...(k+n)Y(k+m)$
vi. If $y(x)=x^n(x)$, then $Y(k)=\delta(k-n)=\begin{cases}1, k=n\\ 0, otherwise\end{cases}$

vii. If $y(x)=e^{\lambda k}(x)$, then $Y(k)=\dfrac{\lambda^k}{k!}$

4.0 Adomian's Polynomials and their evaluation

Adomian polynomials are indispensable in nonlinear analyses by the ADM. Let N be a nonlinear operator acting upon an unknown function$y$. For treating functional equations including such nonlinear terms likeNu, ADM entails decomposition of Nyinto an infinite summation of the Adomian's polynomials, $A_n s$, corresponding to Nas:

$$Ny=\sum_{n=0}^{\infty}A_n, \qquad \ldots(9)$$

where $H_n s$ are classically suggested in Ghorbani (2009) and to be obtained by

$$A_n(y_0,\ldots,y_n)=\frac{1}{n!}\frac{d^n}{dp^n}[F(\textstyle\sum_{i=0}^{n}p^i y_i)]_{p=0} \quad , \quad n\ge 0 \qquad \ldots(10)$$

This gives

$$A_0=F(y_0),$$

$$A_1=\frac{d}{dp}[F(\textstyle\sum_{i=0}^{1}p^i y_i)]_{p=0}=y_1F'(y_0),$$

$$A_2=\frac{1}{2!}\frac{d^2}{dp^2}[F(\textstyle\sum_{i=0}^{2}p^i y_i)]_{p=0}=y_2F'(y_0)+\frac{1}{2!}\,y_1^2F''(y_0),$$

$$A_3=\frac{1}{3!}\frac{d^3}{dp^3}[F(\textstyle\sum_{i=0}^{3}p^i y_i)]_{p=0}=u_3F'(y_0)+y_1y_2F''(y_0)+\frac{1}{3!}\,y_1^3F'''(y_0).$$

5.0 Analysis of the proposed DTM

We illustrate the steps of the Hybrid of DTM and Adomian polynomial by considering the Bratu-Type differential equation, first-order initial value problem (IVP) and second-order IVP.

6.0 Bratu-Type Differential Equation

$$y''(x)-\lambda e^{y(x)}=0,\ y(0)=0,\ y'(0)=0,\ 0<x<1, \qquad \ldots(11)$$

By applying subsection **3.0** and **4.0** on Eqn. (11), we get

$$\frac{(k+1)!}{k!}Y_{k+1}-[A_n]=0 \qquad \ldots(12)$$

$$\frac{(k+2)!}{k!}Y_{k+2}(x)=\lambda A_n \qquad \ldots(13)$$

This leads to the following recurrence relation

$$Y_{k+2}(x)=\frac{k!A_n(x)}{(k+2)!} \qquad \ldots(14)$$

Where $k=0,1,2,\ldots$ and $n=0,1,2,\ldots$

and from initial conditions we get

$$y(0)=0=Y_0(x)=0 \qquad \ldots(15)$$

$$y'(0)=0=Y_1(x)=0 \qquad \ldots(16)$$

Lastly, the series solution is obtained as follows:

$$Y(x)=Y_0x^0+Y_1x^1+\ldots+Y_nx^n \qquad \ldots(17)$$

7.0 RESULTS AND DISCUSSION

The effectiveness of the method for solving Bratu type differential equation is shown above.

Example 1:

We consider the second order homogeneous nonlinear differential equation with exponential source term (Bratu-Type) [Vahidi (2012)]:

$$y''(x) - 2e^{y(x)} = 0,\ y(0) = 0,\ y'(0) = 0,\ 0 < x < 1, \qquad \ldots(18)$$

With the exact solution

$$y(x) = -2\ln(\cos x) \qquad \ldots(19)$$

Now, applying the Hybrid method on Eqn. (18), gives:

$$\frac{(k+2)!}{k!} Y_{k+2}(x) = 2e^{y(x)} \qquad \ldots(20)$$

This leads to the following recurrence relation

$$Y_{k+2}(x) = \frac{k!\, A_n(x)}{(k+2)!} \qquad \ldots(21)$$

Where

$$A_k = e^{Y_k(x)} \qquad \ldots(22)$$

and from initial conditions we get

$$y(0) = 0 = Y_0(x) = 0 \qquad \ldots(23)$$

$$y'(0) = 0 = Y_1(x) = 0 \qquad \ldots(24)$$

Applying the transform initial condition Eqn. (23) and Eqn. (24) in Eqn. (21), we obtain the following

When k = 0, Eqn. (21) becomes

$$Y_2(x) = \frac{0! \times 2 \times e^{Y_0}}{(0+2)!} = 1 \qquad \ldots(25)$$

When k = 1, Eqn. (21) becomes

$$Y_3(x) = \frac{1! \times 2 \times y_1 \times e^{Y_0}}{(1+2)!} = 0 \qquad \ldots(26)$$

When k = 2, Eqn. (21) becomes

$$Y_4(x) = \frac{2! \times 2 \times \left(Y_2 + \frac{1}{2!} \times (Y_1)^2 \right) e^{Y_0}}{(2+2)!} = \frac{1}{6} \qquad \ldots(27)$$

When k = 3, Eqn. (21) becomes

$$Y_5(x) = \frac{3! \times 2 \times \left(Y_3 + Y_1 Y_2 + \frac{1}{3!} \times (Y_1)^3 \right) e^{Y_0}}{(3+2)!} = 0 \qquad \ldots(28)$$

When k = 4, Eqn. (21) becomes

$$Y_6(x) = \frac{4! \times 2 \times \left(Y_4 + Y_1 Y_3 + \frac{1}{2!} \times (Y_2)^2 + \frac{1}{2!} \times (Y_1)^2 + + \frac{1}{4!} \times (Y_1)^4 \right) e^{Y_0}}{(4+2)!} = \frac{2}{45} \quad \ldots(29)$$

$$Y_7(x) = 0 \quad \ldots(30)$$

When k = 6, Eqn. (21) becomes

$$Y_8(x) = \frac{17}{1260} \quad \ldots(31)$$

When k = 7, Eqn. (21) becomes

$$Y_9(x) = 0 \quad \ldots(32)$$

When k = 8, Eqn. (21) becomes

$$Y_{10}(x) = \frac{62}{14175} \quad \ldots(33)$$

We finally obtain the series solution up to seventh term as

$$Y(x) \approx \sum_{k=0}^{10} Y_k x^k \approx x^2 + \frac{1}{6} x^4 + \frac{2}{45} x^6 + \frac{17}{1260} x^8 + \frac{62}{14175} x^{10} \quad \ldots(34)$$

Table 1: Numerical solution of MDTM, RADM and Exact solution

x	Exact solution	Hybrid method (k=10)	RADM (n=12)
0	0	0	0
0.1	0.010016711	0.010016711	0.010016711
0.2	0.040269546	0.040269546	0.040269546
0.3	0.091383312	0.091383311	0.091383285
0.4	0.164458038	0.164458012	0.164457553
0.5	0.261168481	0.261168086	0.261163814
0.6	0.383930339	0.383926662	0.383900215
0.7	0.536171515	0.536146854	0.536023302
0.8	0.722781494	0.722650747	0.722181104
0.9	0.950884887	0.950302575	0.948777491

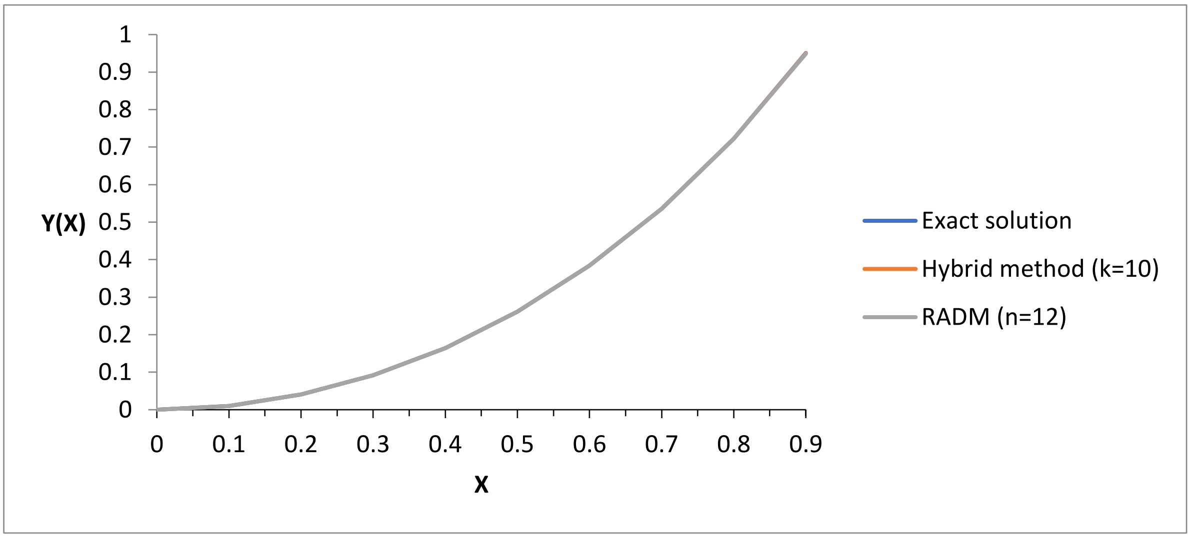

Figure 1: the graph of numerical solution for exact solution, Hybrid method and RADM

Figure 1 shows the graph for exact solution, Hybrid method and RADM that was obtained by using Maple18 software. The trend of the graph for Hybrid method and exact solution is similar but graph for RADM slightly diverges from the exact solution. This shows that the numerical solution for Hybrid method is closer to the exact solution as compared to the RADM.

Table 2: the absolute error of Hybrid method, RADM and the minimum absolute error to the exact solution

x	**Exact solution**	**Hybrid method (k=10)**	**RADM (n=12)**	**Hybrid Error**	**RADM Error**	**Minimum**
0	0	0	0	0	0	0
0.1	0.010016711	0.010016711	0.010016711	1.3704E-15	4.3876E-13	1.37E-15
0.2	0.040269546	0.040269546	0.040269546	6.1359E-12	4.5402E-10	6.14E-12
0.3	0.091383312	0.091383311	0.091383285	8.104E-10	2.6638E-08	8.1E-10
0.4	0.164458038	0.164458012	0.164457553	2.6244E-08	4.8488E-07	2.62E-08
0.5	0.261168481	0.261168086	0.261163814	3.9502E-07	4.6664E-06	3.95E-07
0.6	0.383930339	0.383926662	0.383900215	3.6767E-06	3.0124E-05	3.68E-06
0.7	0.536171515	0.536146854	0.536023302	2.4662E-05	0.00014821	2.47E-05
0.8	0.722781494	0.722650747	0.722181104	0.00013075	0.00060039	0.000131
0.9	0.950884887	0.950302575	0.948777491	0.00058231	0.0021074	0.000582

Example 2:

We introduce the following Bratu-type differential equation (Demir & Zeybek, 2017):

$$y''(x) - e^{2y(x)} = 0,\ y(0) = 0,\ y'(0) = 0,\ 0 < x < 1, \quad \ldots(35)$$

With the exact solution

$$y(x) = \ln\left(\sec(x)\right) \quad \ldots(36)$$

Now, applying the Hybrid method on Eqn. (35), gives:

$$\frac{(k+2)!}{k!} Y_{k+2}(x) = e^{2y(x)} \quad \ldots(37)$$

This leads to the following recurrence relation

$$Y_{k+2}(x) = \frac{k!\,A_n(x)}{(k+2)!} \quad \ldots(38)$$

Where

$$A_k = e^{Y_k(x)} \quad \ldots(39)$$

and from initial conditions we get

$$y(0) = 0 = Y_0(x) = 0 \quad \ldots(40)$$

$$y'(0) = 0 = Y_1(x) = 0 \quad \ldots(41)$$

Applying the transform initial condition Eqn. (40) and Eqn. (41) in Eqn. (37), we obtain the following

When $k = 0$, Eqn. (43) becomes

$$Y_2(x) = \frac{0! \times e^{2Y_0}}{(0+2)!} = \frac{1}{2} \quad \ldots(42)$$

When $k = 1$, Eqn. (37) becomes

$$Y_3(x) = \frac{1! \times 2y_1 \times e^{2Y_0}}{(1+2)!} = 0 \quad \ldots(43)$$

When $k = 2$, Eqn. (37) becomes

$$Y_4(x) = \frac{2! \times 2 \times \left(\frac{1}{2} + (Y_1)^2\right) e^{2Y_0}}{(2+2)!} = \frac{1}{12} \qquad ...(44)$$

When k = 3, Eqn. (37) becomes

$$Y_5(x) = 0 \qquad ...(46)$$

When k = 4, Eqn. (37) becomes

$$Y_6(x) = \frac{1}{45} \qquad ...(47)$$

$$Y_7(x) = 0 \qquad ...(48)$$

When k = 6, Eqn. (37) becomes

$$Y_8(x) = \frac{17}{2520} \qquad ...(49)$$

When k = 7, Eqn. (37) becomes

$$Y_9(x) = 0 \qquad ...(50)$$

When k = 8, Eqn. (37) becomes

$$Y_{10}(x) = \frac{31}{14175} \qquad ...(51)$$

We finally obtain the series solution up to Tenth term as

$$Y(x) \approx \sum_{k=0}^{10} Y_k x^k \approx \frac{1}{2}x^2 + \frac{1}{12}x^4 + \frac{1}{45}x^6 + \frac{17}{2520}x^8 + \frac{31}{14175}x^{10} + \ldots \qquad ...(52)$$

Table 3: Numerical Results of Example 2

x	Exact solution	Hybrid method	Absolute Error
0	0	0	0
0.1	0.005008355	0.00500836	3.98235E-10
0.2	0.020134773	0.02013477	6.06596E-11
0.3	0.045691656	0.04569166	8.49143E-10
0.4	0.082229019	0.08222901	1.27669E-08
0.5	0.13058424	0.13058404	1.97266E-07
0.6	0.19196517	0.19196333	1.83873E-06
0.7	0.268085758	0.26807343	1.2331E-05
0.8	0.361390747	0.36132537	6.53733E-05
0.9	0.475442443	0.47515129	0.000291156
1	0.615626471	0.61448854	0.001137934

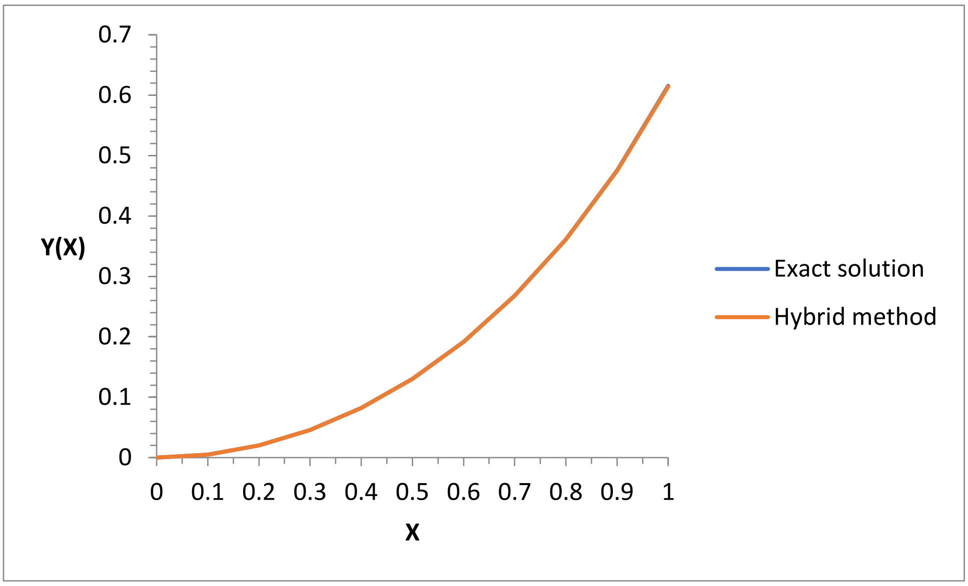

Figure 2: Comparison of the exact solution and the approximate solution by the Hybrid method

4.0 Conclusion

In conclusion, the Differential Transform Method (DTM) is a widely used approach for addressing various types of functional equations. While it has proven effective in many cases, it faces challenges when dealing with exponential nonlinearity, especially in Bratu-Type differential equations. The inability to handle these exponential nonlinearities with existing methods led us to develop a new approach using Adomian's polynomials, which effectively overcomes this limitation. This study has not only successfully demonstrated the feasibility of our modified DTM but also showcased its versatility by applying it to solve Bratu-Type differential equations and nonlinear initial value problems. The results clearly indicate that our Hybrid method aligns closely with exact solutions in series form across various problem scenarios. Notably, it distinguishes itself by eliminating the need for nonlinearity transformation, perturbation, or linearization.

We believe that the outcomes presented in this work significantly expand the practicality and appeal of the Hybrid method. By making it easier to solve nonlinear differential equations without the burden of complex algebraic computations, we recommend this method as a valuable alternative to other semi-analytical approaches for future applications.

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